A Characterization of Community Excess Demand Functions*

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1. INTRODUCTION

Consider a competitive exchange economy with l commodities and m consumers, each of whom has a demand function satisfying his budget equality and maximizing a preference preorder over nonnegative commodity vectors. Each consumer's excess demand function has three properties: (H) homogeneity of degree zero in prices; (B) boundedness from below; and (W) Walras' law, which are preserved under aggregation to the community excess demand function, i.e., are "hereditary." This article deals with a conjecture of Hugo Sonnenschein stating that every candidate community excess demand satisfying the above hereditary properties can be decomposed into a sum of individual excess demand functions, each consistent with preference maximization. Hence the hereditary properties completely characterize the family of community excess demand functions.

This problem was first studied by Sonnenschein [7–9], who used an ingenious argument on the decomposition of multivariate polynomials to

demonstrate the proposition for the case in which nonnuméraire com-
modity demands are polynomials in nonnumériare prices, and the domain
of prices is restricted to a compact set of positive price vectors. The
decomposition carried out in that construction requires a number of
consumers determined by the degree of the polynomials involved; this
nonuniformity makes generalization difficult.

An alternative approach to the decomposition, due to Mantel [4], yielded
a stronger result: Any continuously differentiable function on a compact
set of positive prices satisfying the hereditary properties and a mild
Lipschitz condition on first derivatives can be decomposed into the sum
of $2l$ individual excess demand functions, each consistent with preference
maximization. This argument has two key features. First, it is shown that
the given candidate community excess demand function can be introduced
as a "perturbation" of the demands of l large "Cobb–Douglas" consumers
in such a way that these consumers remain preference maximizers. Second,
it is shown that the negative of the excess demand function of a sum of
"Cobb–Douglas" consumers can be decomposed into individual excess
demand functions of l preference-maximizing consumers. Then, "adding
and subtracting" the demands of l "Cobb–Douglas" consumers allows
the decomposition of the given community excess demand function with
$2l$ consumers.

A further improvement has been obtained by Debreu [2], who established
that any continuous function on a compact set of positive prices satisfying
the hereditary properties can be decomposed into l individual excess
demand functions, each consistent with maximization of a continuous
utility function. This proof uses an elegant geometric decomposition to
show directly that the community excess demand function lies, pointwise,
in the convex cone spanned by l individual utility-maximizing excess
demand functions. This proves to be the key fact; detailed arguments are
required, then, to establish that the individual excess demand functions
obtained in this construction are, in fact, consistent with maximization
of a continuous, strictly convex, monotone utility function.

As Sonnenschein and Mantel have pointed out, these results have
important implications in general equilibrium existence and stability
theory and in econometric demand analysis.

The first result established below grew out of discussions of Debreu's
paper, and is based on a modification of his geometric decomposition.
We drop the hypothesis of continuity of the community excess demand
function, and consequently do not obtain continuity properties for the
preferences of the individual consumers in the decomposition. A revealed
preference argument replaces the more difficult demonstration of con-
tinuous utility maximization. The use of oblique rather than orthogonal

projections in the proof of the decomposition allows the result to be extended to the entire set of positive price vectors.

The second subject discussed concerns how far the above result can be generalized if a further requirement is imposed on the decomposition— namely, that it be compatible with an *a priori* specification of the aggregate endowment vector. Examples show the hereditary conditions are not always sufficient for decomposition; a theorem gives additional conditions under which decomposition is possible.

Our third subject asks whether the Sonnenschein–Mantel–Debreu theorem can be generalized to upper hemicontinuous convex-valued excess demand correspondences. We show that this is not possible; the conclusion is obtained as a corollary of a very general theorem on the structure of excess demand correspondences obtained from classical continuous preference maximization.

2. THE DECOMPOSITION OF COMMUNITY EXCESS DEMAND FUNCTIONS

Let $P = \{ p \in R^l \mid p \gg 0 \}$. A function $f: P \to R^l$ is a candidate community excess demand function if it satisfies the following.

(H) For every $p \in P$ and $\lambda > 0$, $f(\lambda p) = f(p)$ [Homogeneity].

(W) For every $p \in P$, $p \cdot f(p) = 0$ [Walras' law].

(B) There exists $q \gg 0$ such that, for every $p \in P$, $f(p) + q \gg 0$ [bounded below].

Usually a consumer is defined by a nonnegative endowment vector in R^l and a preference preorder on a consumption set, taken here to be the nonnegative orthant of R^l. It follows from standard arguments (see Richter [5]) that any function f satisfying the hereditary properties (H), (W), and (B) and the strong axiom of revealed preference (SARP)[1] can be obtained as the excess demand function of a preference-maximizing consumer of this type. Thus, in this section, we define a preference-maximizing consumer by an "excess demand" function satisfying (H), (W), (B) and SARP.

THEOREM 1. $f: P \to R^l$ *satisfies* (H), (W), *and* (B) *if and only if there exist l preference-maximizing consumers whose individual excess demand functions sum to* f.

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[1] By the SARP in this context of excess demand functions we mean the acyclicity of the relation D defined by $x D y \Leftrightarrow x \neq y$ and $p \cdot x = 0 \geqslant p \cdot y$ for some $p \in P$ for which $x = f(p)$.

Proof. The "if" portion of the theorem is immediate. The "only if" portion is established in two steps. The first step constructs a class of excess demand functions which satisfy SARP. The second step shows that each candidate community excess demand function can be written as the sum of l members of this class.

Let Q denote the intersection of P with a sphere of radius $2 \mid q \mid$ centered at $-q$; i.e.,

$$Q = \{p \in P \mid (p + q) \cdot (p + q) = 4q \cdot q\}. \tag{1}$$

From this construction, one has the following facts: (i) for each $p \in P$, there exists a unique scalar $\lambda > 0$ such that $\lambda p \in Q$; (ii) $P = \{\lambda p \mid p \in Q$ and $\lambda > 0\}$; (iii) $q \in Q$; and (iv) if $p, r \in Q$ and $p \neq r$, then $(r + q) \cdot (p + q) < 4q \cdot q$. Setting $r = q$ implies $p \cdot q \leqslant q \cdot q$, and hence $p \cdot (p + q) = 3q \cdot q - p \cdot q \geqslant 2q \cdot q$. We shall establish the following result to complete the first step of the proof. (The transpose of a vector p is denoted by p^T; I is the identity matrix.)

LEMMA 1. *If $a \in R^l$ is a nonnegative nonzero vector and $\beta: Q \to R$ is a positive real function, then $h: Q \to R^l$ defined by*

$$h(p) = \beta(p) \left(I - \frac{(p + q) \, p^T}{p \cdot (p + q)} \right) a \tag{2}$$

can be extended from Q to P so as to satisfy (H), (W), *and* SARP.

Figure 1 shows the construction of Q and h: $h(p)$ is the (oblique) projection along the direction $p + q$ of the vector $\beta(p) a$ on the hyperplane $T(p) = \{x \in R^l \mid p \cdot x = 0\}$. Since $p \cdot h(p) = 0$, it is immediate that h can be extended to P so that (H) and (W) hold. To prove that SARP holds on P, it suffices to consider $r, p \in Q$. The condition for $h(r)$ to be directly revealed preferred to $h(p)$ is $h(r) \neq h(p)$ (hence $r \neq p$) and

$$0 \geqslant r \cdot h(p) = \beta(p) \left[r \cdot a - \frac{r \cdot (p + q)}{p \cdot (p + q)} p \cdot a \right].$$

But $p \cdot a > 0$ and

$$r \cdot (p + q) - p \cdot (p + q) = (r + q) \cdot (p + q) - (p + q) \cdot (p + q) < 0,$$

implying $0 > r \cdot a - p \cdot a$. Thus, $h(r)$ is directly revealed preferred to $h(p)$ only if r, p stand in the linear order $r \cdot a < p \cdot a$. Hence, the revealed preference relation is acyclic, and SARP holds. This completes the proof of the lemma.

The second step of the theorem proof shows that the candidate community excess demand function can be written as the sum of l individual

FIGURE 1

excess demand functions of the form given in the lemma. Let a^i denote the ith unit vector and $\beta_i(p)$ the ith component of $f(p) + p + q$, which by condition (B) is positive. Then

$$f(p) + p + q = \sum_{i=1}^{l} \beta_i(p)\, a^i, \tag{3}$$

and so $p \cdot f(p) = 0$ implies

$$p \cdot (p + q) = \sum_{i=1}^{l} \beta_i(p)\, p \cdot a^i. \tag{4}$$

Then,

$$f(p) = \sum_{i=1}^{l} \beta_i(p)\, a^i - (p + q)$$

$$= \sum_{i=1}^{l} \beta_i(p)\, a^i - (p + q) \sum_{i=1}^{l} \beta_i(p)\, p \cdot a^i [p \cdot (p + q)]^{-1}$$

$$= \sum_{i=1}^{l} \beta_i(p) \left[I - \frac{(p + q)\, p^T}{p \cdot (p + q)} \right] a^i$$

$$= \sum_{i=1}^{l} h^i(p), \tag{5}$$

where h^i is defined in Eq. (2) with $a = a^i$ and $\beta = \beta_i$.

Finally, we note that, for $p \in Q$,

$$h^i(p) \geqslant -\frac{(p+q)\,\beta_i(p)\,p_i}{p \cdot (p+q)} = -(p+q)\frac{p_i f_i(p) + p_i^2 + p_i q_i}{p \cdot (p+q)}.$$

Conditions (W) and (B) imply $p_i f_i(p) < p \cdot q$. Hence, h^i is uniformly bounded below and satisfies (B). This completes the proof of the theorem.

This result covers the usual case in which P is the domain of the candidate community excess demand function and the demand for some commodity becomes unbounded above as the price of a commodity approaches zero. An analogous result can be established for the alternative case that the community excess demand function is defined and bounded on $\tilde{P} = \{p \in R^l \mid p \geqslant 0 \text{ and } p \neq 0\}$. We say $f: \tilde{P} \to R^l$ is totally bounded (TB) if there exists $q \gg 0$ such that for every $p \in \tilde{P}, f(p) - q \ll 0 \ll f(p) + q$.

THEOREM 2. $f: \tilde{P} \to R^l$ *satisfies* (H), (W), *and* (TB) *if and only if there exist l consumers, each satisfying* SARP, *whose individual excess demand functions satisfy* (H), (W), *and* (TB) *and sum to* f.

Proof. The argument requires only minor modifications of the proof of the first theorem. Define $\tilde{Q}$ in Eq. (1) by replacing P with $\tilde{P}$. Lemma 1 continues to hold with Q replaced by $\tilde{Q}$ provided the vector a is strictly positive.

In the second step of the proof, let a^i denote the ith unit vector, e denote a vector of ones, and α denote a positive scalar. Consider the system of equations

$$f(p) + p + q = \sum_{i=1}^{l} \beta_i(p)[a^i + \alpha e] \tag{6}$$

in the unknowns $\beta(p) = (\beta_1(p),...,\beta_l(p))$, with the solution

$$\beta(p) = \left[I - \frac{\alpha}{1 + \alpha l}\,ee^T\right](f(p) + p + q). \tag{7}$$

Condition (TB) implies that q can be chosen so that $f(p) + p + q \geqslant \delta q \gg 0$ for some $\delta > 0$, and that $\alpha > 0$ can be chosen so that

$$\delta q \gg \frac{\alpha}{1 + \alpha l}\,ee^T(p + 2q) \gg \frac{\alpha}{1 + \alpha l}\,ee^T(f(p) + p + q).$$

For this value of α, $\beta(p) \gg 0$ for $p \in \tilde{Q}$. Write

$$
\begin{aligned}
f(p) &= \sum_{i=1}^{l} \beta_i(p)[a^i + \alpha e] - (p + q) \\
&= \sum_{i=1}^{l} \beta_i(p) \left[I - \frac{(p + q) p^T}{p \cdot (p + q)} \right] (a^i + \alpha e) \\
&= \sum_{i=1}^{l} h^i(p),
\end{aligned}
\tag{8}
$$

where h^i is defined in Eq. (2) with $a = a^i + \alpha e \gg 0$. Then the modification above of Lemma 1 applies, establishing that h^i is an individual excess demand function satisfying SARP.

Equation (7) and condition (TB) on f imply that $\beta(p)$ is uniformly bounded on $\tilde{Q}$. It is then immediate from Eq. (2) that the h^i satisfy (TB). This completes the proof of Theorem 2.

Remark. It is evident from the proof that the individual excess demand functions in the decompositions established in Theorems 1 and 2 share most smoothness properties of the candidate community excess demand functions.

3. DECOMPOSITION OF COMMUNITY TOTAL DEMAND FUNCTIONS

Suppose that, in addition to observing a candidate community excess demand function, one observes the aggregate endowment vector. Do there exist individuals behaving as preorder maximizers, with the nonnegative orthant as a consumption set, whose individual excess demand functions sum to the community excess demand function? An example shows the answer is sometimes negative.[2] Let ω denote an endowment vector, and x denote a total demand vector (e.g., $x - \omega$ is the excess demand vector).

EXAMPLE 1. $l = 2$, $\omega = (1, 1)$, $x^1 = (0, x_2^1)$, $p^1 \cdot x^2 \leqslant p^1 \cdot x^1$ (Fig. 2). An argument by contradiction is given. Suppose there exist n consumers, $i = 1,..., n$ (for some n), with endowments $\omega^i \geqslant 0$ and demands x^{1i}, x^{2i} for the budget with prices p^1, p^2, such that $\omega = \sum_i \omega^i$, $x^j = \sum_i x^{ji}$ for $j = 1, 2$. Then $x_1^1 = 0$ implies $x_1^{1i} = 0$, and hence $p^2 \cdot x^{1i} \leqslant p^2 \cdot x^{2i}$. The strong axiom then requires $p^1 \cdot x^{1i} < p^1 \cdot x^{2i}$, or $p^1 \cdot x^1 < p^1 \cdot x^2$, for a contradiction.

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[2] The following examples are due to D. McFadden; the theorem below sharpens an earlier result obtained by D. McFadden and M. K. Richter.
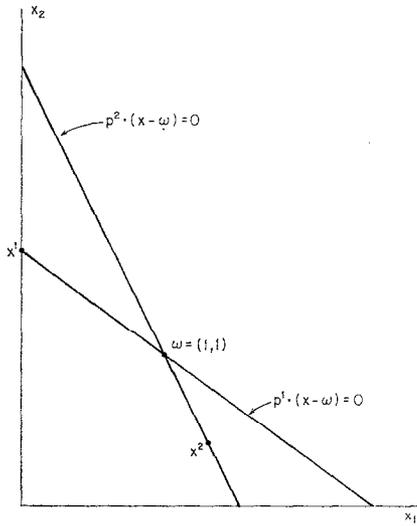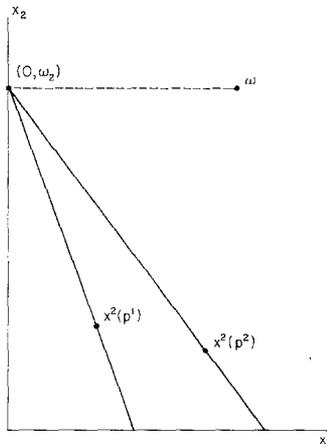
FIGURE 2

FIGURE 3

In view of this example, it is interesting that for $l = 2$ any strictly positive community demand function can be written as the sum of two individual demand functions. Let $\omega = (\omega_1, \omega_2) \geqslant 0$ and $x(p) = f(p) + \omega \gg 0$ be given for $p \in P$. Define consumer 1 to have the endowment $(\omega_1, 0)$ and demand function $x^1(p) = x(p) p_1 \omega_1 / p \cdot \omega$. Define consumer 2 to have the endowment $(0, \omega_2)$ and demand function $x^2(p) = x(p) p_2 \omega_2 / p \cdot \omega$. Note from Fig. 3 that, since $x^2(p) \gg 0$ for

every $p \in P$, the weak axiom, hence the strong holds for $x^2(p)$. The same argument establishes that $x^1(p)$ satisfies the strong axiom.

The next example shows that in the case of three commodities there exist pointwise relatively open sets of aggregate demand functions which cannot be written as the sum of individual demand functions.

EXAMPLE 2. $l = 3$, $\omega = (0, 0, 1)$, $p^1 = (1, 2, 1)$, $p^2 = (2, 1, 1)$. We show (see Fig. 4) that any $x(\cdot)$ satisfying $x_1(p^1) \leqslant \frac{1}{8}$, $x_3(p^1) \leqslant \frac{1}{8}$, and $x_2(p^2) \leqslant \frac{1}{8}$, $x_3(p^2) \leqslant \frac{1}{8}$ cannot be decomposed. The argument is by contradiction. Suppose there exist consumers $i = 1,..., n$ (for some n) with endowments $\omega^i = (0, 0, \alpha_i)$ and demand vectors x^{1i}, x^{2i} at prices p^1, p^2, respectively, with $1 = \sum_i \alpha_i$, $x(p^1) = \sum_i x^{1i} = x^1$, and $x(p^2) = \sum_i x^{2i} = x^2$. For individual i, the strong axiom applied to p^1 and p^2 is equivalent to (a) $p^1 \cdot x^{2i} > p^1 \cdot x^{1i}$, (b) $p^2 \cdot x^{1i} > p^2 \cdot x^{2i}$, or (c) $x^{1i} = x^{2i}$. Furthermore, if two individuals are of the same type, say (a), (b) or (c), then the sum of their demands satisfies the same condition and hence the strong axiom. Then it is sufficient to consider the "composite" consumers a, b, and c. Write x^{1i}, x^{2i} with $i = a, b, c$, for their demands.

We have the inequalities

$$p^1 \cdot x^2 - p^1 \cdot x^1 = \sum_{i=a,b,c} p^1 \cdot (x^{2i} - x^{1i}) > p^1 \cdot (x^{2b} - x^{1b}),$$

and similarly $p^2 \cdot x^1 - p^2 \cdot x^2 > p^2 \cdot (x^{1a} - x^{2a})$. The minimum value,
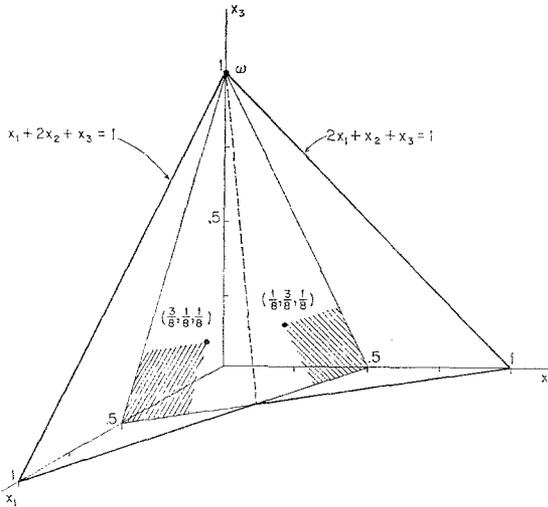

FIGURE 4

$-\alpha_b/2$, of $p^1 \cdot (x^{2b} - x^{1b})$ that could be achieved by varying x^{2b} occurs at $x^{2b} = (\alpha_b/2, 0, 0)$. Similarly, the minimum value of $p^2 \cdot (x^{1a} - x^{2a})$ occurs at $x^{1a} = (0, \alpha_a/2, 0)$. Hence, it is necessary that $p^1 \cdot x^2 > 1 - \alpha_b/2$ and $p^2 \cdot x^1 > 1 - \alpha_a/2$. On the other hand, the constraints $x_1(p^1) \leqslant \frac{1}{8}$, $x_3(p^1) \leqslant \frac{1}{8}$, and $x_2(p^2) \leqslant \frac{1}{8}$, $x_3(p^2) \leqslant \frac{1}{8}$, imply $p^1 \cdot x^2 \leqslant \frac{3}{4}$ and $p^2 \cdot x^1 \leqslant \frac{3}{4}$. Combining these inequalities, $\frac{3}{4} > 1 - \alpha_b/2$, or $\alpha_b > \frac{1}{2}$. Similarly, $\alpha_a > \frac{1}{2}$. But $\alpha_a + \alpha_b = 1 - \alpha_c \leqslant 1$, for a contradiction.

The next result establishes that any candidate aggregate demand function "close" pointwise to a given aggregate endowment can be decomposed. Assume hereafter the aggregate endowment is strictly positive, and scale commodities so that it becomes a vector of ones.

THEOREM 3.   *If $f: P \to R^l$ satisfies (H), (W), and $f(p) + \omega/3l^2 \gg 0$, where $\omega = (1,..., 1)$ is the aggregate endowment, then there exist l consumers with identical endowments ω/l whose demand functions $x^i(p)$ satisfy SARP, (H), $p \cdot x^i(p) = p \cdot \omega/l$, and $x^i(p) \gg 0$, with $f(p) + \omega = \sum_i x^i(p)$.*

Proof.   In the proof of Theorem 1, the following bounds are obtained on individual excess demand functions, where q satisfies $f(p) + q \gg 0$: $h^i(p) \geqslant -(p + q)(p \cdot q + p_i^2 + p_i q_i)/p \cdot (p + q)$, $p \cdot q \leqslant q \cdot q$, $p \cdot q + p \cdot (p + q) = 3q \cdot q$, and $2q \cdot q \leqslant p \cdot (p + q) \leqslant 3q \cdot q$. Then $p_i + q_i \leqslant 2 \mid q \mid$ or $p + q \leqslant 2 \mid q \mid \omega$. Hence,

$$h^i(p) \geqslant -\tfrac{3}{2}(p + q) \geqslant -3 \mid q \mid \omega.$$

Therefore, $h^i(p) + \omega/l \gg 0$ provided $\mid q \mid \leqslant \frac{1}{3}l$. Taking $q = \omega/3l^2$ satisfies this condition. This completes the proof of Theorem 3.

Remark.   The decomposition in this theorem gives the same income to each consumer. It is immediate from the proof that an analogous result could be established for any specified distribution of income.

4. DECOMPOSITION OF COMMUNITY EXCESS DEMAND CORRESPONDENCES

For the next topic we want to treat, it is convenient (but by no means essential) to bring in some continuity considerations which so far have played no role.

A consumer will now be defined as a pair $(\gtrsim, \omega)$ where $\gtrsim$ is a continuous, convex, monotone preference relation[3] on the consumption set $\bar{P}$, the closure of P, and $\omega \in P$ is the initial endowment vector. For the rest

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[3] For definitions, all of them standard, see Debreu [1].

of this paper, pure exchange economies are understood to be composed by consumers with those characteristics. Besides the conditions at the beginning of the text, (H), (B), and (W), excess demand correspondences will now satisfy two more hereditary properties: upper hemicontinuity (u.h.c.) and convex valuedness.

We pose the following question: Can every convex valued u.h.c. correspondence (from P to R^l) which satisfies (H), (B), and (W) be generated as the excess demand correspondence of a pure exchange economy? The next theorem yields a negative answer by showing that these correspondences are in fact subject to strong restrictions.

A subset E of a topological space X is meager (or first category) if it is contained in a countable union of closed sets with empty interior.

THEOREM 4.[4] *If* $f: P \to R^l$ *is the excess demand correspondence of a pure exchange economy* (*with participants as defined above*), *then except for a meager subset of* P *the values of* f *are polyhedra.*

Since, by the Baire Category Theorem (see Royden [6, p. 139]) open subsets of P are not meager, it follows, for example, that if $l > 2$ then $f(p) = \{x \in R^l \mid p \cdot x = 0, \| x \| \leqslant 1\}$ cannot be rationalized in any subset of P with a nonempty interior. We remark, in passing, that there is no hope of retrieving a positive result by dropping the continuity requirement on preferences.

Countable unions of meager sets are meager; therefore, the theorem is an immediate corollary of the following.

I. *Let* $f: P \to R^l$ *be the excess demand correspondence of a consumer* $(\gtrsim, \omega)$. *Then, except for a meager subset of* P, *the values of* f *are segments.*

We remark that points are (degenerate) segments. For $p \in \bar{P}$ let $T(p) = \{x \in R^l : p \cdot x = 0\}$; the smallest, closed, convex cone containing a set $C \subset R^l$ shall be denoted $\hat{C}$. If $E \subset R^l$ is a cone, let Bdry E be its boundary, let $\mathbb{A}E$ be its asymptotic cone (Debreu [1, p. 22]), and let $E^\circ = \{p \in R^l \sim \{0\} : p \cdot x \geqslant 0 \text{ for every } x \in E\}$. E° is a cone, and if E is closed and convex so is E°; if E is a pointed cone (i.e., $x \in E \sim \{0\}$ implies $-x \notin E$), E° has a nonempty interior.

To prove I we shall need the following.

II. *If* C *is a nonempty, closed, convex set bounded below and such that* $C + P \subset C$ *and* $0 \notin C$, *then* $\hat{C}$ *is pointed* (*i.e., if* $x \in \hat{C} \sim \{0\}$, *then* $-x \notin \hat{C}$).

Proof. For any $x \in R^l \sim \{0\}$, $\bar{x}$ will denote the ray through x. By definition of the asymptotic cone, if $x \in \hat{C} \sim \{0\}$ and $\bar{x} \cap C = \varnothing$ then

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[4] This result is due to Andreu Mas-Colell.

$x \in \mathbb{A}C$. Suppose that for some $y \in \hat{C} \sim \{0\}$, $-y \in \hat{C}$. Since $\mathbb{A}C = \bar{P}$, either $y \notin \mathbb{A}C$ or $-y \notin \mathbb{A}C$. Without loss of generality, let $y \notin \mathbb{A}C$; then $\bar{y} \cap C \neq \varnothing$ and therefore (since $0 \notin C$ and $C + \bar{P} \subset C$) $y \notin -\bar{P}$, which implies $-y \notin \mathbb{A}C = \bar{P}$, so $-\bar{y} \cap C \neq \varnothing$. But $\bar{y} \cap C \neq \varnothing$ and $-\bar{y} \cap C \neq \varnothing$ implies $0 \in C$, a contradiction.

III. *If $\Gamma \subset R^l$ is a nonempty, closed, convex pointed cone, then $J = \{p \in \text{Bdry } \Gamma^\circ \mid T(p) \cap \Gamma \text{ is a ray}\}$ is dense in* Bdry Γ°.

Proof. We can take $q \in \text{Int } \Gamma^\circ$ and let $M = \{x \in R^l \mid q \cdot x = 1\}$. Then $M \cap \Gamma$ is a compact convex subset of M. The density of $J = \{p \in \text{Bdry } \Gamma^0 \mid T(p) \cap \Gamma \cap M \text{ is a point}\}$ in Bdry Γ° follows then, for example, from the differentiability a.e. on $T(q)$ of the support function of $M \cap \Gamma$ (see Katzner [3, pp. 204–210]).

Let $f: P \to R^l$ and $(\gtrsim, \omega)$ be as in the statement of I. For $\epsilon > 0$, let $A_\epsilon \subset P$ consist of the elements $p \in P$ for which $f(p)$ contains some triangle of area $\geqslant \epsilon$. Because of the u.h.c. of f on P, $A\hat{\epsilon}$ is closed. We show the following.

IV. *A_ϵ has empty interior.*

Proof. Let $p \in A_\epsilon$ and, for any $x \in f(p)$, consider

$$C = \{y \in \bar{P}: y \gtrsim x + \omega\} - \{\omega\}.$$

Suppose first that $0 \notin C$. Note that $p \in \text{Bdry } \hat{C}^\circ$ and that if $q \in (\text{Bdry } \hat{C}^\circ) \cap P$ then $f(q) \subset C \cap T(q)$. Hence, by II and III we can find $\langle p_n \rangle \to p$, $p_n \in P$, such that $f(p_n)$ is a segment which implies $p_n \notin A_\epsilon$. This concludes the proof of IV since, if $0 \in C$ (which implies $0 \in f(p)$), we can approximate p arbitrarily closely by a p^* such that $T(p^*)$ intersects the interior of C, implying $f(p) \cap f(p^*) = \varnothing$, so $0 \notin f(p^*)$, and we are in the first case.

To complete the proof of I note that, by IV and the remark preceding it, $A = \bigcup_{n=1}^\infty A_{1/n}$ is meager and· if $p \notin A$, then $f(p)$ has to be a segment.

5. FURTHER QUESTIONS

This paper has established that candidate community excess demand functions in an economy with l commodities can be decomposed into the sum of l individual excess demand functions, each consistent with preference maximization. This conclusion has been shown not to hold universally when consideration is extended to candidate community total demand functions or excess demand correspondences.

These results suggest several further questions which we have not attempted to answer.

(i) Suppose the hypothesis of continuity of the community excess demand function is added in Theorems 1 and 2. Can one obtain the result of Debreu that decomposition is possible with individual excess demand functions consistent with maximization of a continuous utility function? There is no obvious obstacle to such a demonstration; however, the extended domains of the excess demand functions in these theorems cause considerable technical difficulties in adapting the arguments of Debreu. We should add that, if one wants continuous and monotone utility functions, the difficulties may be substantial.

(ii) Suppose a candidate community total demand function is specified as a function of prices *and* the aggregate endowment (or aggregate income). Under what conditions, including specification of the income distribution, is decomposition possible with individual demand functions of prices and individual endowments (or individual incomes) that are consistent with preference maximization?[5]

(iii) In light of Theorem 4, is there a set of sufficient conditions, short of single-valuedness, which would make possible the decomposition of a convex valued u.h.c. excess demand correspondence?

(iv) Suppose in each of the questions posed in this paper that one can introduce a competitive profit-maximizing production sector, satisfying the usual technological assumptions and having specified rules for distribution of profits. Can one now decompose a broader class of community demand functions?

(v) Suppose in one of the questions above that a specified demand relation cannot be decomposed. Under what conditions are there decomposible demand relations arbitrarily "close" (in some sense) to the given one? In view of Theorem 4, this question is particularly interesting for the case of correspondences.

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[5] Unpublished results of Hugo Sonnenschein suggest that decomposition may be possible under conditions analogous to those in Theorem 3 provided the domain of aggregate income is sufficiently restricted.

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