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INSTANTANEOUS AND NON-INSTANTANEOUS ADJUSTMENT TO EQUILIBRIUM IN TWO-SECTOR GROWTH MODELS (*)

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The purpose of this paper is to analyze some local stability properties of steady states in the two sector growth model introduced by Uzawa [3], [4]. In the model one distinguishes between a set of variables which is always adjusted to fulfill some « momentary equilibrium » conditions and a variable (aggregate capital) in whose dynamic path one is interested. This abstraction rules out short run inefficiency and short run disequilibrium behavior. It is postulated partly to simplify the analysis partly for lack of an adequate theory; it is justified by the fact that, presumably, if the short run rates of accomodation of the variables are sufficiently high relative to the pace of capital accumulation all the qualitative conclusions of the instantaneously adjusted model (I. M. from now on) remain valid (1).

We want to investigate situations in which a variable is not instantaneously adjusted. In Part I we consider the case where labour does not shift instantaneously between sectors, while in Part II capital is the factor that does not move instantaneously. In Part III, as in Uzawa [4], the wage-rental ratio is rigid in the short-run and unemployment of factors is possible. Let us observe that the significance of the questions raised in this paper can be viewed from two standpoints; either as proper of an economy with short run disequilibrium behavior or as corresponding to an economy with possibly inefficient short-run equilibrium (although inefficiency is ruled out for steady states).

Our plan is to formulate conditions which guarantee local stability of a steady state *regardless of speeds of adjustment*. In particular we are interested in the validity of this statement: *Consider a steady-state of an economy subject alternatively to the following two dynamic specifications. In the first, short run equilibrium is instantaneous, in the second some particular variable is, momentarily, fixed. Then*

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(1) The term « presumably » is needed because this is not, in general, a true statement. However it can be checked that it holds in the two sector growth models.

the local stability of the steady state under the first specification implies its local stability under the second specification. Of course this is a very strong property which cannot be expected to be generally true. We prove that it holds in the case where capital is not instantaneously shiftable between sectors. In spite of the apparent symmetry it *does not hold* when labour is the factor which is not instantaneously movable. As was already pointed out by Uzawa [4] it does not hold either when the wage-rental ratio is given in the short-run. However, one can derive, in the last two cases, fairly broad stability conditions. We observe that the elasticity of substitution conditions (for local stability) seem to generalize more easily to situation with short run disequilibrium than the capital intensity conditions.

We consider only the case with a uniform saving ratio. The calculations have been worked out, too, for the classical saving assumption, and the results are given in footnotes. The analysis and proofs are presented in such a way as to take maximum advantage of the existing results for the I. M.

Finally, one word concerning the restriction of the analysis to local stability. The reason for it is that we want to use fairly general adjustment equations. It seems that in order to obtain global stability results [2] one would need more specific forms of the adjustment equations, at the cost of losing generality in the corresponding local results.

Introduction

The momentary equilibrium of the two-sector model with instantaneous adjustment is characterized by:

$$F_i\,(K_i\,,\,L_i) \qquad (i = 1\,,\,2)$$

$$L_1 + L_2 = L$$

$$K_1 + K_2 = K \tag{1}$$

$$\frac{\partial F_1}{\partial K_1} = p\,\frac{\partial F_2}{\partial K_2}\,; \quad \frac{\partial F_1}{\partial L_1} = p\,\frac{\partial F_2}{\partial L_2}$$

$$s\,(F_1 + pF_2) = pF_2 \qquad (0 < s < 1)$$

Its dynamic path is given by:

$$\frac{\dot{L}}{L} = n$$

$$\dot{K} = F_2$$

[2] Except in Part III where the extension of the analysis is quite straightforward.

We assume the F_i's are linear homogeneous, and

$$\frac{\partial F_i}{\partial K_i} > 0 \; ; \quad \frac{\partial F_i}{\partial L_i} > 0 \; ; \quad \frac{\partial^2 F_i}{\partial K_i^2} < 0 \; ; \quad \frac{\partial^2 F_i}{\partial L_i^2} < 0 \qquad (i = 1, 2)$$

When it is convenient we also assume that the Inada Conditions hold.

Consider the dynamic system:

$$(A) \begin{cases} \dot{X}_1 = G_1 (X_1 , X_2) \\ \dot{X}_2 = G_2 (X_1 , X_2) \end{cases}$$

with G_1, G_2 continuously differentiable. Let $X = (\hat{X}_1 , \hat{X}_2)$ be a stationary point of (A). If $\left[\frac{\partial \dot{X}_2}{\partial X_2}\right]_{\hat{X}} \neq 0$, $\left[\frac{d\dot{X}_1}{dX_1}\right]_{\substack{\dot{X}_2 = 0 \\ \hat{X}}}$ is well defined.

We state as a lemma a well known stability result:

Lemma 1: If $\left[\frac{\partial \dot{X}_1}{\partial X_1}\right]_{\hat{X}} < 0$, $\left[\frac{\partial \dot{X}_2}{\partial X_2}\right]_{\hat{X}} < 0$, $\left[\frac{d\dot{X}_2}{dX_1}\right]_{\substack{\dot{X}_2 = 0 \\ \hat{X}}} < 0$,

the system (A) is locally stable at $\hat{X}$.

This follows since under the stated conditions the Jacobian matrix the system at $\hat{X}$ is a Hicksian matrix which — for the two variables cases — is sufficient for local stability.

Part I

We will assume in this part that labor does not move instantaneously and that all markets are cleared. This means that in the short run the wage rate may differ between sectors. We will further assume that the relative labor force in every sector changes over time in the same direction as the wage differential. Formally, let

$$m = \frac{L_1}{L}$$

$$k_i = \frac{K_i}{L_i} \qquad (2)$$

$$f_i (k_i) = F_i (k_i , 1)$$
$$w_1 = f_1 - k_1 f'_1 \qquad (3)$$
$$w_2 = p (f_2 - k_2 f'_2)$$

$$\omega_i = \frac{f_i}{f'_i} - k_i \qquad (i = 1 , 2)$$

For $0 < m < 1$, $k > 0$ *given,* the momentary equilibrium of the economy is determined by:

$$m k_1 + (1 - m) k_2 = k \tag{5}$$

$$f'_1 (k_1) = p f'_2 (k_2) \tag{6}$$

$$s [m f_1 (k_1) + p (1 - m) f_2 (k_2)] = p (1 - m) f_2 (k_2) \tag{7}$$

Equations (6) and (7) reduce to

$$s m (\omega_1 + k_1) = (1 - s) (1 - m) (\omega_2 + k_2) \tag{8}$$

Under the assumptions in (1), (5) and (8) have a unique solution in k_1 and k_2 (3).

The path of $m(t)$, $k(t)$ is described by the following dynamic equations

$$\begin{cases} \dot{m} = cH(m, k), \ \text{sign}\ \dot{m} = \text{sign}\ (w_1 - w_2) \qquad c > 0 & (9) \\ \dot{k} = (1 - m) f_2 - nk & (10) \end{cases}$$

We assume that H is continuously differentiable, and the system (9)-(10) has a unique solution for every (m_o , k_o) which is continuous with respect to initial conditions. The Inada conditions insure that the solution stays in the nonspecialization region $[k(t) > 0$, $1 > m(t) > 0]$. Expressions for $\frac{\partial K_1}{\partial m}$, $\frac{\partial K_1}{\partial k}$ are derived in A-1 of the Appendix. Define

$\sigma_i (m , k) = \frac{dk_i}{d\omega_i} \frac{\omega_i}{k_i}$, the elasticities of substitution. It is clear that the steady states of the model (5)-(10) coincide with the steady states of the model (1). Suppose $(\hat{m} , \hat{k})$ is a steady state. For short let $\hat{x} \equiv (\hat{m} , \hat{k})$. We will study now the stability properties of this model. If at $\bar{x} = (\bar{m} , \bar{k})$, $\dot{m} = 0$, then:

a) $$\left[\frac{\partial \dot{m}}{\partial m}\right]_{\bar{x}} < 0$$

Proof: In $\bar{x}$,

$$\omega_1 = \omega_2 = \omega$$

From (8),

$$m = \frac{(1 - s) [\omega + k_2 (\omega)]}{s [\omega + k_1 (\omega)] + (1 - s) [\omega + k_2 (\omega)]} \tag{11}$$

(3) In the (k_1 , k_2) space (8) gives a monotonic increasing relation starting at the origin and (5) is a decreasing one.

From Uzawa [7] we know that, if $\omega_1 = \omega_2 = \omega$, k is a function of ω and

$$\frac{dk}{d\omega} > 0 \tag{12}$$

Combining (11) and (12) and manipulating:

$$\text{sign}\left[\frac{dm}{dk}\right]_{\dot{m}=0} = \text{sign}\left[\frac{\omega + \sigma_2 k_2}{\omega + k_2} - \frac{\omega + \sigma_1 k_1}{\omega + k_1}\right]_{\bar{x}} \tag{13}$$

where all the terms are functions of (m, k).

However
$$\left[\frac{\partial \dot{m}}{\partial m}\right]_{\bar{x}} = -\frac{\left[\frac{\partial \dot{m}}{\partial k}\right]_{\bar{x}}}{\left[\frac{dm}{dk}\right]_{\dot{m}=0}} \tag{14}$$

and, by A.2 in the Appendix:

$$\text{sign}\left[\frac{\partial \dot{m}}{\partial k}\right]_{\bar{x}} = \text{sign}\left[\frac{d\omega_1 \partial k_1}{dk_1 \partial k} - \frac{d\omega_2 \partial k_2}{dk_2 \partial k}\right]_{\bar{x}} = $$
$$= \text{sign}\left[\frac{\omega + \sigma_2 k_2}{\omega + k_2} - \frac{\omega + \sigma_1 k_1}{\omega + k_1}\right]_{\bar{x}} \tag{15}$$ (4)

This ends the proof.

b) If $\sigma_1(\hat{m}, \hat{k}) \geq 1$ or (5) $\sigma_1(\hat{m}, \hat{k}) \cdot k_1 \geqq \sigma_2(\hat{m}, \hat{k}) \cdot k_2$,

then $\left[\frac{\partial \dot{k}}{\partial k}\right]_{\hat{x}} < 0$ where $\hat{x} = (\hat{m}, \hat{k})$ is a steady state.

Proof: From (10)

$$\left[\frac{\partial \dot{k}}{\partial k}\right]_{\hat{x}} = (1 - m) f'_2(k_2) \left[\frac{\partial k_2}{\partial k} - \frac{\omega + k_2}{k}\right]_{\hat{x}}$$

From A-3 in the Appendix,

$$\left[\frac{\partial k_2}{\partial k}\right]_{\hat{x}} = \left[\frac{\omega + k_2}{\Omega(m, k)\, m k_1 + (1 - m) k_2 + (1 - m)\,\omega}\right]_{\hat{x}} \tag{17}$$

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(4) When $\dot{m} = 0$, $\text{sign}\,\frac{\partial \dot{m}}{\partial \alpha} = \text{sign}\,\frac{\partial(\omega_1 - \omega_2)}{\partial \alpha}$.

(5) For example: $\sigma_1(\hat{m}, \hat{k}) \geq \sigma_2(\hat{m}, \hat{k})$ and $k_1(\hat{m}, \hat{k}) \geq k_2(\hat{m}, \hat{k})$.

where

$$\Omega(m, k) = \frac{\dfrac{\omega}{\sigma_2 k_2} + 1}{\dfrac{\omega}{\sigma_1} + k_1} (\omega + k_1) \tag{18}$$

and $\Omega(m, k) \geq 1$ if $\sigma_1 \geq 1$ or if $\sigma_1 k_1 \geq \sigma_2 k_2$.

Therefore, under these conditions, and taking (5) into account,

$$\left[\frac{\partial k_2}{\partial k}\right]_{\hat{x}} < \left[\frac{\omega + k_2}{k}\right]_{\hat{x}} \text{ or } \left[\frac{\partial \dot{k}}{\partial k}\right]_{\hat{x}} < 0 .$$

Combining Lemma 1, a) and b) we can state the following proposition:

Proposition 1:

If the steady state $\hat{k}$ *corresponding to the model* (1) *with uniform saving rate and instantaneous adjustment of labor is locally stable and either* 1) $\sigma_1(\hat{m}, \hat{k}) \geq 1$ *or* 2) $\sigma_1(\hat{m}, \hat{k}) \cdot k_1 \geq \sigma_2(\hat{m}, \hat{k}) \cdot k_2$, *the steady state* $[m(\hat{k}), \hat{k}]$ *corresponding to the model* (5)-(10) *will be locally stable.*

In particular, using the results of Uzawa [4] and Drandakis [1], we can assert that the steady state $(\hat{m}, \hat{k})$ in the model (5)-(10) will be locally stable if

1) $\sigma_1(\hat{m}, \hat{k}) \geq 1$ or

2) $k_1(\hat{m}, \hat{k}) \geq k_2(\hat{m}, \hat{k})$ and $\sigma_1(\hat{m}, \hat{k}) \geq \sigma_2(\hat{m}, \hat{k})$.

As the following example shows, the stability of an I. M. steady state does not necessarily yield its stability in the N. I. M. (5)-(10) [6].

Let $\quad f_1(k_1) = \left[\dfrac{3}{1 + 2^{24}}\right]^{1/23} \left[\dfrac{2^{24}}{1 + 2^{24}} k_1^{-23} + \dfrac{1}{1 + 2^{24}}\right]^{-1/23}$

$\quad f_2(k_2) = 3/2\ k_2^{1/2}$

$\quad s = 1/2, \quad n = 9/14 .$

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(6) Under the classical savings assumption we have found that the local stability of a steady-state equilibrium in the I. M. implies the local stability of this equilibrium in the N. I. M.

A steady state for this economy is given by:

$$\hat{k} = 7/5 \ , \ \hat{m} = 2/5$$

corresponding to:

$$\hat{k}_1 = 2 \ , \ \hat{k}_2 = 1 \ , \hat{f}_1 = 1 \ , \hat{f}_2 = 3/2 \ , \hat{f'}_2 = 3/4 \ , \hat{f'}_1 = 1/3 \ ,$$
$$\hat{p} = 4/9 \ , \ \hat{\omega} = 1 \ , \ \hat{\sigma}_1 = 1/24 \ , \ \hat{\sigma}_2 = 1 \ .$$

The capital intensity condition is fulfilled, $\hat{k}_1 > \hat{k}_2$; however if we substitute those values in (17) we find:

$$\left[\frac{\partial k_2}{\partial k}\right]_{\hat{x}} = \frac{2}{6/5 + 24/26 \cdot 1/5} > \frac{2}{7/5} = \frac{\omega + k_2}{k}$$

By (16)
$$\left[\frac{\partial \dot{k}}{\partial k}\right]_{(2/5\,,\,7/5)} > 0 \ .$$

Therefore the behavior of the system at (2/5, 7/5) is cyclic. There is always a value of c for which the trace of the Jacobian at (2/5, 7/5) is positive, i. e. the cycles will be divergent.

Part II

In this part we will assume that capital does not move instantaneously or that it is not shiftable at all. Competitive conditions prevail in each sector. The short-run rate of return on capital, r_i, may differ between sectors. We will assume that investment functions are such that the direction of movement of the relative amount of capital between sectors accords with the current rate-of-return differential (7). Under those conditions the steady states of the model coincide with the steady states of a model with perfect mobility of capital.

In order to make the treatment quite simple we will use a notation which will bring out the formal analogy with the models treated in Part I.

$$\text{Call } u = \frac{K_i}{K} \ , \ \tau_i = \frac{L_i}{K_i} \ , \ f_i(\tau_i) = F_i(1 \ , \ \tau_i) \ ,$$

$$r_1 = f_1 - \tau_1 f'_1 \ , \ r_2 = p\,(f_2 - \tau_2 f'_2) \ , \ w_1 = f'_1 \ ,$$

$$w_2 = pf'_2 \ , \ \bar{\omega}_1 = \frac{r_1}{w_1} = 1/\omega_1$$

(7) An interesting analysis of a similar situation has been given by Inada [2].

Observe that:

$$\sigma_i = \frac{dk_i\omega}{d\omega k_i} = \frac{d\tau_i\overline{\omega}}{d\overline{\omega}\tau_i} \tag{19}$$

u and τ $(= 1/k)$ are state variables.

For given $0 < u < 1$, $\tau > 0$ the momentary equilibrium of the economy is *determined* by:

$$u\tau_1 + (1 - u)\tau_2 = \tau \tag{20}$$

$$f'_1(\tau_1) = pf'_2(\tau_2) \tag{21}$$

$$(1 - s)\,[uf_1 + (1 - u)\,pf_2] = uf_1 \tag{22}$$

(21) and (22) reduce to:

$$su\,(\overline{\omega}_1 + \tau_1) = (1 - s)\,(1 - u)\,(\overline{\omega}_2 + \tau_2) \tag{23}$$

The path of $u(t)$, $\tau(t)$ is described by the following dynamic equations:

$$\dot{u} = cH(u, \tau); \quad \text{sign}\,\dot{u} = \text{sign}\,\{r_1 - r_2\} \qquad c > 0 \tag{24}$$

$$\dot{\tau} = n\tau - (1 - u)\,f_2(\tau_2) \tag{25}$$

We assume that H is continuously differentiable and (24)-(25) have a unique solution for every (u_o, τ_o) which is continuous with respect to initial conditions. Suppose $(\hat{u}, \hat{\tau}) \equiv \hat{x}$ is a steady-state equilibrium of the system (20)-(25). Suppose that at $\overline{x} = (\overline{u}, \overline{\tau})$, $\dot{u} = 0$. Then:

a) $$\left[\frac{\partial \dot{u}}{\partial u}\right]_{\overline{x}} < 0$$

Proof: Equations (20), (23) and (24) are formally identical with equations (5), (8) and (9). Taking into account (19) and that

$$\text{sign}\,\frac{d\tau}{d\overline{\omega}} = \text{sign}\,\frac{dk}{d\omega}$$

a simple relabelling of symbols in A-1 and A-2 of the Appendix yields the result.

b) $$\left[\frac{\partial \dot{\tau}}{\partial \tau}\right]_{\hat{x}} < 0, \ \hat{x} \text{ is a steady state.}$$

Proof: Since at $\hat{x}$, $\dot{\tau} = 0$,

$$\text{sign}\left[\frac{\partial \dot{\tau}}{\partial \tau}\right]_{\hat{x}} = \text{sign}\left[\frac{\partial\,(\dot{\tau}/\tau)}{\partial \tau}\right]_{\hat{x}} = \text{sign}\left[-\,(1 - u)\, f'_2\,(\tau_2)\,\frac{\partial \tau_2}{\partial \tau}\right]$$

by differentiation of (25). By A-1 of the Appendix, after relabelling, $\frac{\partial \tau_2}{\partial \tau} > 0$. Therefore $\left[\frac{\partial \dot{\tau}}{\partial \tau}\right]_{\hat{x}} < 0$ q.e.d.

Combining a), b) and Lemma 1 we may state (recall that $\tau = 1/k$) the following:

Proposition 2:

If the steady state k *corresponding to the model* (1) *with a uniform rate of savings and instantaneous shiftability of capital is locally stable, then the steady state* $[u\,(\hat{k})\,,\,\hat{k}]$ *corresponding to the model* (20)-(25) *is locally stable* (8).

Therefore, using the results of Uzawa [4] and Drandakis [1], this will be the case if

1) $\sigma_1\,(\hat{u}\,,\,\hat{\tau}) \geq 1$ or

2) $k_1\,(\hat{u}\,,\,\hat{\tau}) \geq k_2\,(\hat{u}\,,\,\hat{\tau})$.

Part III

We will assume in this part that the wage rental ratio is rigid in the short run and that its rate of adjustment is inversely related to the quantity of labor unemployed and directly related to the quantity of unused capital. The state variables are k and ω . This model has been briefly considered by Uzawa [4] whose general approach we use.

Let $m_i = \frac{L_i}{L}$; given $0 < \omega < \infty$, $k > 0$ the momentary equilibrium is determined by:

$$s\,(m_1 f_1 + m_2 f_2) = p m_2 f_2 \tag{26}$$

$$f_i = (\omega + k_i)\, f'_i \qquad (i = 1\,,\,2) \tag{27}$$

$$f'_1 = p f'_2 \tag{28}$$

$$k_1 m_1 + k_2 m_2 \leq k \tag{29}$$

$$m_1 + m_2 \leq 1 \tag{30}$$

(8) Under the classical savings assumption, again, the local stability of a steady-state equilibrium in the I. M. guarantees the local stability of this equilibrium in the N. I. M.

The model is closed when either (29) or (30) holds with equality. (26)-(28) reduces to:

$$\frac{1-s}{s}\ \frac{\omega + k_2(\omega)}{\omega + k_1(\omega)} = \frac{m_1}{m_2} \tag{31}$$

Let $k(\omega)$ designate the — uniquely determined — capital labor ratio that makes expressions (24) and (30) hold with equality, $\frac{dk}{d\omega} > 0$ (Uzawa [4]). If $k < k(\omega)$ there is labor unemployment $(m_1 + m_2 < 1)$. If $k > k(\omega)$ there is capital underutilization $(m_1 k_1 + m_2 k_2 < k)$.

The dynamic path of the model is described by:

$$\begin{cases} \dot{\omega} = cH(\omega, k);\ \text{sign}\ \dot{\omega} = \text{sign}\ [k - k(\omega)] \quad , \quad c > 0 & (32) \\ \dot{k} = m_2 f_2(k_2) - nk & (33) \end{cases}$$

We assume that H is continuously differentiable and (32)-(33) have a unique solution for every (ω_o, k_o), which is continuous with respect to initial conditions.

Define $\Psi(\omega) = \frac{1}{n} s f'_2[k_2(\omega)]\,[k(\omega) + \omega]$.

When $k \geq k(\omega)$ $n\Psi(\omega) = m_2 f_2$ and (33) may be expressed as (see Uzawa [4]):

$$\dot{k} = n\Psi(\omega) - nk\ ,\ \textit{if}\ k \geq k(\omega) \tag{34}$$

$$\dot{k} = n\,\Psi(\omega)\,\frac{k}{k(\omega)} - nk\ ,\quad \text{if}\ k < k(\omega) \tag{35}$$

The steady states of this model coincide with the steady states of a model with instantaneous adjustment of ω. Let $\hat{x}$ be a steady state of (32)-(33), then:

a) $\left[\frac{\partial \dot{\omega}}{\partial \omega}\right]_{\hat{x}} < 0$. This follows from the fact that $\frac{dk(\omega)}{d\omega} > 0$.

Since the system (32)-(33) is not differentiable at $\hat{x}$, in order to study local stability at $\hat{x}$ we will rely on phase-diagram analysis. The phase diagram can be drawn as in Figure 1. In a steady state $\Psi(\hat{\omega}) = k(\hat{\omega})$. We observe at once that if $\Psi(\hat{\omega})$ intersects $k(\hat{\omega})$ from above the corresponding steady state will be locally stable if $\left[\frac{d\,\Psi(\omega)}{d\omega}\right]_{\hat{x}} > 0$.

If $\hat{x}$ is locally stable in the I. M., (34) [for $k = k(\omega)$] yields the conclusion that $\Psi(\omega)$ intersects k (ω) from above.

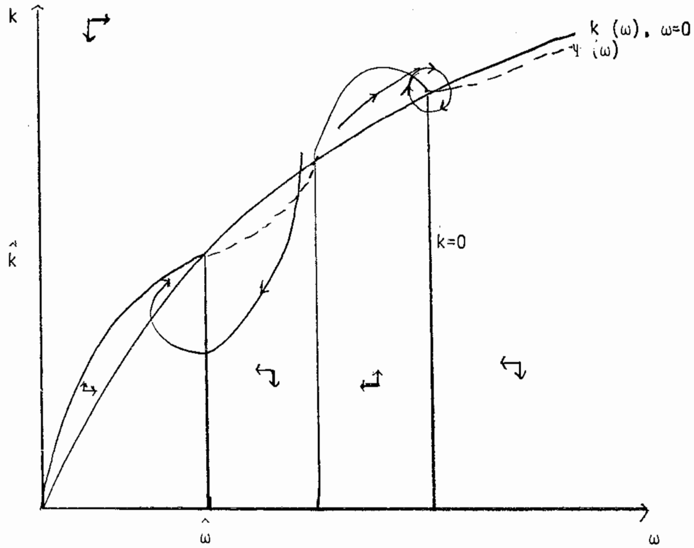

Figure 1.

We can state the following proposition.

Proposition 3:

A steady state $(\hat{\omega}, \hat{k})$ of the model (26) - (33) is locally stable if $\sigma_1 (\hat{\omega}, \hat{k}) \geq 1$ [9].

Proof: $\sigma_1 (\hat{\omega}, \hat{k}) \geq 1$ is a sufficient condition for the stability of the I. M. We only need to prove $\left[\frac{d\,\Psi(\omega)}{d\omega}\right]_{\hat{w}} > 0$ and the result follows.

The logarithmic differentiation of $\Psi(\omega)$ gives:

$$\frac{1}{\Psi}\frac{d\Psi}{d\omega} = -\frac{1}{k_2(\omega) + \omega} + \frac{\frac{dk(\omega)}{d\omega} + 1}{k(\omega) + \omega} \tag{36}$$

We consider four cases:

1) *If* $\sigma_1 \geq 1$, $k_2 > k_1$, (36) is strictly positive and the result follows.

2) *If* $\sigma_1 \geq 1$, $\sigma_2 \geq 1$, then $\frac{dk(\omega)}{d\omega} \geq \frac{k(\omega)}{\omega}$ (37)

(See Drandakis [1]). Therefore, again, (36) is strictly positive and the result follows.

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(9) Under the classical savings assumption the same result holds.

3) If $\sigma_1 > 1$, $k_2 \leq k_1$, $\partial_2 < 1$; $\Psi(\omega)$ intersects $k(\omega)$ from above so that in a small neighbourhood around $\hat{\omega}$, if $\omega' < \hat{\omega}$ then $\Psi(\omega') > k(\omega')$, $\sigma_1(\omega') > 1$, $\sigma_2(\omega') < 1$. Therefore $n\Psi(\omega') = m_2 f_2(k_2)$ and $m_1 + m_2 = 1$. Denote the last two equalities by (38).
Differentiating (31) and taking (38) into account we find:

$$\left[\frac{\partial m_2}{\partial \omega}\right]_{w'} = \left[\frac{s(1-m_2)\left(1+\dfrac{dk_1}{d\omega}\right) - (1-s)\, m_2 \left(1+\dfrac{dk_2}{d\omega}\right)}{(1-s)(k_2+\omega) + s(k_1+\omega)}\right]_{w'}$$

The numerator reduces to [after using (34)]:

$$\frac{1+\dfrac{\sigma_1 k_1}{\omega'}}{k_1+\omega'} - \frac{1+\dfrac{\sigma_2 k_2}{\omega'}}{k_2+\omega'}$$

which is positive since $\sigma_1 > 1$ and $\sigma_2 < 1$.
In this case [10], since:

$$\left[\frac{n \cdot d\Psi}{d\omega}\right] = \left[\frac{dm_2 f_2}{d\omega}\right]_{\hat{w}} = \lim_{w' \to w} \left[f_2 \frac{\partial m_2}{\partial \omega} + m_2 f'_2 \frac{dk_2}{d\omega}\right]_{w'}$$

(where the minus sign in the middle term means left hand derivative), it follows that:

$$\left[\frac{d\Psi}{d\omega}\right]_{\hat{w}} > 0$$

4) If $\sigma_1 = 1$, $k_2 \geq k_1$, $\sigma_2 < 1$, the same result is obtained by a judicious use of ε's and δ's in the argument above.

[10] By the assumptions in (1), $\psi(\omega)$ is continuously differentiable.

APPENDIX

A-1

Differentiating totally the system (5)-(7) and using (8) we find:

$$\frac{\partial k_1}{\partial k} = \frac{1}{\Delta}\left[-pf''_2 (1-m)(1-s) f_2 + (1-m)(1-s) p (f'_2)^2\right] =$$

$$= \frac{1}{A}\,\frac{1 + \dfrac{\omega_2}{\sigma_2 + k_2}}{\omega_2 + k_2}$$

$$\frac{\partial k_2}{\partial k} = \frac{1}{\Delta}\left[-f''_1 (1-m)(1-s) f_2 + smf'_1 f'_2\right] =$$

$$= \frac{1}{A}\,\frac{1 + \omega_1/\sigma_1 k_1}{\omega_1 + k_1}$$

Where

$$\Delta = \begin{vmatrix} m & (1-m) & 0 \\ f''_1 & -pf''_2 & -f'_2 \\ -smf'_1 & (1-m)(1-s)pf'_2 & (1-m)(1-s)f_2 \end{vmatrix} > 0$$

and

$$A = \frac{\Delta}{f'_2 f_1 sm} = \frac{m}{\sigma_2 k_2}\left[\frac{\omega_2 + \sigma_2 k_2}{\omega_2 + k_2}\right] + \frac{1-m}{\sigma_1 k_1}\left[\frac{\omega_1 + \sigma_1 k_1}{\omega_1 + k_1}\right]$$

taking into account that

$$-\frac{f''_i(k_i)}{f'_i(k_i)} = \frac{\omega_1}{\sigma_i k_i}\,\frac{1}{\omega_i + k_i}$$

Clearly

$$\frac{\partial k_1}{\partial k} > 0\,, \quad \frac{\partial k_2}{\partial k} > 0\,.$$

A-2

Substituting the expressions from A-1, we get

$$\left[\frac{d\omega_1}{dk_1}\,\frac{\partial k_1}{\partial k} - \frac{d\omega_2}{dk_2}\,\frac{\partial k_2}{\partial k}\right]_{w_1 = w_2} =$$

$$= \frac{1}{A}\left[\frac{\omega}{\sigma_1 k_1}\left(-\frac{f''_2}{f'_2} + \frac{f'_2}{f_2}\right) - \frac{\omega}{\sigma_2 k_2}\left(-\frac{f''_1}{f'_1} + \frac{f'_1}{f_1}\right)\right]$$

Manipulating, this expression reduces to:

$$\frac{1}{A}\left[\frac{\omega}{\sigma_1 k_1 \sigma_2 k_2}\,\frac{\omega + \sigma_2 k_2}{\omega + k_2} - \frac{\omega}{\sigma_1 k_1 \sigma_2 k_2}\,\frac{\omega + \sigma_1 k_1}{\omega + k_1}\right],$$

which establishes the claim.

A-3

Taking the expression for $\frac{\partial k_2}{\partial k}$ in A-1 and substituting A into it we have:

$$\frac{\partial k_2}{\partial k} = \frac{1}{\frac{mk_1}{k_2} \frac{\omega_2/\sigma_2 + k_2}{\omega_1/\sigma_1 + k_1} \frac{\omega_1 + k_1}{\omega_2 + k_2} + (1 - m)}$$

In $\hat{x}$, $\omega_1 = \omega_2 = \omega$, yielding:

$$\frac{\partial k_2}{\partial k} = \frac{\omega + k_2}{\Omega(m, k)\, mk_1 + (1 - m)\, k_2 + (1 - m)\, \omega}$$

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