Mathematical Programming 6 (1974) 229–233. North-Holland Publishing Company

SHORT COMMUNICATION

A NOTE ON A THEOREM OF F. BROWDER *

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Received 30 May 1973
Revised manuscript received 21 January 1974

A particular case of a mathematical theorem of F. Browder on the behavior of the fixed point set of a mapping under variations of a parameter has recently found applications in programming theory in connection with the abstract (non-linear) complementarity problem (see Eaves, [2, 3]). Two relevant extensions of Browder's result are provided: The first asserts that, under smoothness assumptions, the connected set of fixed points one gets from Browder's theorem is "generically" an arc; the second gives a generalization to the case where the mapping is an upper hemicontinuous contractible valued correspondence.

Let X be a convex, open subset of $\mathbf{R}^n$, $I = [0, 1]$, and $F : X \times I \to X$ a continuous function such that $F(X \times I) \subset K$, where $K \subset X$ is compact. Define $C_F = \{(x, t) \in X \times I: x = F(x, t)\}$. It is a particular case of a theorem of Browder [1, Theorem 2, p. 186]:

Theorem 1. *There is a component T of C_F such that $T \cap X \times \{0\} \neq \emptyset$ and $T \cap X \times \{1\} \neq \emptyset$.*

This result has recently found applications in programming theory in connection with the abstract (nonlinear) complementarity problem (see [2, 3, 4]). Our purpose in this note is to put on record two extensions of the theorem.

The first says that when F is required to be twice differentiable, then, except for a "negligible" (i.e., nowhere dense) set of functions, every component with the property in (1) is (up to diffeomorphism) a segment. In other words, not only one does not have to worry about the component not being arcwise connected, but also the possibility of it branching off can be ruled out. This would seem to solve quite satisfactorily a problem posed by Eaves [2, p. 79].

* Research for this work has been supported by N.S.F. Grants G.S.-27226 and G.S.-3274.

The second extension, prompted by a question of Saigel [7, p. 11], shows that, as it should be expected from the Eilenberg–Montgomery fixed-point theorem [5], the result holds when the hypothesis of F being a continuous function is replaced by "F is an upper hemi-continuous (u.h.c.) correspondence such that for every $(x, t) \in X \times I$, $F(x, t)$ is contractible".

Let $A \subset X$ be open and such that $\bar{A} \subset X$. Let $\mathcal{F}$ be the set of twice continuously differentiable functions $F: X \times I \to A$ endowed with the topology of uniform convergence of the values of the function and its first partial derivatives.

Theorem 2. *There is an open and dense set* $\mathcal{F}' \subset \mathcal{F}$ *such that for every* $F \in \mathcal{F}'$, *any component* T *of* $C_F = \{(x, t) \in X \times I\colon x \in F(x, t)\}$ *with* $T \cap X \times \{0\} \neq \emptyset$ *is diffeomorphic to a segment.*

Proof. For $F \in \mathcal{F}$, let $\hat{F}: X \times I \to \mathbf{R}^n$ be given by $\hat{F}(x, t) = F(x, t) - x$. Applying R. Thom Transversality Theorem (see, for example, Sternberg [9, p. 65]) we conclude the existence of an open and dense set $\mathcal{F}' \subset \mathcal{F}$ such that for every $F \in \mathcal{F}'$, 0 is a regular value of $\hat{F}$ and $\hat{F} \mid X \times \{0\}$.[1] Let $F \in \mathcal{F}'$, then $\hat{F}^{-1}(0)$ is a C^1 one-dimensional manifold (see, for example [6, Lemma 4, p. 13]). Any C^1 connected one-dimensional manifold is (up to diffeomorphism) a segment or a circle [6, Appendix]. Let $T \subset \hat{F}^{-1}(0)$ be a component intersecting $X \times \{0\}$, then T is a connected C^1 one-dimensional manifold which cannot be a circle (if $F(x, 0) = x$, $(x, 0) \in T$ and T is a circle, then rank $D(\hat{F} \mid X \times \{0\})(x, 0) = 0$ and so 0 would not be a regular value of $\hat{F} \mid X \times \{0\}$). Therefore, T is a (closed) segment.

Theorem 3. *If* $F: X \times I \to X$ *is an upper hemi-continuous correspondence such that for every* $(x, t) \in X \times I$, $F(x, t)$ *is contractible, then Theorem 1 holds.* [2]

[1] $F \mid X \times \{0\}$ denotes the restriction of F to $X \times \{0\}$; a $v \in \mathbf{R}^m$ is said to be a regular value of a C^1 function $g: E \to \mathbf{R}^m$, $E \subset \mathbf{R}^n$ if rank $Dg(x) = m$ whenever $g(x) = v$.

[2] A correspondence $g: Y \to Z$ is an upper hemi-continuous (or, also, upper semi-continuous) if for every $y \in Y$ and open set $O \subset Z$ with $g(y) \subset O$ there is an open $U \subset Y$ with $y \in U$ and $g(U) \subset O$. A set $E \subset \mathbf{R}^n$ is contractible if (endowing E with the relative topology) there is a homotopy $\varphi: E \times I \to E$ such that $\varphi \mid E \times \{0\}$ is the identity on E and $\varphi \mid E \times \{1\}$ is a constant map (see [8, p.25]).

Remark. Of course, the convexity of X is not essential. Presumably, it could be replaced by the weaker hypothesis of the Eilenberg–Montgomery Theorem [5]. For the case "F u.h.c. and convex valued", Theorem 3 is well known (see [7]).

The graph of a function or correspondence $f : Z \to Y$, Z, $Y \subset \mathbf{R}^m$ is denoted $G(f)$. For any $A \subset \mathbf{R}^n$, let $B_\epsilon(A)$ be the ϵ-neighborhood of the set A. By Theorem 1 and standard continuity arguments, Theorem 3 holds if for every $\epsilon > 0$ there is a continuous function $h : K \times I \to K$ such that $G(h) \subset B_\epsilon(G(F|K))$. Therefore, since K can be assumed to be a compact, convex polyhedron, Theorem 3 follows from the following lemma (see for definitions [8, Chs. 1 and 3]).

Lemma. *Let $Y \subset \mathbf{R}^n$ be open and $Z \subset \mathbf{R}^n$ convex. If $F : Y \to Z$ is a u.h.c. correspondence such that for every $y \in Y$, $F(y)$ is contractible, then for every compact (topological) polyhedron $W \subset Y$ and $\epsilon > 0$ there is a continuous function $f : W \to Z$ such that $G(f) \subset B_\epsilon(G(F \mid W))$.*

Proof. We proceed by induction; the lemma is clearly true for a 0-dimensional polyhedron (i.e., a finite set of points). Suppose it is true for $(m-1)$-dimensional ones and let $W \subset Y$ be m-dimensional.

For every $x \in W$ let $\varphi_x : F(x) \times I \to F(x)$ be a homotopy between the identity map and a constant map; take $\delta_x < \epsilon/4$ such that if $\|(z, t) - (z', t')\| < \delta_x$, $(z, t), (z', t') \in F(x) \times I$, then $\|\varphi_x(z, t) - \varphi_x(z', t')\| < \epsilon/4$. By the u.h.c. of F there is, for every $x \in W$, a $\gamma_x < \epsilon/4$ such that if $x' \in B_{\gamma_x}(x) \cap W$, then $B_{\gamma_x}(F(x')) \subset B_{\delta_x/3}(F(x))$. Let

$$\mathcal{B} = \{B_{\gamma_{x_1}/2}(x_1), \ldots, B_{\gamma_{x_N}/2}(x_N)\}$$

be a finite covering of W and take $0 < \rho < \frac{1}{2} \min \{\gamma_{x_1}, \ldots, \gamma_{x_N}\}$.

Let H be a triangulation of W finer than $\mathcal{B}$ [8, Theorem 3.14, p. 125]. [3] Denote by W' the polyhedron originated by the $(m-1)$-dimensional skeleton of H. By the induction hypothesis there is a continuous function $f' : W' \to Z$ with $G(f) \subset B_\rho(G(F \mid W'))$.

Take an arbitrary m-simplex $L \in H$. To conclude the proof, it suffices

[3] To be rigorous, a triangulation is a pair (H', ψ'), where H' is a simplicial complex and ψ a homeomorphism between H' and W. However, no confusion will arise for identifying H' with its image on W.

to show that $f' \mid \text{Bdry } L$ can be extended to a continuous function $f: L \to Z$ in such a manner that $G(f) \subset B_\epsilon(G(F))$.

Let $\bar{x}$ be the barycenter of L. For every $x \in L \sim \{\bar{x}\}$, $(y(x), t(x)) \in \text{Bdry } L \times I$ shall denote the "polar" coordinates of x, i.e., the unique $(y, t) \in \text{Bdry } L \times I$ such that $x = t\,\bar{x} + (1-t)\,y$. Those coordinate functions are continuous on $L \sim \{\bar{x}\}$. Let $A_1 = \{x \in L\colon t(x) \leqslant \frac{1}{2}\}$, $A_2 = L \sim A_1$.

For some i, $L \subset B_{\gamma x_i/2}(x_i)$. By continuity there is $0 < \lambda < \frac{1}{2}$ such that if $x, x' \in A_1$ and $\|x-x'\| < \lambda$, then $\|(z, 2t(x))-(z', 2t(x'))\| < \delta_{x_i}$ for every $z \in B_{\delta_{x_i}/3}(f'(y(x)))$, $z' \in B_{\delta_{x_i}/3}(f'(y(x')))$. Define

$$A_1^1 = \{x \in A_1\colon t(x) \geq \lambda/4\}, \quad A_1^2 = A_1 \sim A_1^1 ,$$

and let $\{v_1, ..., v_M\}$ be the vertices of a simplicial subdivision of A_1^1 with mesh $< \lambda/4$. By construction, for every $v \in A_1$, $B_{\delta_{x_i}/3}(f'(y(v))) \cap F(x_i) \neq \emptyset$. Hence for every v_j we can pick a

$$f''(v_j) \in \varphi_{x_i}(B_{\delta_{x_i}/3}(f'(y(v_j))) \cap F(x_i) \times \{2t(v_j)\})$$

and define $f'' : A_1^1 \to Z$ by linear extension. Finally, let $f: L \to Z$ be given by

$$f(v) = \begin{cases} \varphi_{x_i}(F(x_i) \times \{1\}) & \text{if } v \in A_2 , \\ f''(v) & \text{if } v \in A_1^1 , \\ (1-4t(v)/\lambda) f'(y(v)) + 4t(v)/\lambda\, f''(\frac{1}{4}\lambda\bar{x} + (1-\frac{1}{4}\lambda)\, y(v)) & \text{if } v \in A_1^2 . \end{cases}$$

It is immediate that f is well defined and continuous.

It remains to show that $G(f) \subset B_\epsilon(G(F \mid W)$. Since $L \subset B_{\epsilon/4}(x_i)$, it suffices to prove $f(L) \subset B_{\epsilon/2}(F(x_i))$. This is seen to be true by the following easily checked arguments: If $x \in A_2$, then $f(x) \in F(x_i)$; if $x \in A_1^1$, then $f(x)$ belongs to a simplex $T \subset Z$ with mesh $< \epsilon/4$ and with every vertex in $F(x_i)$; if $x \in A_1^2$, then $f(x)$ is within $\epsilon/4$ of such a simplex.

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