On the measure of Voronoi cells *

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Abstract

We study the measure of a typical cell in a Voronoi tessellation defined by n independent random points drawn from a density f in $\mathbb{R}^d$. In particular, we prove that the asymptotic distribution of the measure–with respect to $d\mu = f(x)dx$–of the cell centered at a point $x \in \mathbb{R}^d$ is independent of x and the density f. We determine all moments of the asymptotic distribution and show that the distribution becomes more concentrated as d becomes large. In particular, we show that the variance converges to zero exponentially fast in d. We also obtain a bound independent of the density for the rate of convergence of the diameter of a typical Voronoi cell.

Keywords: random pointset, Voronoi tessellation, stochastic geometry

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1 Introduction

Let $X_1, \ldots, X_n$ be independent, identically distributed random vectors taking values in $\mathbb{R}^d$. We denote the common distribution of the X_i by μ. We assume throughout the paper that μ is absolutely continuous with respect to the Lebesgue measure λ and denote the density of μ by f. Hence, $\mu(A) = \int_A f(x)dx$ for all Lebesgue measurable sets $A \subset \mathbb{R}^d$.

The X_i define the random subsets $S_1, \ldots, S_n$ of $\mathbb{R}^d$ such that S_i contains all points in $\mathbb{R}^d$ that are strictly closer to X_i than any other point in $\{X_1, \ldots, X_n\} \setminus \{X_i\}$. Formally,

$$S_i = \left\{ x \in \mathbb{R}^d : \|x - X_i\| < \min_{j=\{1,\ldots,n\}\setminus\{i\}} \|x - X_j\| \right\} .$$

$\{S_1, \ldots, S_n\}$ is a so-called *Voronoi tessellation* and the S_i are the *Voronoi cells*. Here $\|\cdot\|$ denotes the usual Euclidean norm.

In this paper we are interested in the measure of a "typical" Voronoi cell. In particular, we study the conditional distribution of the random variable $\mu(S_1)$ conditioned on the event that $X_1 = x$ for some x in the support of μ. Equivalently, we study $M_n(x)$ where for any $x \in \mathbb{R}^d$, $M_n(x)$ is the μ-measure of the Voronoi cell S_1 with nucleus x associated with the point process $\{x, X_2, \ldots, X_n\}$.

Note that since $\sum_{j=1}^n \mu(S_j) = 1$ and $\mu(S_1), \ldots, \mu(S_n)$ are identically distributed, we have $n\mathbb{E}\left[\mu(S_1))\right] = 1$. In Theorem 1 below we prove that, for μ-almost all x, we have $n\mathbb{E}M_n(x) \to 1$. We also show that $n^2\mathbb{E}[M_n(x)^2]$ converges to a limit that is independent of x and the distribution μ. In fact, we prove that for μ-almost all x, $nM_n(x)$ has a limiting distribution that only depends on the dimension. We show that the limiting distribution becomes more concentrated as the dimension d grows. In particular, the variance converges to zero exponentially fast in d.

Finally, we study the diameter $\text{diam}(S_1)$ of the Voronoi cell centered at X_1. We show that for μ-almost all x, conditional on $X_1 = x$, $\text{diam}(S_1)$ converges to zero at a rate of $n^{-1/d}$.

Throughout the paper, $B_{x,r}$ denotes the closed ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$.

Motivation and related work

Nearest-neighbor-based estimators are among the most basic and popular tools in addressing classification, regression, density estimation and other problems in nonparametric statistics, see Biau and Devroye [2] for a recent survey. Many of these estimators build on a Voronoi partition of the space based on data drawn from an unknown density. In statistics it is important to understand the behavior of the estimators under as few assumptions as possible on the distribution of the data. In particular, the measure and the diameter of the

Voronoi cells give valuable information about the properties of nearest-neighbor estimators. The distribution-free results derived in this paper were motivated by and contribute to the understanding of such estimators.

The measure of a "typical" cell in a Voronoi tessellation has been mostly studied in the case when the points are drawn from a homogeneous Poisson process. Asymptotically, as $n \to \infty$, this is equivalent to the special case of drawing n points from the uniform distribution μ on (say) the unit ball. The study of the (Lebesgue) measure of Voronoi cells dates back to at least Gilbert [7] who derived formulas and numerical estimates for the second and third moments the measure of a Voronoi cell when $d = 2$ or 3. See also Brakke [3, 4], Hayen and Quine [8].

Our notion of the distribution of a typical cell is analogous to the so-called "Palm distribution" of the volume of a Voronoi cell in stochastic geometry—Chiu, Stoyan, Kendall, and Mecke [5], Møller [12], Møller and Stoyan [13].

Brakke [3, 4], Hayen and Quine [8], Heinrich et al. [9], Heinrich and Muche [10], Zuyev [16] study characteristics of "typical" cells in a Voronoi tessallation of a homogeneous Poisson process, including the second moment of the volume.

The results of the paper may be easily extended to inhomogeneous multi-dimensional Poisson processes. This extension is relevant for modelling mobile communication networks, see Baccelli et al. [1].

For a survey and comprehensive treatment of Voronoi tessellations, we refer to Okabe, Boots, Sugihara, and Chiu [14].

2 Convergence of the first two moments

Theorem 1 below establishes the asymptotic value of the first and second moments of the measure of a typical cell centered at a point x. The remarkable feature is that the asymptotic values are independent of both the density f and the point x (for μ-almost all x) and only depend on the dimension d. In fact, in Theorem 2 we show that the limit distribution is also independent of f and x. We emphasize that both theorems hold without any assumption on the density f.

The asymptotic second moment is expressed in terms of a random variable W defined as follows. Let Y be a random vector uniformly distributed in $B_{0,1}$. Define $\overline{1} = (1, 0, 0, \ldots, 0) \in \mathbb{R}^d$ and let $\overline{B} = B_{\overline{1},1} \bigcup B_{Y,\|Y\|}$. Introduce the random variable

$$W = \frac{\lambda(\overline{B})}{\lambda(B_{0,1})} \tag{1}$$

(recall that λ denotes the Lebesgue measure) and let

$$\alpha(d) \stackrel{\text{def}}{=} \mathbb{E}\left[\frac{2}{W^2}\right] .$$

Theorem 1 *Let μ have a density f. Then, as $n \to \infty$,*

(i)

$$n\mathbb{E}M_n(x) \to 1 \quad \textit{for } \mu\textit{-almost all } x .$$

(ii)

$$n^2\mathbb{E}[M_n(x)^2] \to \alpha(d) \quad \textit{for } \mu\textit{-almost all } x .$$

In Section 4 we obtain estimates for the asymptotic conditional second moment $\alpha(d)$. In particular, in Theorem 3 we show that for all dimensions, $1 \le \alpha(d) \le 1 + 6(3/4)^{d/2}$ and therefore the asymptotic variance of $\mu(S_1)$ (conditioned on $X_1 = x$) decreases to zero exponentially in d.

Lebesgue density theorem. The proofs use a version of the Lebesgue density theorem that we recall first.

Let $\mathcal{B}$ be the class of all closed balls of $\mathbb{R}^d$ containing the origin. We say that x is a *Lebesgue point* for f if for all sequences $B_k \in \mathcal{B}$ with $\lambda(B_k) \downarrow 0$,

$$\lim_{k\to\infty} \frac{\int_{x+B_k} f(y)dy}{\lambda(B_k)} = f(x) .$$

Let A be the set of all $x \in \mathbb{R}^d$ such that $f(x) > 0$ and x is a Lebesgue point for f. Then $\mu(A) = 1$ by Wheeden and Zygmund [15, pp. 106–108]. See also Devroye and Györfi [6], Chapter 2.

Proof of part (i). Observe that

$$\begin{aligned}\mathbb{E}M_n(x) &= \mathbb{P}\left\{X_{n+1} \in S_1 \mid X_1 = x\right\} \\ &= \mathbb{P}\left\{\cap_{i=2}^n \{X_i \notin B_{X_{n+1},\|X_{n+1}-x\|}\}\right\} \\ &= \mathbb{E}\left[(1 - Z(x))^{n-1}\right] ,\end{aligned}$$

where $Z(x) = \mu(B_{X,\|X-x\|})$. (Recall that both X and X_{n+1} are independent of $X_1, \ldots, X_n$ and have the same distribution.) First note that

$$n\mathbb{E}\left\{(1 - Z(x))^{n-1}\right\} \to 1$$

whenever

$$\lim_{z\downarrow 0} z^{-1}\mathbb{P}\left\{\mu(B_{X,\|X-x\|}) \le z\right\} = 1 \ . \tag{2}$$

To see this, suppose that (2) holds and for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\mathbb{P}\{Z(x) \le z\} - z| < z\epsilon$ whenever $z \in (0,\delta)$. But then

$$\begin{aligned}
& n\mathbb{E}\left\{(1-Z(x))^{n-1}\right\} \\
&= n\int_0^1 \mathbb{P}\left\{(1-Z(x))^{n-1} \ge t\right\} dt \\
&= n(n-1)\int_0^1 \mathbb{P}\{Z(x) \le z\}(1-z)^{n-2}dz \\
&\qquad \text{(by a change of variables)} \\
&= n(n-1)\int_0^\delta \mathbb{P}\{Z(x) \le z\}(1-z)^{n-2}dz + n(n-1)\int_0^\delta \mathbb{P}\{Z(x) \le z\}(1-z)^{n-2}dz \\
&= n(n-1)\int_0^\delta z(1+\rho(z))\mathbb{P}\{Z(x) \le z\}(1-z)^{n-2}dz + O(n^2(1-\delta)^n) \\
&\qquad \text{(where } |\rho(z)| < \epsilon\text{)} \\
&= 1 + O(\epsilon) + O(n^2(1-\delta)^n) \ .
\end{aligned}$$

The intuitive reason of why (2) should hold is that for any x, $\mu(B_{x,\|X-x\|})$ is uniformly distributed on $[0,1]$ and that $\mu(B_{X,\|X-x\|})$ is stochastically close to $\mu(B_{x,\|X-x\|})$ when $\|X - x\|$ is small in a sense specified below. The rest of the proof establishes the convergence (2).

By the Lebesgue density theorem, it suffices to prove (2) for all Lebesgue points x with $f(x) > 0$. Fix such a point x. Since for any sequence $B_k \in \mathcal{B}$ with $\lambda(B_k) \downarrow 0$ we have

$$\frac{\mu(x+B_k)}{\lambda(x+B_k)} \to f(x) \ ,$$

for any $\epsilon \in (0,1)$ we can find $\delta > 0$ (possibly depending on x) such that $\|v-x\| \le \delta$ implies

$$\left|\frac{\mu(B_{v,\|v-x\|})}{\lambda(B_{v,\|v-x\|})} - f(x)\right| \le \epsilon f(x)$$

and

$$\left|\frac{\mu(B_{x,\|v-x\|})}{\lambda(B_{x,\|v-x\|})} - f(x)\right| \le \epsilon f(x) \ .$$

This also implies that for any v with $\|v-x\| \geq \delta$,

$$\begin{aligned} \mu(B_{v,\|v-x\|}) &\geq \mu(B_{v^*,\delta}) \geq (1-\epsilon)f(x)\lambda(B_{0,\delta}) \\ \text{and} \quad \mu(B_{x,\|v-x\|}) &\geq \mu(B_{x,\delta}) \geq (1-\epsilon)f(x)\lambda(B_{0,\delta}) \end{aligned} \tag{3}$$

where v^* is the unique point on the surface of $B_{x,\delta}$ and on the line segment $(x, v]$. Take $z > 0$ so small that $z < (1-\epsilon)f(x)\lambda(B_{0,\delta})$. We again use the fact that $\mu(B_{x,\|X-x\|}) = U$ has the uniform distribution on $[0, 1]$. We rewrite

$$\mu(B_{X,\|X-x\|}) = \frac{\mu(B_{X,\|X-x\|})}{\lambda(B_{X,\|X-x\|})} \cdot \frac{\lambda(B_{x,\|X-x\|})}{\mu(B_{x,\|X-x\|})} \cdot \mu(B_{x,\|X-x\|}) \ .$$

If $\|X-x\| \leq \delta$, then the product of the first two factors is sandwiched between

$$\frac{1-\epsilon}{1+\epsilon} \quad \text{and} \quad \frac{1+\epsilon}{1-\epsilon} \ .$$

By (3), we have $\|X-x\| \leq \delta$ if either $\mu(B_{X,\|X-x\|}) \leq z$ or $\mu(B_{x,\|X-x\|}) \leq z$, and in that case,

$$\mu(B_{X,\|X-x\|}) \geq \frac{1-\epsilon}{1+\epsilon}\mu(B_{x,\|X-x\|})$$

and

$$\mu(B_{X,\|X-x\|}) \leq \frac{1+\epsilon}{1-\epsilon}\mu(B_{x,\|X-x\|}) \ .$$

Thus,

$$\begin{aligned} \mathbb{P}\{\mu(B_{X,\|X-x\|}) \leq z\} &\leq \mathbb{P}\left\{\frac{1-\epsilon}{1+\epsilon}\mu(B_{x,\|X-x\|}) \leq z\right\} \\ &\leq \mathbb{P}\left\{\frac{1-\epsilon}{1+\epsilon}U \leq z\right\} \\ &= \min\left\{z\frac{1+\epsilon}{1-\epsilon}, 1\right\} \ . \end{aligned}$$

Similarly, for $0 < z \le 1$,

$$
\begin{aligned}
\mathbb{P}\{\mu(B_{X,\|X-x\|}) \le z\} &= \mathbb{P}\{\mu(B_{X,\|X-x\|}) \le z, \|X-x\| \le \delta\} \\
&\ge \mathbb{P}\left\{\frac{1+\epsilon}{1-\epsilon}\mu(B_{x,\|X-x\|}) \le z, \|X-x\| \le \delta\right\} \\
&= \mathbb{P}\left\{\mu(B_{x,\|X-x\|}) \le \frac{1-\epsilon}{1+\epsilon}z, \|X-x\| \le \delta\right\} \\
&= \mathbb{P}\left\{\mu(B_{x,\|X-x\|}) \le \frac{1-\epsilon}{1+\epsilon}z\right\} \\
&= \mathbb{P}\left\{U \le \frac{1-\epsilon}{1+\epsilon}z\right\} \\
&= \frac{1-\epsilon}{1+\epsilon}z \ .
\end{aligned}
$$

This proves (2) and part (i) of Theorem 1.

Proof of part (ii). Similarly to the proof of part (i), observe that

$$
\begin{aligned}
\mathbb{E}[M_n(x)^2] &= \mathbb{P}\left\{X_{n+1} \in S_1, X_{n+2} \in S_1 \mid X_1 = x\right\} \\
&= \mathbb{P}\left\{\cap_{i=2}^n \{X_i \notin B_{X_{n+1},\|X_{n+1}-x\|} \cup B_{X_{n+2},\|X_{n+2}-x\|}\}\right\} \\
&= \mathbb{E}\left[(1-Z_2(x))^{n-1}\right] \ ,
\end{aligned}
$$

where

$$
Z_2(x) = \mu(B_{X,\|X-x\|} \cup B_{X',\|X'-x\|})
$$

with X and X' independent and distributed with respect to μ. In analogy with the argument of part (i), in order to prove that

$$
\lim_{n\to\infty} n^2 \mathbb{E}\left\{(1-Z_2(x))^{n-1}\right\} = \alpha(d) \ ,
$$

it suffices to show that, for μ-almost all x,

$$
\lim_{z\downarrow 0} z^{-2}\mathbb{P}\left\{\mu(B_{X,\|X-x\|} \cup B_{X',\|X'-x\|}) \le z\right\} = \alpha(d)/2.
$$

The rough idea of the proof is as follows. The approximate equalities are made rigorous below. For small z,

$$
\begin{aligned}
& z^{-2}\mathbb{P}\left\{\mu(B_{X,\|X-x\|}\cup B_{X',\|X'-x\|})\le z\right\} \\
= \quad & z^{-2}\mathbb{P}\left\{\frac{\mu(B_{X,\|X-x\|}\cup B_{X',\|X'-x\|})}{\max\{\mu(B_{X,\|X-x\|}),\mu(B_{X',\|X'-x\|})\}}\max\{\mu(B_{X,\|X-x\|}),\mu(B_{X',\|X'-x\|})\}\le z\right\} \\
\underset{z\to 0}{\sim} \quad & z^{-2}\mathbb{P}\left\{\frac{\mu(B_{X,\|X-x\|}\cup B_{X',\|X'-x\|})}{\max\{\mu(B_{X,\|X-x\|}),\mu(B_{X',\|X'-x\|})\}}\max\{\mu(B_{x,\|X-x\|}),\mu(B_{x,\|X'-x\|})\}\le z\right\} \\
\underset{z\to 0}{\sim} \quad & z^{-2}\mathbb{P}\left\{W\max\{U_1,U_2\}\le z\right\} \\
& \text{(where } U_1,U_2 \text{ are i.i.d. uniform, independent of } W\text{)} \\
= \quad & z^{-2}\mathbb{P}\left\{WU^{1/2}\le z\right\} \\
= \quad & z^{-2}\mathbb{E}\left[\min(z^2/W^2,1)\right]\underset{z\to 0}{\sim}\mathbb{E}\left[\frac{1}{W^2}\right] .
\end{aligned}
$$

To prove the desired limit formally, as before, by the Lebesgue density theorem, we may assume that $x\in\mathbb{R}^d$ is such that $f(x)>0$ and x is a Lebesgue point for f. A key point of the proof uses coupling. Let (Y_1,Y_2) be the canonical reordering of (X,X') such that

$$\|Y_2-x\|\ge\|Y_1-x\| .$$

and introduce $M=\max(\|X-x\|,\|X'-x\|)=\|Y_2-x\|$. Define the random variable N by

$$N=\begin{cases}1 & \text{if } Y_1=X'\\ 2 & \text{if } Y_2=X' .\end{cases}$$

Then set $V_2=Y_2$ and let V_1 be uniformly distributed on $B_{x,\|V_2-x\|}$ such that V_1 is maximally coupled with Y_1 given Y_2. From Doeblin's coupling argument,

$$\mathbb{P}\{Y_1\ne V_1 \mid Y_2\}=\frac{1}{2}\int|f_{Y_1}(v)-f_{V_1}(v)|dv ,$$

where f_{Y_1}, f_{V_1} are the conditional densities of Y_1 and V_1 given Y_2.

Choose $\delta>0$ so small that for $M\le\delta$, we have, simultaneously,

$$\frac{\mu(B_{x,M})}{\lambda(B_{x,M})}\in[f(x)(1-\epsilon),f(x)(1+\epsilon)] ,$$

$$\frac{\mu(B_{X,\|X-x\|}\cup B_{X',\|X'-x\|})}{\lambda(B_{X,\|X-x\|}\cup B_{X',\|X'-x\|})}\in[f(x)(1-\epsilon),f(x)(1+\epsilon)] ,$$

and
$$\frac{\mu(B_{X,M})}{\lambda(B_{X,M})}\left|\frac{\lambda(B_{x,M})}{\mu(B_{x,M})}-\frac{1}{f(x)}\right|+\frac{1}{\lambda(B_{x,M})f(x)}\int_{B_{x,M}}|f(v)-f(x)|dv\le\epsilon\ .$$
Such a δ exists by three applications of the Lebesgue density theorem. (Recall that x is a Lebesgue point.) Since

$$f_{Y_1}(v)=\frac{f(v)}{\mu(B_{x,\|Y_2-x\|})}\mathbb{1}_{v\in B_{x,M}}\quad\text{and}\quad f_{V_1}(v)=\frac{1}{\lambda(B_{x,M})}\mathbb{1}_{v\in B_{x,M}}\ ,$$

(where $\mathbb{1}$ denotes the indicator function) we have, writing $B=B_{x,M}$,

$$\begin{aligned}\int|f_{Y_1}(v)-f_{V_1}(v)|dv &= \int_B\left|\frac{f(v)}{\lambda(B)}\frac{\lambda(B)}{\mu(B)}-\frac{1}{\lambda(B)}\right|dv\\ &\le \frac{1}{\lambda(B)}\int_B f(v)\left|\frac{\lambda(B)}{\mu(B)}-\frac{1}{f(x)}\right|dv+\frac{1}{\lambda(B)}\int_B\left|\frac{f(v)}{f(x)}-1\right|dv\\ &= \frac{\mu(B)}{\lambda(B)}\left|\frac{\lambda(B)}{\mu(B)}-\frac{1}{f(x)}\right|+\frac{1}{f(x)}\frac{1}{\lambda(B)}\int_B|f(v)-f(x)|\,dv\\ &\le \epsilon\end{aligned}$$

if $M\le\delta$, by choice of δ. Finally, define a pair of random variables (V,V'), both taking values in $\mathbb{R}^d$, as follows.

$$(V,V')=\begin{cases}(V_1,V_2) & \text{if}\quad N=2\\ (V_2,V_1) & \text{if}\quad N=1\ .\end{cases}$$

The argument above implies that

$$\mathbb{P}\{(V,V')\neq(X,X')\mid M\}\le\mathbb{1}_{M>\delta}+\mathbb{1}_{M\le\delta}\frac{\epsilon}{2}\ .$$

Since
$$(\mu(B_{x,\|X-x\|}),\mu(B_{x,\|X'-x\|}))\stackrel{\mathcal{L}}{=}(U,U'),$$
where U,U' are independent uniform $[0,1]$ random variables, we have,

$$\mu(B_{x,M})\stackrel{\mathcal{L}}{=}\max(U,U')\stackrel{\mathcal{L}}{=}\sqrt{U}\ .$$

By construction, V_1 is uniform on $B_{x,\|Y_2-x\|}$, so that, given Y_2,

$$\frac{\lambda(B_{Y_2,\|Y_2-x\|}\cup B_{Y_1,\|Y_1-x\|})}{\lambda(B_{Y_2,\|Y_2-x\|})}\stackrel{\mathcal{L}}{=}W\ ,$$

where W was defined in (1). To complete the argument, set

$$B_X = B_{X,\|X-x\|}, \quad B_{X'} = B_{X',\|X'-x\|}, \quad M = \max(\|X-x\|, \|X'-x\|) \ .$$

Then

$$\begin{aligned}\mu(B_X \cup B_{X'}) &= \frac{\mu(B_X \cup B_{X'})}{\lambda(B_X \cup B_{X'})} \cdot \frac{\lambda(B_X \cup B_{X'})}{\lambda(B_{x,M})} \cdot \frac{\lambda(B_{x,M})}{\mu(B_{x,M})} \cdot \mu(B_{x,M}) \\ &\stackrel{\text{def}}{=} r_1 \cdot r_2 \cdot r_3 \cdot r_4.\end{aligned}$$

Note that

$$r_1 \in [f(x)(1-\epsilon), f(x)(1+\epsilon)]$$

when $M \le \delta$, and similarly,

$$r_3 \in \left[\frac{1}{f(x)(1+\epsilon)}, \frac{1}{f(x)(1-\epsilon)}\right]$$

when $M \le \delta$. When $(X, X') = (V, V')$, we have

$$r_2 = \frac{\lambda(B_X \cup B_{X'})}{\lambda(B_{x,M})} \stackrel{\mathcal{L}}{=} W$$

with W independent of

$$r_4 = \mu(B_{x,M}) \stackrel{\mathcal{L}}{=} \sqrt{U} \ .$$

Thus, since for small enough z, $\mu(B_X \cup B_{X'}) \le z$ also $\mu(B_{x,M}) \le z$ implies $M \le \delta$ (as argued in the proof of (2)), for such z, we have

$$\mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, (X, X') = (V, V')\} \le \mathbb{P}\left\{\frac{1-\epsilon}{1+\epsilon} W\sqrt{U} \le z\right\} \ .$$

Write

$$\mathbb{P}\{\mu(B_X \cup B_{X'}) \le z\} = \mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, M \le \delta\} + \mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, M > \delta\} \ .$$

Clearly,

$$\mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, M > \delta\} = 0$$

for z small enough. For such a z, we have

$$\begin{aligned}\mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, M \le \delta\} &= \mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, M \le \delta, (X, X') \ne (V, V')\} \\ &\quad + \mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, M \le \delta, (X, X') = (V, V')\} \\ &\stackrel{\text{def}}{=} q_1 + q_2.\end{aligned}$$

We have

$$
\begin{aligned}
q_1 &\le \mathbb{P}\{\mu(B_X) \le z\}\mathbb{P}\{\mu(B_{X'}) \le z\} \sup_{\rho\le\delta} \mathbb{P}\{(X,X') \ne (V,V') \mid M \le \rho\} \\
&= z^2(1+o(1)) \sup_{\rho\le\delta} \mathbb{P}\{(X,X') \ne (V,V') \mid M \le \rho\} \\
&\qquad \text{(according to (2))} \\
&= z^2(1+o(1))\epsilon \qquad \text{(by the choice of } \delta) \ .
\end{aligned}
$$

Also,

$$
\begin{aligned}
q_2 &\le \mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, M \le \delta\} \\
&\le \mathbb{P}\left\{ \frac{1-\epsilon}{1+\epsilon} W\sqrt{U} \le z \right\} \\
&\le \left(\frac{1+\epsilon}{1-\epsilon}\right)^2 \mathbb{E}\left\{\frac{1}{W^2}\right\} z^2 \\
&= \left(\frac{1+\epsilon}{1-\epsilon}\right)^2 z^2 \alpha(d)/2 \ .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\mathbb{P}\{\mu(B_X \cup B_{X'}) \le z\} \\
&\ge \mathbb{P}\{\mu(B_X \cup B_{X'}) \le z, M \le \delta, (X,X') = (V,V')\} \\
&\ge \mathbb{P}\left\{ \frac{1+\epsilon}{1-\epsilon} \frac{\lambda(B_V \cup B_{V'})}{\lambda(B_{x,M})} \mu(B_{x,M}) \le z, M \le \delta, (X,X') = (V,V') \right\} \\
&= \mathbb{P}\left\{ \frac{1+\epsilon}{1-\epsilon} \frac{\lambda(B_V \cup B_{V'})}{\lambda(B_{x,M})} \mu(B_{x,M}) \le z, (X,X') = (V,V') \right\} \\
&= A - B,
\end{aligned}
$$

where

$$\begin{aligned}
A &= \mathbb{P}\left\{\frac{1+\epsilon}{1-\epsilon}\frac{\lambda(B_V \cup B_{V'})}{\lambda(B_{x,M})}\mu(B_{x,M}) \le z\right\} \\
&= \mathbb{P}\left\{\frac{1+\epsilon}{1-\epsilon}W\sqrt{U} \le z\right\} \\
&= \mathbb{P}\left\{U \le \left(\frac{z(1-\epsilon)}{W(1+\epsilon)}\right)^2\right\} \\
&= \mathbb{E}\left\{\min\left\{\left(\frac{z(1-\epsilon)}{W(1+\epsilon)}\right)^2, 1\right\}\right\} \\
&= (1-o(1))\left(\frac{1-\epsilon}{1+\epsilon}\right)^2 z^2 \frac{\alpha(d)}{2}
\end{aligned}$$

(by the dominated convergence theorem) and

$$\begin{aligned}
B &= \mathbb{P}\left\{\frac{1+\epsilon}{1-\epsilon}\frac{\lambda(B_V \cup B_{V'})}{\lambda(B_{x,M})}\mu(B_{x,M}) \le z, (X,X') \ne (V,V')\right\} \\
&\le \mathbb{P}\left\{\mu(B_{x,M}) \le z, (X,X') \ne (V,V')\right\} \\
&= \mathbb{P}\left\{\mu(B_{x,M}) \le z, M \le \delta, (X,X') \ne (V,V')\right\} \\
&\le \mathbb{P}\left\{\mu(B_X \cup B_{X'})\frac{1-\epsilon}{1+\epsilon}\frac{1}{2^d} \le z, M \le \delta, (X,X') \ne (V,V')\right\} \\
&= O(1)z^2\epsilon \qquad\qquad \text{(by the choice of } \delta\text{)}.
\end{aligned}$$

Since ϵ was arbitrary, this concludes the proof of Part (ii) of Theorem 1.

3 Convergence in distribution

In the next result (Theorem 2) we determine the asymptotic distribution of $M_n(x)$. We do this by determining the asymptotic moments of the limiting distribution. Once again, the limit is the same for μ-almost all x.

In order to describe the asymptotic moments, for any positive integer k define the random variable

$$W_k = \frac{\lambda(B_{\overline{1},1} \cup B_{Y_1,\|Y_1\|} \cup \cdots \cup B_{Y_{k-1},\|Y_{k-1}\|})}{\lambda(B_{0,1})} ,$$

where $Y_1, \ldots, Y_{k-1}$ are independent random variables distributed uniformly in $B_{0,1}$. Note that

$$1 = W_1 \leq W_2 \leq \cdots \leq \frac{\lambda(B_{0,2})}{\lambda(B_{0,1})} = 2^d.$$

Now we may define a non-negative random variable Z with moments

$$\mathbb{E}[Z^k] = \mathbb{E}\left[\frac{k!}{W_k^k}\right]$$

for $k \geq 1$. We may use Carleman's condition to verify that the distribution of Z is uniquely defined. Indeed, note that $\mathbb{E}[Z^k] \leq k!$ and therefore

$$\sum_{k=1}^{\infty} (\mathbb{E}[Z^k])^{-1/(2k)} \geq \sum_{k=1}^{\infty} (k!)^{-1/(2k)} = \infty ,$$

and Carleman's condition is satisfied. Note that if E is an exponentially distributed random variable with mean 1, then

$$\mathbb{E}\left[E^k\right] = k! \geq \mathbb{E}[Z^k] \geq k!/2^{dk} = \mathbb{E}\left[\left(\frac{E}{2^d}\right)^k\right] .$$

We also have

$$\mathbb{E}\left[e^{sZ}\right] \leq \sum_{k=0}^{\infty} s^k = \frac{1}{1-s}$$

for $0 < s < 1$ and

$$\mathbb{E}\left[e^{sZ}\right] \geq \sum_{k=0}^{\infty} \left(\frac{s}{2^d}\right)^k = \frac{1}{1-s/2^d}$$

for $0 < s < 2^d$.

The next theorem establishes the convergence announced above.

Theorem 2 *Let μ have a density f. Then, for μ-almost all x, the random variable $nM_n(x)$ converges, in distribution, to Z.*

Note that for the case of a Voronoi tessallation of $\mathbb{R}^d$ defined by a Poisson point process of constant intensity, Zuyev [16] describes the distribution of the volume of the so-called "fundamental region" of the cell containing the origin, conditional on having a point at the origin, as a mixture of Gamma distributions. The fundamental region contains the Voronoi cell. Since this distribution equals the limit for the uniform density and our result is density free, the random variable Z described here is stochastically dominated by the same mixture of Gamma random variables.

In the case of $d = 1$ it is easily seen that Z is distributed as $(E_1 + E_2)/2$, where E_1 and E_2 are independent exponentially distributed random variables with mean 1.

Since the proof of Theorem 2 is an extension of that of part (ii) of Theorem 1, we only sketch the arguments.

Sketch of proof of Theorem 2. By the moment method, it suffices to show that for all Lebesgue points $x \in \mathbb{R}^d$ with $f(x) > 0$, and for all $k \geq 1$, we have

$$\mathbb{E}\left[(nM_n(x))^k\right] \to \mathbb{E}[Z^k] \; .$$

As we argued in the case $k = 2$ in the proof of Theorem 1,

$$\mathbb{E}\left[n^k M_n(x)^k\right] = n^k \mathbb{E}\left[(1 - Z_k(x))^{n-1}\right],$$

where

$$Z_k(x) \stackrel{\text{def}}{=} \mu(B_{X_1,\|X_1-x\|} \cup \cdots \cup B_{X_k,\|X_k-x\|})$$

is such that

$$\frac{\mathbb{P}\left\{Z_k(x) \leq z\right\}}{z^k} \underset{z\to 0}{\sim} \frac{\mathbb{P}\left\{W_k U^{1/k} \leq z\right\}}{z^k} \; ,$$

where U is uniform $[0, 1]$, independent of W_k. Here we use the fact that

$$\max_{1\leq i\leq k} \mu(B_{0,\|X_i\|}) \stackrel{\mathcal{L}}{=} \max_{1\leq i\leq k} U_i \stackrel{\mathcal{L}}{=} U^{1/k}$$

with the U_i being independent and uniform on $[0, 1]$. The equivalence $\underset{z\to 0}{\sim}$ above can be proved by the same arguments as detailed in the proof of part (ii) of Theorem 1.

As in the proof of Theorem 1, in order to show that

$$\mathbb{E}\left[n^k M_n(x)^k\right] \to \mathbb{E}\left[\frac{k!}{W_k^k}\right] \; ,$$

it suffices to show that

$$\lim_{z\downarrow 0} z^{-k}\mathbb{P}\left\{\mu(B_{X_1,\|X_1-x\|} \cup \cdots \cup B_{X_k,\|X_k-x\|}) \leq z\right\} = \mathbb{E}\left[\frac{1}{W_k^k}\right] \; .$$

By the approximation above,

$$z^{-k}\mathbb{P}\left\{\mu(B_{X_1,\|X_1-x\|} \cup \cdots \cup B_{X_k,\|X_k-x\|}) \leq z\right\} \underset{z\to 0}{\sim} z^{-k}\mathbb{P}\left\{W_k U^{1/k} \leq z\right\} \underset{z\to 0}{\sim} \mathbb{E}\left[\frac{1}{W_k^k}\right] \; .$$

4 Some values of $\alpha(d)$

In this section we investigate the asymptotic second moment $\alpha(d)$. Since the limiting first moment equals 1, we must have that $\alpha(d) \geq 1$. On the other hand, for all d, we have $\alpha(d) \leq 2$. To see this, simply note that

$$\alpha(d) = \mathbb{E}\left[\frac{2}{W_2^2}\right] \leq \mathbb{E}\left[\frac{2}{W_1^2}\right] = 2\ .$$

Large values of d

Here we show that for large values of d, $\alpha(d)$ approaches 1 exponentially fast.

Theorem 3 *For all d, $1 \leq \alpha(d) \leq 1 + 6(3/4)^{d/2}$.*

Proof. Define

$$A = B_{0,1}, \quad B = B_{\bar{1},1}, \quad \text{and} \quad C = B_{Y,\|Y\|}\ ,$$

where Y is uniformly distributed on A. Recall the definition of the random variable W:

$$W = \frac{\lambda(B \cup C)}{\lambda(B)} \quad \text{and let} \quad U = \frac{\lambda(C)}{\lambda(B)}\ .$$

Observe that U is uniformly distributed on $[0,1]$. We write

$$\begin{aligned}
\alpha(d) - 1 &= \mathbb{E}\left[\frac{2}{W^2} - \frac{2}{(1+U)^2}\right] \\
&= 2\mathbb{E}\left[\frac{1}{W^2}\left(1 - \left(\frac{W}{1+U}\right)^2\right)\right] \\
&\leq 2\mathbb{E}\left[1 - \left(\frac{W}{1+U}\right)^2\right]\ ,
\end{aligned}$$

since $W \geq 1$. We have that

$$2\left[1 - \left(\frac{W}{1+U}\right)^2\right] \leq 2$$

and

$$2\left[1-\left(\frac{W}{1+U}\right)^2\right] = 2\left[\frac{(1+U-W)(1+U+W)}{(1+U)^2}\right]$$
$$\leq 4\left[\frac{1+U-W}{1+U}\right]$$
$$= 4\left[\frac{\lambda(B\cap C)}{\lambda(B)+\lambda(C)}\right] \; .$$

Thus

$$\alpha(d) - 1 \leq 2\mathbb{E}\left[\mathbb{1}_{Y\in B}\right] + 4\mathbb{E}\left[\frac{\lambda(B\cap C)}{\lambda(B)+\lambda(C)}\mathbb{1}_{Y\notin B}\right] \; .$$

We finish the proof by showing that

$$\mathbb{E}\left[\mathbb{1}_{Y\in B}\right] \leq (3/4)^{d/2} \tag{4}$$

and

$$\mathbb{E}\left[\frac{\lambda(B\cap C)}{\lambda(B)+\lambda(C)}\mathbb{1}_{Y\notin B}\right] \leq (3/4)^{d/2} \; . \tag{5}$$

For (4), note that

$$\mathbb{E}\left[\mathbb{1}_{Y\in B}\right] = \frac{\lambda(A\cap B)}{\lambda(B)} \leq \frac{\lambda(B_{b,\sqrt{3/4}})}{\lambda(B)} = (3/4)^{d/2} \; ,$$

where $b = (1/2, 0, 0, \ldots 0) \in \mathbb{R}^d$, since $A\cap B \subset B_{b,\sqrt{3/4}}$ (see Figure 1.)

For (5), we bound

$$\frac{\lambda(B\cap C)}{\lambda(B)+\lambda(C)}\mathbb{1}_{Y\notin B} \leq \sup_{y\notin B} \frac{\lambda(B\cap B_{y,\|y\|})}{\lambda(B)+\lambda(B_{y,\|y\|})}$$
$$\leq \frac{\lambda(B\cap B_{a,1})}{\lambda(B)}$$
$$\text{(where } a = (1/2, \sqrt{3/4}, 0, 0, \ldots, 0) \in \mathbb{R}^d\text{)}$$
$$= \frac{\lambda(A\cap B)}{\lambda(B)}$$
$$\leq (3/4)^{d/2}$$

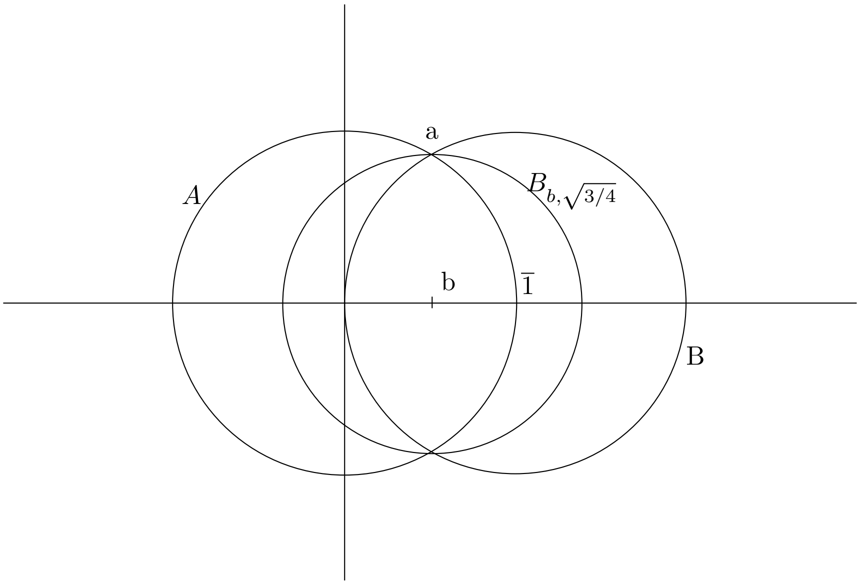

Figure 1: The sets A, B, and $B_{b,\sqrt{3/4}}$.

by arguing as above. That the supremum reached by placing Y (thus, y) at a is clear in two steps. First, the intersection $B \cap B_{y,\|y\|}$ can only grow by replacing y by $y/\|y\|$ since

$$B_{y,\|y\|} \subset B_{y/\|y\|,1}.$$

Next, of all the points on the surface of $B_{\bar{1},1}$, but outside $B_{\bar{1},1}$, the intersection $\lambda(B \cap B_{y/\|y\|,1})$ is maximized by placing y at a.

Small values of d

It is not difficult to see that $\alpha(1) = 3/2$. Indeed, according to (1),

$$\overline{B} = B_{\bar{1},1} \cup B_{Y,\|Y\|} = B_{1,1} \cup B_{Y,|Y|} = \begin{cases} B_{1,1} & \text{if} \quad Y \geq 0 \\ B_{1,1} \cup B_{Y,|Y|} & \text{if} \quad Y < 0 \end{cases},$$

we have

$$W = \frac{\lambda(\overline{B})}{\lambda(B_{0,1})} \stackrel{\mathcal{L}}{=} \begin{cases} 1 & \text{with probability } 1/2, \\ 1+U & \text{with probability } 1/2, \end{cases}$$

where U is uniform $[0,1]$. Hence,

$$\alpha(1) = \mathbb{E}\left[\frac{2}{W^2}\right] = \frac{1}{2}\left(\frac{2}{1} + \mathbb{E}\left[\frac{2}{(1+U)^2}\right]\right) = 1 + \mathbb{E}\left[\frac{1}{(1+U)^2}\right] = 3/2 \ .$$

Previous work has considered the variance of the Lebesgue measure of the Voronoi cell containing the origin defined by a homogeneous Poisson process, conditioned on the fact that a point falls in the origin. From these results, we deduce values of $\alpha(d)$ for $d = 2, 3$. Indeed Gilbert [7], Brakke [3], and Hayen and Quine [8] showed that $\alpha(2) \approx 1.2801760409267$ while Gilbert [7] and Brakke [4] showed $\alpha(3) \approx 1.179032437845$.

5 On the diameter of a typical cell

Here we prove that, independently of the density, for μ-almost all $x \in \mathbb{R}^d$, conditional on $X_1 = x$, the diameter of the Voronoi cell centered at X_1 converges to zero, in probability, at a rate of $n^{-1/d}$ in the sense specified in Theorem 4. More precisely, for any $x \in \mathbb{R}^d$, let $D_n(x)$ be the diameter of the Voronoi cell S_1 with nucleus x associated with the point process $\{x, X_2, \ldots, X_n\}$. Then we have the following:

Theorem 4 *Let μ have a density f. Then for μ-almost all $x \in \mathbb{R}^d$,*

$$\lim_{t\to\infty} \limsup_{n\to\infty} \mathbb{P}\left\{n^{1/d} D_n(x) \geq t\right\} = 0 \ .$$

Proof. A set $C \subset \mathbb{R}^d$ is a cone of angle $\theta \in (0, \pi/2)$ centered at 0 if there exists an $x \in \mathbb{R}^d$ with $\|x\| = 1$ such that

$$C = \left\{ y \in \mathbb{R}^d : \frac{(x, y)}{\|y\|} \geq \cos(\theta/2) \right\} \ .$$

Let γ_d be the minimal number of cones $C_1, \ldots, C_{\gamma_d}$ of angle $\pi/4$ centered at 0 such that their union covers $\mathbb{R}^d$. (For example, $\gamma_2 = 8$.) Let $R_{n,j}$, $j = 1, \ldots, \gamma_d$, be the distance between x and the nearest neighbor among $X_2, \ldots, X_n$ belonging to $x + C_j$ (i.e., the cone C_j translated by x). Define $R_{n,j} = \infty$ if no such point exists.

We bound the diameter of the Voronoi cell S_1 by observing that

$$D_n(x) \leq \sqrt{2} \max_{j=1,\ldots,\gamma_d} R_{n,j} \ .$$

To see this, consider an arbitrary point $y \in S_1$ and let $x + C_j$ be a cone (among the covering cones) containing y. If ξ_j is the nearest neighbor of x among $X_2, \ldots, X_n$ belonging to $x + C_j$, then $\|y - x\| \leq \|y - \xi_j\|$ and therefore $\|y - x\| \leq R_{n,j}/(2\cos(\pi/4))$. Thus, the diameter of S_1 is at most $\max_{j=1,\ldots,\gamma_d} R_{n,j}/\cos(\pi/4)$.

A simple extension of the Lebesgue density theorem implies that if $A = B_{0,1}$ is the unit ball centered at the origin, then for μ-almost all $x \in \mathbb{R}^d$,

$$\min_{j=1,\ldots,\gamma_d} \frac{\int_{x+r[C_j \cap A]} f}{\lambda(r[C_j \cap A])} \to f(x) \quad \text{as} \quad r \downarrow 0 \;, \tag{6}$$

where $r[C_j \cap A] = \{rx : x \in C_j \cap A\}$. Thus, for μ-almost all x, there exists $R(x) > 0$ such that for all $0 < r \leq R(x)$,

$$\min_{j=1,\ldots,\gamma_d} \int_{x+r[C_j \cap A]} f \geq r^d \frac{f(x)}{2} \lambda(C_1 \cap A) \;.$$

If $f(x) = 0$ or x does not satisfy (6), set $R(x) = 0$. For any $t > 0$, we have

$$\begin{aligned}
\left\{D_n(x) > tn^{-1/d}\right\} &\subset \left\{\max_{j=1,\ldots,\gamma_d} R_{n,j} > \frac{tn^{-1/d}}{\sqrt{2}}\right\} \\
&\subset \bigcup_{j=1}^{\gamma_d} \left\{x + \frac{tn^{-1/d}}{\sqrt{2}}[C_j \cap A] \text{ has no point among } X_2, \ldots, X_n\right\} \;.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathbb{P}\left\{n^{1/d} D_n(x) \geq t\right\} \\
&\leq \sum_{j=1}^{\gamma_d} \mathbb{P}\left\{x + \frac{tn^{-1/d}}{\sqrt{2}}[C_j \cap A] \text{ has no point among } X_2, \ldots, X_n\right\} \;.
\end{aligned}$$

We bound the probability of each event in the union as follows.

$$\begin{aligned}
&\mathbb{P}\left\{x + \frac{tn^{-1/d}}{\sqrt{2}}[C_j \cap A] \text{ has no point among } X_2, \ldots, X_n\right\} \\
&\leq \mathbb{P}\left\{x + \min\left(R(x), \frac{tn^{-1/d}}{\sqrt{2}}\right)[C_j \cap A] \text{ has no point among } X_2, \ldots, X_n\right\} \\
&= \left(1 - \mu\left(x + \min\left(R(x), \frac{tn^{-1/d}}{\sqrt{2}}\right)[C_j \cap A]\right)\right)^{n-1} \\
&\leq \left(1 - \left(\min\left(R(x), \frac{tn^{-1/d}}{\sqrt{2}}\right)\right)^d \lambda(C_1 \cap A) \frac{f(x)}{2}\right)^{n-1} \\
&\leq \exp\left(-(n-1) \min\left(R(x)^d, \frac{t^d n^{-1}}{\sqrt{2^d}}\right) \left(\lambda(C_1 \cap A) \frac{f(x)}{2}\right)\right)
\end{aligned}$$

and the theorem follows since $R(x)^d f(x) > 0$ for μ-almost all x.

Note that it follows from the proof that, for μ-almost every x, there exists a positive constant $C(x)$ such that

$$\mathbb{P}\left\{n^{1/d} D_n(x) \geq t\right\} \leq \gamma_d \exp(-(n-1)C(x))$$

and therefore the diameter $D_n(x)$ has an exponentially bounded tail. This phenomenon has been observed before for the Poisson-Voronoi tessellation, see, e.g, Maier, Mayer, and Schmidt [11].

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