Existence of sparsely supported correlated equilibria*

Fabrizio Germano
fabrizio.germano@upf.edu

Gábor Lugosi
lugosi@upf.es

Departament d'Economia i Empresa
Universitat Pompeu Fabra
Ramon Trias Fargas 25-27
08005 Barcelona, Spain

April 2006

Abstract

We show that every N-player $K_1 \times \cdots \times K_N$ game possesses a correlated equilibrium with at least $\prod_{i=1}^N K_i - 1 - \sum_{i=1}^N K_i(K_i - 1)$ zero entries. In particular, the largest N-player $K \times \cdots \times K$ games with unique fully supported correlated equilibrium are two-player games.
Keywords Correlated equilibrium; support; finite games.

1 The result

Consider an N-player $K_1 \times \cdots \times K_N$ normal form game $\gamma = (N, S, \{\gamma^i\}_{i=1}^N)$, where, for each player $i = 1, \ldots, N$, S_i is a set of pure strategies with

*We thank an anonymous referee for useful comments. The first author acknowledges financial support from the Spanish Ministry of Science and Technology, grant SEJ2004-03619, and in form of a Ramón y Cajal fellowship. The second author acknowledges support by the PASCAL Network of Excellence under EC grant no. 506778, as well as from Spanish Ministry of Science and Technology and FEDER, grant BMF2003-03324. Both authors also acknowledge financial support from BBVA grant "Aprender a jugar."

$K_i = \#S_i \geq 2$ and $\gamma^i : S \to I\!R$ is a payoff function defined on the set of pure strategy profiles $S = \times_{i=1}^N S_i$. Denote by $\Delta(S)$ the set of probability distributions on S and by $S_{-i} = \times_{j \neq i} S_j$ the set of pure strategy profiles of the players other than i. Given $s^i \in S_i$ and $s^{-i} \in S_{-i}$, we sometimes write $s = (s^{-i}, s^i)$ for a generic element of S.

A probability distribution $p = (p(s))_{s \in S} \in \Delta(S)$ is a *correlated equilibrium* of the game γ if it satisfies, for all $i = 1, \ldots, N$, and all $s^i, t^i \in S_i$ with $s^i \neq t^i$,

$$\sum_{s^{-i} \in S_{-i}} p(s^{-i}, s^i) \left(\gamma^i(s^{-i}, s^i) - \gamma^i(s^{-i}, t^i) \right) \geq 0.$$

We refer to these $\sum_{i=1}^N K_i(K_i - 1)$ inequalities as the *incentive constraints*. The notion of correlated equilibrium was introduced by Aumann [2] as a rich generalization of Nash equilibrium. The set $\mathcal{C} \subset \Delta(S)$ of correlated equilibria is a nonempty convex polytope defined by the incentive constraints, as well as the nonnegativity constraints, $p(s) \geq 0$, $s \in S$, and the constraint $\sum_{s \in S} p(s) = 1$, the latter two guaranteeing $p \in \Delta(S)$. Recall that a correlated equilibrium that is a product measure is also a Nash equilibrium.

The purpose of this note is to point out that games with many players have sparsely supported correlated equilibria. More precisely, the main result is the following.

Theorem 1 *Any N-player $K_1 \times \cdots \times K_N$ game possesses a correlated equilibrium with at least $\prod_{i=1}^N K_i - 1 - \sum_{i=1}^N K_i(K_i - 1)$ zero entries.*

To better understand the result, consider an N-player $2 \times \cdots \times 2$ game. There are 2^N pure strategy profiles, yet the theorem implies the existence of a correlated equilibrium with at least $2^N - 1 - 2N$ zeros in its support. Thus, there always exists a correlated equilibrium concentrated on an *exponentially small fraction* of pure strategy profiles.

The result also has implications for the (non)existence of games with unique fully supported correlated equilibria.

Corollary 1 *For $N \geq 3$, there exist no $K \times \cdots \times K$ games with unique fully supported correlated equilibrium.*

To see this, notice that, for any $N \geq 3$ and $K \geq 2$, we have

$$K^{N-1} - N(K-1) \geq 1.$$

Hence $K^N - NK^2 + NK = K(K^{N-1} - N(K-1)) \geq 2$ since $K \geq 2$. Therefore,

$$\prod_{i=1}^{N} K_i - 1 - \sum_{i=1}^{N} K_i(K_i - 1) = K^N - NK^2 + NK - 1 \geq 1,$$

which means that, for $N \geq 3$, there always exists a correlated equilibrium with at least one zero entry and which cannot be fully supported.

This is to be contrasted with Nitzan [4], who shows that the set of two-player $K \times K$ games possessing a unique fully supported correlated equilibrium has positive measure for any K. It complements Nitzan's results by showing that not only do such games have zero measure as soon as $N \geq 3$, but that such games simply cannot exist in these "remaining" cases.

Finally, recall that if an N-player game has a unique correlated equilibrium, then it must also be a Nash equilibrium and hence a product measure. For simplicity, consider again $2 \times \cdots \times 2$ games. The correlated equilibrium has at least $2^N - 1 - 2N$ zero entries and thus at most $2N + 1$ atoms, and, since it is a product measure, there are at most $\log_2(2N + 1)$ non-degenerate marginal distributions. This implies the following fact suggesting that large games with a unique correlated equilibrium must be quite "degenerate" in some sense.

Corollary 2 *Consider an N-player $2 \times \cdots \times 2$ game with a unique correlated equilibrium. At this equilibrium there are at most $\log_2(2N + 1)$ players who use a non-degenerate mixed strategy, all others play a pure strategy.*

2 The proof

To simplify notation, set $d = \prod_{i=1}^{N} K_i - 1$, $m = \sum_{i=1}^{N} K_i(K_i - 1)$, and assume, without loss of generality, $d > m$. We identify a probability distribution $p \in \Delta(S)$ over the set of pure strategy profiles with a $(d+1)$-vector p of non-negative components p_j, satisfying $\sum_{j=1}^{d+1} p_j = 1$. Each incentive constraint takes the form of a linear inequality, which can be written as $C_k p \geq 0$, $k = 1, \ldots, m$.

Fix $q^0 \in \mathcal{C}$, which exists since $\mathcal{C}$ is nonempty, (e.g., Aumann [2], Hart and Schmeidler [3]), and consider the affine subspace,

$$H_0 = \{p \in I\!R^{d+1} : \sum_{j=1}^{d+1} p_j = 1 \text{ and } C_k p = C_k q^0, \quad k = 1, \ldots, m\}.$$

By the dimension theorem, (e.g., Artin [1]), H_0 has dimension at least $d - m$, and any point in H_0 satisfies all the incentive constraints defining $\mathcal{C}$. Moreover, since $H_0 \subset I\!R^{d+1}$ is defined by $m+1$ equalities, there exist $d-m$ entries whose values can be set arbitrarily, yet the system of $m+1$ equations defining H_0 has a solution with these restricted values in the $d-m$ entries. In particular, there exists $\overline{q}^0 \in H_0$ in which $d-m$ entries are equal to, say, -1, and, without loss of generality, we can assume $\overline{q}^0$ to be of the form

$$\overline{q}^0 = (-1, \ldots, -1; p_{d-m+1}, \ldots, p_{d-m_1}; p_{d-m_1+1}, \ldots, p_{d+1}), \qquad m_1 \leq m,$$

where the first $d-m$ entries are -1's, the next $m-m_1$ entries, $p_{d-m-1}, .., p_{d-m_1}$, are nonpositive, and the remaining m_1+1 entries, $p_{d-m_1+1}, .., p_{d+1}$, are positive (m_1+1 is thus the number of positive entries of $\overline{q}^0$).

Consider now the line segment $L \subset H_0$ between q^0 and $\overline{q}^0$. It intersects the union of hyperplanes, $\cup_{j=1}^{d-m_1}\{p : p_j = 0\}$, at least once and at most $d - m_1$ times. Take the first intersection encountered when moving from q^0 ($\in \mathcal{C}$) towards $\overline{q}^0$ along L and denote the point of intersection by $q^1 \in \mathcal{C}$. Let $N_1 \subset \{1, \ldots, d - m_1\}$ be the set of nonnegativity constraints holding with equality at q^1. Set $\#N_1 = n_1$ and notice that, since $d - m > 0$, we have $n_1 > 0$. If $d - m - n_1 \leq 0$, then we are done, since we have found a point $q^1 \in \mathcal{C}$ with at least $n_1 \geq d - m$ zero entries. If, however, $d - m - n_1 > 0$, then repeating the procedure (at most $d-m$ times) will eventually lead to a point in $\mathcal{C}$ with the desired property. More specifically, starting with $\ell = 1$, consider the following.

PROCEDURE: Suppose $q^\ell \in \mathcal{C}$ is given, together with the corresponding set $N_\ell \subset \{1, \ldots, d - m_\ell\}$, and numbers n_ℓ and m_ℓ, and suppose $d - m - n_\ell > 0$. Consider the affine subspace

$$H_\ell = \{p \in I\!\!R^{d+1} : p_j = 0, j \in N_\ell, \ \sum_{j=1}^{d+1} p_j = 1, \ \ C_k p = C_k q^\ell, k = 1, \ldots, m\}.$$

By the dimension theorem, H_ℓ has dimension at least $d - m - n_\ell$, and any point in H_ℓ satisfies: the nonnegativity constraints in N_ℓ with equality, the constraint $\sum_{j=1}^{d+1} p_j = 1$, and all the incentive constraints defining $\mathcal{C}$. Therefore there exists $\overline{q}^\ell \in H_\ell$ in which $d - m - n_\ell$ entries are equal to, say, -1, and, again without loss, the positive entries coincide with the last $m_{\ell+1}+1$ entries, for some $0 \leq m_{\ell+1} \leq m$; (it is always possible to relabel the coordinates and the matrix C defining the incentive constraints at each iteration ℓ).

Next, as before, consider the line segment from q^ℓ ($\in \mathcal{C}$) to $\overline{q}^\ell$. Again, the segment is entirely contained in H_ℓ and it eventually leads to a (first) intersection, say at $q^{\ell+1} \in \mathcal{C}$, with one or more of the hyperplanes defining $\cup_{j=1, j\notin N_\ell}^{d-m_{\ell+1}} \{p : p_j = 0\}$. Once again, the point $q^{\ell+1}$ implies a corresponding set $N_{\ell+1} \subset \{1, \ldots, d - m_{\ell+1}\}$ of nonnegativity constraints holding with equality at $q^{\ell+1}$, as well as numbers $n_{\ell+1} = \#N_{\ell+1}$ ($> n_\ell$) and $m_{\ell+1}$ ($\leq m$), where $m_{\ell+1} + 1$ is the number of positive entries of $\overline{q}^\ell$.

If $d - m - n_{\ell+1} \leq 0$, then, again, we are done; otherwise, repeat the above procedure with $\ell = \ell + 1$. Notice that while $d - m - n_\ell > 0$, repeating the procedure always yields a new point $\overline{q}^\ell$ with at least one entry equal to -1, which in turn yields a point $q^{\ell+1} \in \mathcal{C}$ with at least one additional entry equal to zero, thus $n_{\ell+1} > n_\ell$. Therefore, there exists $\overline{\ell} \leq d-m$, such that repeating the procedure $\overline{\ell}$ times, eventually yields an affine space $H_{\overline{\ell}}$ of dimension greater or equal to zero, and a point $q^{\overline{\ell}}$ satisfying at least $d-m$ nonnegativity constraints with equality, the constraint $\sum_{j=1}^{d+1} p_j = 1$, as well as all the incentive constraints defining $\mathcal{C}$. In other words, it eventually yields a point $p = q^{\overline{\ell}} \in \mathcal{C}$ with at least $d-m$ zero entries, which completes the proof.

References

[1] M. Artin. *Algebra.* Prentice-Hall, Englewood Cliffs, NJ, 1991.

[2] R.J. Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1: 67-96, 1974.

[3] S. Hart and D. Schmeidler. Existence of correlated equilibria. *Mathematics of Operations Research*, 14: 18-25, 1989.

[4] N. Nitzan. Tight correlated equilibrium. Center for the Study of Rationality, Hebrew University of Jerusalem, Discussion Paper # 394, 2005.