JOURNAL OF ECONOMIC THEORY 10, 117-122 (1975)

A Further Result on the Representation of Games by Markets

I. Introduction

In a recent, very interesting sequence of papers Billera and Bixby [1-4] have studied the problem of characterizing the class of games without side payments which can be generated by pure exchange markets with a finite number m of commodities, consumption sets equal to the nonnegative orthant of R^m, and concave, continuous utility functions. For games with side payments the problem was posed and solved by Shapley and Shubik [8].

Billera and Bixby show that games originated by markets with the above characteristics have convex, compact attainable sets and are "totally balanced" in a certain sense (all the definitions will be given below). Conversely, they prove that every such game, which, in addition, is polyhedral (i.e., the attainable sets are polyhedra), is representable by markets with the described properties. Furthermore, they conjecture this to be the general situation. If the allowable class of markets is enlarged to include either infinitely many commodities, or general consumption sets, or production, then Billera [1] shows that every totally balanced game (with convex, compact attainable sets) has a market representation in this extended class.

By relying heavily on the machinery built, and the results obtained, by Billera and Bixby we show in this note how it is possible to obtain market representations in the nonextended sense (finite number of commodities, consumption sets equal to the nonnegative orthant of some R^m, no production) for a very large class (an "open and dense" one) of totally balanced games. The result we give falls short of a full characterization in that the games are required to satisfy a regularity condition (which we call slackness) describable by saying that the (nonredundant) inequalities defining the balancedness condition should hold strictly.

The main open problems in this field are: (i) to prove the full characterization theorem (or to provide a counterexample), (ii) to characterize games coming from pure exchange markets with quasiconcave utility (see [7]), (iii) to sharpen available results by, for example, bounding the number of commodities needed for representing N-players games.

The terminological conventions and definitions (except for the *Slackness* one, which we are proposing) are taken from Billera and Bixby [4].

2. Terminological Conventions

The set of players is $N = \{1,..., n\}$; $2^N = \{S \subseteq N : S \neq \phi\}$. For every $S \in 2^N$ let $R^S = \{x \in R^n : x_i = 0 \text{ for } i \notin S\}$; R^S can be naturally identified with $R^{\#(S)}$. If $x \in R^n$, then $x_S \in R^S$ is given by "$x_{Si} = x_i$ if $i \in S$; $x_{Si} = 0$, otherwise." Also, $R_+{}^n = [0, \infty)^n$, $R_+{}^S = R^S \cap R_+{}^n$.

For $A \subset B \subset R^m$, $\text{Int}_B A$ denotes the interior of A relative to B, and co A the convex hull of A.

Let $\Delta = \{p \in R_+{}^n : \sum_{i=1}^n p_i = 1\}$. Given a convex set $A \subset R_+{}^n$ such that $A = C - R_+{}^n$, for some compact set C, we define the support function of A, $g_A : \Delta \to R$ by $g_A(p) = \sup\{px : x \in A\}$; $g_A(p)$ will also be denoted $g(p; A)$. Here, support functions will only be defined for sets with the described properties. Then, g_A is convex and continuous and for every $p \in \Delta$ the sup. is attained (i.e., there is $x \in A$ such that $px = g_A(p)$); moreover, $g_{A+B} = g_A + g_B$.

3. Definitions

A (n-person cooperative) *game* (without side payments) is a correspondence $V : 2^N \to R^n$ such that, for every $S \in 2^N$, $V(S) \subset R^n$ is of the form $V(S) = C - R_+{}^S$, where $C \subset R_+{}^S$ is nonempty, compact, and convex. The *subgame* of V on $T \in 2^N$ is the restriction of V to 2^T.

Given the games $V_1, V_2 : 2^N \to R^n$ and $x \in R^n$, we can define the games $V_1 + x$, $V_1 \cap V_2$, $V_1 + V_2$ by letting, respectively, $(V_1 + x)(S) = V_1(S) + x_S$, $(V_1 \cap V_2)(S) = V_1(S) \cap V_2(S)$, $(V_1 + V_2)(S) = V_1(S) + V_2(S)$; $V_1 \cap V_2$ is the *intersection* and $V_1 + V_2$ the *sum* of the games V_1, V_2.

A game V is *balanced* if $\sum_{S \subset N} \delta_S V(S) \subset V(N)$ whenever $\delta_S \geqslant 0$, $S \in 2^N$, are such that $\sum_{i \in S} \delta_S = 1$ for every $i \in N$. A game V is *totally balanced* if every subgame of V is balanced.

A game V is *balanced with slack* if $\sum_{S \subset N} \delta_S V(S) \subset \text{Int}_{R^N} V(N)$ whenever $\delta_S \geqslant 0$, $S \in 2^N$, are such that $\delta_N = 0$ and $\sum_{i \in S} \delta_S = 1$ for every $i \in N$. A game V is *totally balanced with slack* if every subgame of V is balanced with slack.

Remark 1. Clearly, the slackness condition is in the nature of a nondegeneracy assumption. With respect to the Hausdorff metric the totally balanced with slack games constitute an open, dense subset of the totally balanced ones; this follows from Lemma 1 below, whose conclusion could be taken as an alternative definition of slackness.

In games with side payments the condition is equivalent to the core having, relative to the Pareto surface, a nonempty interior or, alternatively, to the set of utility allocations blocked only by N being nonempty; without side payments the condition is sufficient, but not necessary, for the latter properties to hold. Hypotheses of the same nature as the slackness one appear in other contexts (for example, in Green [5] or Neuefeind's [6] work on dynamic mechanism leading to the core).

An *n-trader, m-commodity simple market* is a collection $\{(u_i, \omega_i): i \in N\}$, where, for every i, $u_i : [0, 1]^m \to R$ is a continuous, concave function and $\omega^i \in [0, 1]^m$; moreover, without loss of generality, we assume

$$\sum_i \omega^i \leqslant (1,..., 1) \in [0, 1]^m.$$

Remark 2. Every result would hold if we were to require, simply, "for every i, $u_i : R_+{}^m \to R$ is a continuous, concave, nondecreasing function and $\omega^i \in R_+{}^m$."

Given a market $\{(u_i, \omega^i) : i \in N\}$ a game V is defined by letting

$$V(S) = \left\{x \in R^S: x_i \leqslant u_i(y^i),\ y^i \in [0, 1]^m,\ i \in S;\ \sum_{i \in S} y^i \leqslant \sum_{i \in S} \omega^i\right\}.$$

A game $V : 2^N \to R^n$ is a *simple market game* if, for some integer m, it can be generated (represented) in the above manner by an n-person, m-commodity simple market.

A basic result of Billera and Bixby is: *every simple market game is totally balanced* [3, Theorem 2.1].

We prove:

THEOREM. *Every game $V : 2^N \to R^n$ that is totally balanced with slack is a simple market game (i.e., it has a simple market representation).*

4. LEMMATA

The following lemmata are either straightforward facts or results of Billera and Bixby. Let $e = (1,..., 1) \in R^n$.

LEMMA 1. *If the game $V : 2^N \to R^n$ is balanced with slack, then, for some $\epsilon > 0$, the game V_ϵ defined by "$V_\epsilon(N) = V(N) - \{\epsilon e\}$, $V_\epsilon(S) = V(S)$ otherwise" is balanced with slack.*

Proof. It is sufficient to consider nonnegative collections $\{\delta_S : S \in 2^N\}$ with $\delta_N = 0$. Since the set of such collections which solve the equations

$$\text{“}\sum_{i \in S} \delta_S = 1, \quad i \in N\text{”}$$

is compact the result follows by the slackness assumption and the continuity of $\sum_S \delta_S V(S)$ with respect to the δ_S's.

LEMMA 2. *If V is a simple market game and $x \in R^n$, then $V + x$ is a simple market game.*

Proof. Billera and Bixby [3, Proposition 2.2].

LEMMA 3. *The intersection of two simple market games is a simple market game.*

Proof. Billera and Bixby [3, Proposition 3.4].

LEMMA 4. *The sum of two simple market games is a simple market game.*

Proof. Follows from Billera and Bixby [2, Lemma 2.2].

LEMMA 5. *Let V be a balanced game such that $V(N)$ is of the form $V(N) = C - R_+{}^n$, where C is a polyhedron, then there is a simple market game V' such that $V'(N) = V(N)$ and $V'(S) \supset V(S)$ for all $S \in 2^N$.*

Proof. Billera and Bixby [3, Theorem 3.6].

LEMMA 6. *Let V be a game and $x \in V(N)$, then there is a simple market game V' such that $V'(N) = V(N)$ and $x_S \in V'(S)$ for all $S \in 2^N$.*

Proof. By Billera and Bixby [2, Theorem 2.3] there is a simple market game $\hat{V}$ such that $\hat{V}(N) = V(N)$ and $\hat{V}$ has a simple market representation $\{(u_i, \omega^i) : i \in N\}$ with every u_i nondecreasing. Let y^i, $i \in N$, be such that $u_i(y^i) \geqslant x_i$, for every i, and $\sum_i y^i = \sum_i \omega^i$; then the game induced by the market $\{(u_i, y^i) : i \in N\}$ has the desired properties.

LEMMA 7. *If a subgame $V' : 2^T \to R^n$, $T \in 2^N$, of a game is a simple market game, then for any $x \in R^n$ there is an extension $V'' : 2^N \to R^n$* of V' *(i.e., $V''(S) = V'(S)$ for all $S \in 2^T$) such that V'' is a simple market game and $x_S \in V''(S)$ for every $S \in 2^N \sim 2^T$.*

Proof. Follows from [3, Lemma 3.7].

5. Proof of the Theorem

By Lemmata 3 and 7 it suffices to prove:

If the game $V : 2^N \to R^n$ is balanced with slack, then there is a simple market game $V' : 2^N \to R^n$ such that $V'(S) \supset V(S)$ for all $S \in 2^N$ and $V'(N) = V(N)$.

The facts not already exploited by Billera and Bixby which are brought to bear in the proof are: (i) the compactness of the effective domain of the support functions of the convex sets defining the game (to take advantage of this the slackness condition is essential); (ii) the additivity property of market games (Lemma 4).

For every $p \in \Delta$ ($\subset R^n$), $\delta > 0$, and $0 \leqslant j \leqslant n$ define $p^{\delta j} \in \Delta$ by

$$\text{“}p_j^{\delta j} = (p_j + \delta)/(1 + \delta) \qquad \text{and} \qquad p_i^{\delta j} = p_i/(1 + \delta) \qquad \text{if} \quad i \neq j\text{”};$$

then let $P(p, \delta) = \{p^{\delta 1}, \ldots, p^{\delta n}\}$ and note that, for any $p \in \Delta$ and $\delta > 0$, $p \in \operatorname{Int}_\Delta (\operatorname{co} P(p, \delta))$.

Let $a \in R^n$ be an upper bound for $V(N)$ (i.e., for every $x \in V(N)$, $x \leqslant a$) and $\epsilon > 0$ be such that V_ϵ is a balanced game (see Lemma 1). For every $(x, p) \in V(N) \times \Delta$ such that $px = g(p, V(N))$ pick a $\delta(x, p) > 0$ small enough to guarantee that $g(q, V_\epsilon(N)) \leqslant qx$ for every $q \in P(p, \delta(x, p))$; its existence follows by the continuity of $g_{V_\epsilon(N)}$ and the fact that $g(p, V_\epsilon(N)) < g(p, V(N))$. Define now a game $V^{x,p} : 2^N \to R^n$ by “$V^{x,p}(N) = \{ y \in R^n : y \leqslant a, qy \leqslant qx$ for every $q \in P(p, \delta(x, p))\}$ and $V^{x,p}(S) = V(S)$, otherwise.” Obviously, $V^{x,p}(N)$ is of the form $V^{x,p}(N) = C - R_+{}^n$ where C is a (compact) polyhedron. Moreover, since

$$V_\epsilon(N) \subset V^{x,p}(N), \; V^{x,p} \text{ is balanced.}$$

The collection $\{\operatorname{Int}_\Delta(\operatorname{co} P(p, \delta(x, p)) \colon (x, p) \in V(N) \times \Delta; px = g(p, V(N))\}$ constitutes an open covering of the compact set Δ (for every $p \in \Delta$ there is $x \in V(N)$ such that $px = g(p, V(N))$). Let $\{(x^1, p^1), \ldots, (x^M, p^M)\}$ determine a finite subcover.

For every $1 \leqslant h \leqslant M$, there is, by Lemma 5, a simple market game $V_1{}^h$ such that $V_1{}^h(N) = V^{x^h, p^h}(N)$ and $V_1{}^h(S) \supset V^{x^h, p^h}(S)$ for $S \in 2^N \sim \{N\}$; also, by Lemmata 2 and 6, there is a simple market game $V_2{}^h$ such that $V_2{}^h(N) = V(N) - \{x^h\}$ and $0 \in V_2{}^h(S)$ for every $S \in 2^N$. Let $V^h = V_1{}^h + V_2{}^h$; then (Lemma 4) V^h is a simple market game such that $V^h(S) \supset V(S)$ for every $S \in 2^N$. Finally, take $V' = \bigcap_{h=1}^M V^h$; again (Lemma 3) V' is a simple market game and $V'(S) \supset V(S)$ for every $S \in 2^N$. We claim that $V'(N) \subset V(N)$.

Let $y \notin V(N)$; then (by the separating hyperplane theorem) $py > g(p, V(N))$ for some $p \in \Delta$. For some $1 \leqslant h \leqslant M$

$$p \in \text{co } P(p^h, \delta(x^h, p^h))$$

which implies $g(p, V_1{}^h(N)) = px^h$. We have

$$g(p, V^h(N)) = g(p, V_1{}^h(N)) + g(p, V_2{}^h(N)) = px^h + g(p, V(N)) - px^h$$
$$= g(p, V(N) < py.$$

Hence $y \notin V'(N) \subset V^h(N)$. Q.E.D.

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Received: April 10, 1974

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* I am indebted to L. Billera for his comments and to R. Mantel and D. Schmeidler for very helpful conversations on the topics discussed here. I want to thank an unknown referee, who was very helpful on matters of terminology and notation. Of course, I alone bear responsibility.