

## ON THE CONTINUITY OF EQUILIBRIUM PRICES IN CONSTANT-RETURNS PRODUCTION ECONOMIES\*

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### 1. Introduction

The purpose of this paper is to extend the theory on the local uniqueness of equilibrium prices initiated by Debreu (1970) [and pursued by E. and H. Dierker (1972), Smale (1974a), Hazewinkel (1972), Delbaen (1971), Hildenbrand (1972), Fuchs (1974), and others] to production economies of the constant returns variety. The problem is the familiar one: good theoretical reasons for the uniqueness of equilibria being unavailable, one settles for expecting that, at least, equilibria will be locally unique and will vary continuously with perturbations of the economy, i.e., that economies will be ‘regular’ (as  $e_a$ , or  $e_b$ , not  $e_c$ , in fig. 1). Still, if multiplicity of equilibria is possible, non-regular economies (as  $e_c$  in fig. 1) are unavoidable and the best possible result is to establish their exceptionality (that is to say, to justify giving to regular economies its name) by showing, and this is certainly intuitive (see fig. 1), that they are associated with degeneracies and constitute a ‘negligible’ subset of economies.

The notion of ‘negligibility’ we will use is topological [by contrast to measure theoretical, see E. and H. Dierker (1972)]; namely, the space of economies will be a topological space and negligible sets will be the ones which closures have empty interiors. In a production context, where only a limited number of commodities may be factors of production, initial endowments redistributions (if at all allowable) do not give enough parameter variability and so, consumers’ or producers’ characteristics other than initial endowments have to be regarded as variable; the space of economies becomes then infinite-dimensional and forces the topological definition of negligible.

Different approaches to production have been considered by Fuchs (1974) and Smale (1974b). However, constant returns to scale have so far been excluded from the theory – in the case of Fuchs (1974), because supply functions are assumed to exist, and, in the case of Smale (1974b), because the class of constant

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returns economies which fit his model (the linear activity case) constitute, on the whole, a negligible subset of his space of economies. Constant returns to scale is a central phenomenon in economics (indeed, it has even been argued that a correctly specified model should always exhibit constant returns) and, clearly, the theory should encompass them.

An attempt in that direction will be made in the present paper. To this end we confine ourselves to the class of economies with constant returns; we feel this is justified, partly because of the non-constant-returns case has been taken care of in the above references and partly, and mainly, in order to avoid throwing away with the exceptional set of economies the whole set of constant-returns ones (which would certainly eliminate the problem but would hardly solve it!).

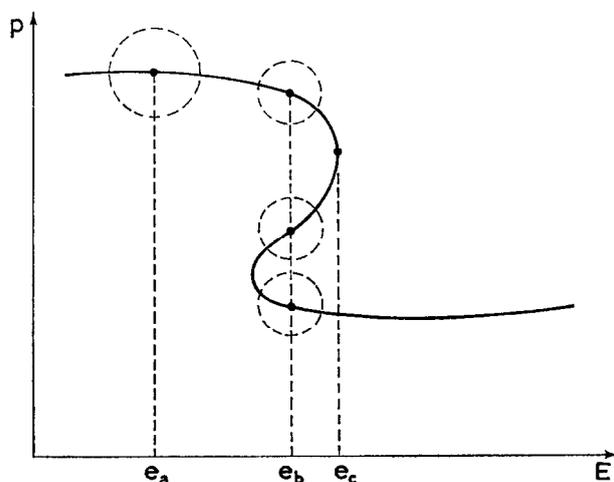


Fig. 1

The difficulty with constant returns is that, while the theory essentially necessitates smoothness assumptions, they constitute an endogenous (in the sense of internal to the theory) source of non-smoothness. Clearly, production sets which are (pointed) cones do not yield supply functions and their boundaries are not smooth. Much more seriously, unless the cone is polyhedral, it cannot be expressed as an intersection of a finite number of manifolds with smooth boundary [and so, it falls outside the model in Smale (1974b), or in Mas-Colell (1973)]. The origin constitutes, in the non-polyhedral case, a fundamental singularity (see fig. 2). In the present analysis non-polyhedral production cones will be allowed.

We will make convexity assumptions throughout; this should be contrasted with Smale (1974a, b) where they are not made.

Consumers will be described by demand functions-initial endowments pairs

and producers by production sets. The production theory envisioned in this paper is of the activity analysis family and it is a generalization of the linear activity model. An elementary activity is specified by a list of commodities (the possible inputs or outputs), a normalization convention, and an isoquant, i.e., a convex set described by its support function (which we require to be smooth, in a certain sense). The use of support functions then permits the definition of (topological) spaces of elementary activities in a natural manner. We emphasize that our model allows for a space of activities in which each one is of the linear variety.

There is another endogenous source of non-smoothness which is disregarded here, namely, constraints (the nonnegativity ones, for example) in the consumption sets. The problems associated with them have been studied by Smale (1974b) and his treatment of consumers is distinctly superior to the one here; our justification for being sketchy on this point is that we want to focus attention on the problems specifically associated with production.



Fig. 2

Besides the deficiency mentioned in the last paragraph the main limitations of this paper are two: (i) no purely intermediate commodities are allowed, i.e., every commodity can, conceivably, be either possessed as initial endowment or consumed by some consumer; however, the much more drastic assumption that every commodity can be an initial endowment (or the one that every commodity can be consumed) is not made; (ii) we concern ourselves only with price-income equilibria in an arbitrarily large but a priori given compact sets of strictly positive prices and incomes. Additional comments on these points will be given in the main body of the text.

Concerning the mathematics, we use the differential topology methods made standard for the treatment of those problems by Debreu (1970) and Smale (1974a). The specific results appealed to do not go beyond those already employed by Fuchs (1974) and Smale (1974a, b).

This is not a paper on the existence of competitive equilibria and, in fact, the non-emptiness of the set of equilibria does not follow from our assumptions. The mathematical tools appropriate to our problem are closer in spirit to the counting of equations and unknowns of the old days, than to the fixed-point theorems of the existence proofs.

Section 2 describes the model and states the results; section 3 contains the proof.

## 2. The model and results

The commodity space is  $\mathbf{R}^l$ ;  $\mathbf{R}_+^n = \{x \in \mathbf{R}^n : x \geq 0\}$ ,  $\mathbf{R}_{++}^n = \{x \in \mathbf{R}^n : x \gg 0\}$ ,  $S = \{p \in \mathbf{R}_{++}^l : \|p\| = 1\}$ ,  $P = \mathbf{R}_{++}^l$ . A coordinate subspace of  $\mathbf{R}^n$  is a subspace defined by conditions of the form  $\{x^i = \dots = x^j = 0\}$ .

### 2.1. Consumers

There are  $N$  consumers.

- (I) *There are given  $N+1$  sets  $\Omega \subset \mathbf{R}^{lN}$ ,  $X^i \subset \mathbf{R}^l$ ,  $1 \leq i \leq N$ , such that:*  
 (i) *for every  $1 \leq i \leq N$ ,  $X^i$  is the strictly positive orthant of a coordinate subspace of  $\mathbf{R}^l$ , (ii)  $\bigcap_{i=1}^N X^i \neq \phi$ , (iii)  $\Omega$  is non-empty and convex, (iv)  $\{\sum_{i=1}^N (x^i + \omega^i) : x^i \in X^i, (\omega^1, \dots, \omega^N) \in \Omega\}$  is an open subset of  $\mathbf{R}^l$ .*

The consumption set of the  $i$ th consumer is  $X^i$ ,  $\Omega$  is the set of allowable  $N$ -tuples of initial endowments (not necessarily a product). In substance, assumption (iv) says that every commodity can be either consumed or possessed as initial endowment by some consumer, i.e., there are not purely intermediate commodities. Assumption (ii) says that there is a common vector (a 'common commodity') in all the consumption sets; it is devised to avoid degeneracies (of the sort: the world is composed of two continents unknown one to the other).

For every  $i$ , let  $L(X^i)$  be the minimal coordinate subspace containing  $X^i$  and denote by  $\pi_i: \mathbf{R}^l \rightarrow L(X^i)$  the perpendicular projection map.

For every  $i$ ,  $\mathcal{H}^i$  denotes the set of  $C^1$  functions  $h^i: P \times \mathbf{R}_{++} \rightarrow X^i$  which are 'homogeneous', i.e.,  $h^i(\lambda p, \lambda t) = h^i(\pi_i(p), t)$  for all  $\lambda \in \mathbf{R}_{++}$  (the values of  $h^i$  depend only on the budget set) and satisfy Walras' Law, i.e.,  $ph^i(p, t) = t$  for all  $(p, t) \in P \times \mathbf{R}_{++}$ . Endow  $\mathcal{H}^i$  with the topology of uniform convergence in compact sets of the functions and their first partial derivatives.

- (II) *For every  $i$ , there is given an open subset  $\mathcal{H}^i$  of  $\mathcal{H}^i$ .*

Every  $h^i \in \mathcal{H}^i$  is a demand function possible for the  $i$ th consumer. The implicit important restrictions of formalizing consumers by  $C^1$  demand functions are two: (i) the analysis is put into a convexity framework; this is in keeping with the general approach of this paper; (ii) (non-negativity) constraints as sources of non-smoothness of demand are disregarded; this is a simplification made in order to focus attention on the sources of non-smoothness specific to production. In a preference approach the present analysis is consistent with the specification of consumer  $i$  by a smooth, convex preference relation defined on  $X^i$  and such that the closures in  $\mathbf{R}^l$  of the indifference hypersurfaces are contained in  $X^i$  [see Debreu (1972)]. In fact, the results of this paper can be readily applied to a model where consumers are formalized in this manner.

Let  $\mathcal{E} = \prod_{i=1}^N \mathcal{H}^i \times \Omega$ ;  $\mathcal{E}$  is the space of consumers, its members will be denoted by  $e = (h, \omega) = (h^1, \dots, h^N, \omega^1, \dots, \omega^N)$ .

### 2.2. Technology

The technological and productive possibilities of the economy are described by means of production sets  $Y \subset \mathbf{R}^l$  [see Debreu (1959, ch. 3)].

How do we define an appropriate space of constant-returns production sets? The following example and remarks may help to motivate the procedure we shall employ. Suppose that among the  $l = 5$  commodities the first is an output while the others are inputs; as an example of an elementary activity consider the Cobb–Douglas production function  $x_1 = x_2^{\frac{1}{2}}x_3^{\frac{1}{3}}$ ; normalizing by putting  $x_1 = 1$  we obtain the unit product isoquant  $\{(1, x_2, \dots, x_5) : 1 = x_2^{\frac{1}{2}}x_3^{\frac{1}{3}}\}$ . Let  $a = (1, 0, 0, 0, 0)$  and observe that the fourth and fifth commodities do not enter in the activity so that once normalized only the second and third matter. Let  $E \subset \mathbf{R}^l$  be the coordinate subspace defined by  $x_1 = x_4 = x_5 = 0$ . Given  $E$  and  $a$  the convex set  $C = \{(x_2, x_3) \in E : -x_2^{\frac{1}{2}}x_3^{\frac{1}{3}} \geq 1\}$  does completely describe the activity since the unit isoquant is simply the frontier of  $C + \{a\}$  (see fig. 3). We have therefore identified the activity, given a fixed  $E$  and  $a$ , with a convex set; it is then natural to identify the space of activities with a space of convex sets and by describing those by their support functions, we can in turn identify the activities with a space of functions.

We now proceed more formally.

For any coordinate subspace  $E \subset \mathbf{R}^l$  with  $\dim E \geq 1$  denote by  $\mathcal{F}_E$  the space of  $C^2$  functions  $f: P \rightarrow \mathbf{R}$  which are convex, homogeneous of degree 1, and such that  $Df(p) \in E$  everywhere in their domains<sup>1</sup> (i.e., every  $f \in \mathcal{F}_E$  is of the form  $f' \circ \pi$ , where  $f'$  is defined on  $E \cap P$  and  $\pi$  is the projection on  $E$ ). With the obvious operations of addition and scalar multiplication  $\mathcal{F}_E$  is then a cone; the subset  $\mathcal{F}'_E$  of linear functions in  $\mathcal{F}_E$  [i.e.,  $f \in \mathcal{F}'_E$  if and only if there is  $b \in E$ , such that  $f(p) = pb$ ] is a linear space;  $\mathcal{F}_E$  is endowed with the topology of uniform convergence in compact sets of the function and their first partial derivatives.

An ‘activity’ will be specified by a triple  $(E, a, f)$ , such that  $E \subset \mathbf{R}^l$  is a coordinate subspace with  $\dim E \geq 1$ ,  $a \in \mathbf{R}^l_+$  and  $f \in \mathcal{F}_E$ . It is understood that the production set originated by an activity is

$$Y_{(E,a,f)} = \text{clo.} \{x \in \mathbf{R}^l : x \leq \lambda(Df(p) + a) \text{ for some } p \in P \text{ and } \lambda \in \mathbf{R}_+\}.$$

The description of an activity as the triple  $(E, a, f)$  is a natural one: by taking  $f$  as a support function one defines a convex subset  $C$  of  $E$ , i.e.,

$$\begin{aligned} C &= \{x \in E : px \leq f(p) \text{ for all } p \in P\} \\ &= \text{clo.} \{x \in E : x \leq Df(p) \text{ for some } p \in P\}; \end{aligned}$$

<sup>1</sup>We regard  $Df(p)$  indistinctly as a linear function or as a vector.

<sup>2</sup>clo. stands for closure.

then we convene that a production set  $Y_{(E,a,f)}$  is generated by letting it be the minimal closed cone containing  $(C + \{a\}) - \mathbf{R}_+^l$ ; see fig. 3. Thus, the description of an activity can embody both a normalization rule (if  $\dim E \leq l-1$  and  $a \neq 0$ ) and restrictions of the type 'such and such commodities are not inputs or outputs of the activity'.

(III) *There is given a collection  $\{(E_j, a_j, \mathcal{B}_j) : j \in M\}$ ,  $1 \leq \#M < \infty$ , such that, for every  $j \in M$ ,  $E_j$  is a coordinate subspace with  $\dim E_j \geq 1$ ,  $a_j \in \mathbf{R}_+^l$ , and  $\mathcal{B}_j \subset \mathcal{F}_{E_j}$  is open relative to some subcone of  $\mathcal{F}_{E_j}$  which contains  $\mathcal{F}'_{E_j}$ . Let  $\mathcal{B} = \prod_{j \in M} \mathcal{B}_j$  and, for  $f \in \mathcal{B}$ ,  $Y_f = \sum_{j \in M} Y_{(E_j, a_j, f_j)}$ .*

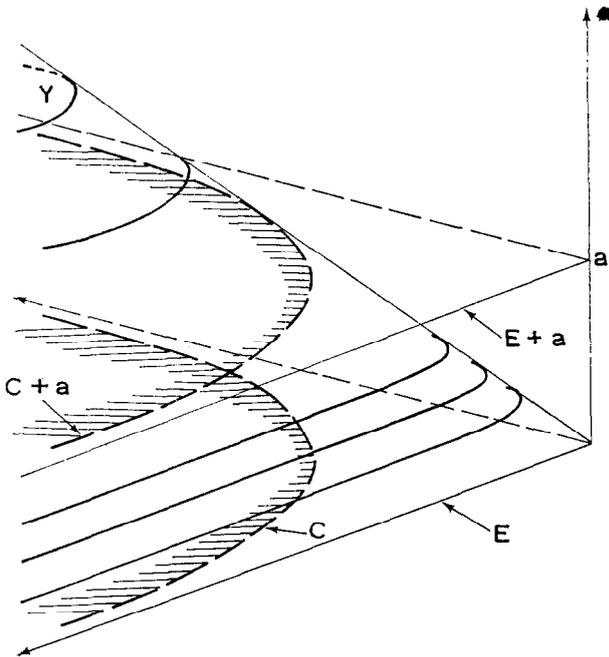


Fig. 3

The condition on  $\mathcal{B}_j$  implies that if  $f_j \in \mathcal{B}_j$  and  $b \in E_j$ , then, for  $\bar{\varepsilon} > 0$  small enough, if  $\varepsilon < \bar{\varepsilon}$ , the function defined by  $f_j(p) + \varepsilon bp$  also belongs to  $\mathcal{B}_j$ . If the reader finds the condition too involved, he can simply take ' $\mathcal{B}_j$  is an open subset of  $\mathcal{F}_{E_j}$ '. However, a lot of generality is gained, at no expense, with the first formulation; in particular, it includes the model where the activities are constrained to be linear [hence the results here generalize those in Mas-Colell (1973)].

The kind of isoquants retained in this paper are the ones in fig. 4 (three specimens are drawn; they could touch the boundary). The isoquants of fig. 5

could be incorporated with full generality easily. The spirit of this paper is that, given constant returns to scale, isoquants as the ones in fig. 6 reveal the existence of more elementary activities (drawn in the figure), the combination of which gives rise to the flat areas.

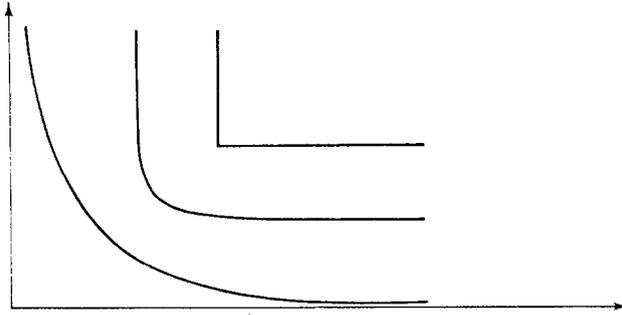


Fig. 4

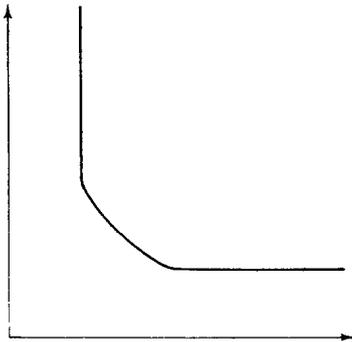


Fig. 5

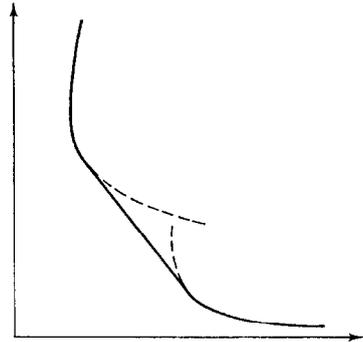


Fig. 6

### 2.3. Equilibrium and the equilibrium correspondence

Given an  $e = (h, \omega) \in \mathcal{E}$  and a (production) set  $Y \subset \mathbf{R}^l$  a  $(p, w) \in S \times \mathbf{R}_{++}^N$  is a competitive price-income equilibrium for  $e$  and  $Y$  if:

- (i)  $\sum_{i=1}^N (h^i(p, w^i) - \omega^i) \in Y$ ,
- (ii)  $py \leq 0$  for all  $y \in Y$ ,
- (iii)  $p\omega^i = w^i$  for every  $i$ .

Note that if  $(p, w)$  and  $(p, w')$  are competitive price-income equilibria for  $e$  and  $Y$ , then  $w = w'$ , so the notion of competitive price equilibrium is unambiguously defined.

For a fixed  $Y$  define the equilibrium correspondence (empty values are allowed)  $W^Y: \mathcal{E} \rightarrow S \times \mathbf{R}_{++}^N$  by

$$W^Y(e) = \{(p, w) \in S \times \mathbf{R}_{++}^N : (p, w) \text{ is an equilibrium for } e \text{ and } Y\}.$$

Define also  $W: \mathcal{E} \times \mathcal{B} \rightarrow S \times \mathbf{R}_{++}^N$  by

$$W(e, f) = W^{Y_f}(e).$$

We are interested in the local uniqueness and the stability under perturbations of  $W^Y$  and  $W$ .

Let  $H \subset S \times \mathbf{R}_{++}^N$ . We say that  $W^Y$  is regular at  $e \in \mathcal{E}$  with respect to  $H$  if, for every  $(p, w) \in W^Y(e) \cap H$ , there are open neighborhoods of  $e$  and  $(p, w)$ ,  $\mathcal{O} \subset \mathcal{E}$ ,  $V \subset S \times \mathbf{R}_{++}^N$ , respectively, and a continuous function  $\varphi: \mathcal{O} \rightarrow V$  such that  $(p', w') \in W^Y(e')$ ,  $(e', (p', w')) \in \mathcal{O} \times V$  if and only if  $(p', w') \in \varphi(e')$ . Say that  $W$  is regular at  $(e, f)$  with respect to  $H$  if the analogous property holds.

All throughout our analysis we will be concerned only with price equilibria in an a priori given compact set  $H$ . In the approach taken here the extension of the results from  $H$  to  $S \times \mathbf{R}_{++}^N$  hinges on the excess demand functions having an appropriate boundary behavior. Conditions for this can be given [also, a finer topology on  $\mathcal{F}^i$  would be required; see Mas-Colell (1973)]. However, they are both very restrictive (in a production context) and foreign to the nature of the mathematical problem and methods involved (purely local ones), so we prefer to avoid them. A fully satisfactory treatment would allow for zero equilibrium prices and would call for (i) a different approach to the modeling of the consumption side of the economy, (ii) the existence of purely intermediate commodities.

#### 2.4. A proposition

Note that, by definition, given  $H$  compact,  $W^Y$  and  $W$  are regular with respect to  $H$  on an open set of their domains.

*Proposition.* Given  $H \subset S \times \mathbf{R}_{++}^N$ , compact, there is an open, dense set  $\mathcal{B}' \subset \mathcal{B}$  such that

- (i) if  $f \in \mathcal{B}'$  then  $W^{Y_f}$  is regular with respect to  $H$  on an open, dense set  $\mathcal{E}_f \subset \mathcal{E}$ ;
- (ii) if  $f \in \mathcal{B}'$  and  $e \in \mathcal{E}_f$ , then  $W$  is regular at  $(e, f)$  with respect to  $H$ .

*Remark 1.* Intuitively, the proposition says that ‘most’ economies are as  $e_a$  and  $e_b$  in fig. 1. We point out that, without any change in the proofs, the definition of regularity can be strengthened by replacing ‘differentiable’ for ‘continuous’.

*Remark 2.* It is clear that the proposition includes pure exchange economies as a special case. As already indicated without assumption I(ii) on  $(X^1, \dots, X^N)$  the proposition is open to trivial and obvious counterexamples. The same is true of assumption (iv) on the  $X^i$ 's and  $\Omega$ . We hasten to add that this is not due to very substantive reasons but to the fact that, as it stands in the text, the problem is not properly posed if either of the two assumptions may not hold.

*Remark 3.* The proposition is not vacuous; except for special choices of  $Y$ ,  $\mathcal{H}$ , and  $\mathcal{B}$ , the correspondences  $W^Y$  and  $W$  will be non-empty on an open set.

### 3. Proof of the proposition

We will prove only part (i). Once this has been done the proof of (ii) is simply a routine matter (reiterated application of the implicit-function theorem). The avoidance of a formal proof will save notation.

Let  $H \subset S' \times (1/r, r)^N$ , where  $r > 0$  and  $S'$  is the intersection of  $S$  with a convex, open cone in  $\mathbf{R}^l$  such that  $\text{clo. } S' \subset S$ . Denote  $\mathcal{H}_c^i = \{h^i \mid S' \times (1/r, r)^N : h^i \in \mathcal{H}^i\}^3$  and analogously for  $\mathcal{H}_c^i$ . Endow  $\mathcal{H}_c^i$  with the topology of uniform convergence of the function and their first partial derivatives; then  $\mathcal{H}_c^i$  becomes a second countable space.

Let

$$T(\mathcal{H}_c^i) = \{v \mid S' \times (1/r, r) : v : P \times \mathbf{R}_{++} \rightarrow L(X^i) \text{ s.t. for all } (p, t) \in P \times \mathbf{R}_{++}, \text{ and } \lambda \in \mathbf{R}_{++}, v(\lambda p, \lambda t) = v(\pi_i(p), t) \text{ and } pv(p, t) = 0\};$$

then  $T(\mathcal{H}_c^i)$  is a closed subspace of the Banach space of  $C^1$  functions on  $S' \times (1/r, r)$  with bounded partial derivatives [and norm  $\sup_{(p,t)} \{\|h^i(p, t)\| + \|Dh^i(p, t)\|\}$ ; see Abraham and Robbin (1967, p. 24)];  $\mathcal{H}_c^i$  is simply a translation of an open subset of this subspace and therefore it is itself a manifold which tangent space (at every point) is precisely  $T(\mathcal{H}_c^i)$ .

Let  $\mathcal{E}' = \prod_{i=1}^N \mathcal{H}_c^i \times \Omega \equiv \mathcal{H}_c \times \Omega$ . Since only the values of  $h \in \prod_{i=1}^N \mathcal{H}^i$  on an open neighborhood of the compact set  $H$  are relevant for the equilibrium problem and a dense subset of  $\prod_{i=1}^N \mathcal{H}_c^i$  is extendable to a dense subset of  $\prod_{i=1}^N \mathcal{H}^i$ , it is clear that it suffices to prove the proposition for the situation where  $\mathcal{E}, S \times \mathbf{R}_{++}^N$ , are replaced throughout by  $\mathcal{E}', S' \times (1/r, r)^N$ ; in particular, we have now  $W : \mathcal{E}' \times \mathcal{B} \rightarrow S' \times (1/r, r)^N$ ,  $W^Y : \mathcal{E}' \rightarrow S' \times (1/r, r)^N$ . Also, by extending it if necessary, we can assume that  $\Omega$  is open relative to its linearity space;  $\mathcal{E}' \times S' \times (1/r, r)^N$  is then a second countable  $C^1$  manifold.

Define  $\Psi^1 : \mathcal{E}' \times S' \times (1/r, r)^N \rightarrow S$ ,  $\Psi^2 : \mathcal{E}' \times S' \times (1/r, r)^N \rightarrow \mathbf{R}^l$ ,  $\Psi^3 : \mathcal{E}' \times S' \times (1/r, r)^N \rightarrow \mathbf{R}^{N-1}$  by, respectively,  $\Psi^1(h, \omega, p, w) = p$ ,  $\Psi^2(h, \omega, p, w) = \sum_{i=1}^N (h^i(p, w^i) - \omega^i)$ ,  $\Psi^3(h, \omega, p, w) = (p\omega^1 - w^1, \dots, p\omega^{N-1} - w^{N-1})$ . Let  $\Psi :$

<sup>3</sup> $g \mid J$  designates the restriction of the function  $g$  to the set  $J$ .

$\mathcal{E}' \times S' \times (1/r, r)^N \rightarrow S \times \mathbf{R}^l \times \mathbf{R}^{N-1}$  be given by  $\Psi = (\Psi^1, \Psi^2, \Psi^3)$ . For  $e \in \mathcal{E}'$ ,  $\Psi_e: S' \times (1/r, r)^N \rightarrow S \times \mathbf{R}^l \times \mathbf{R}^{N-1}$  is the induced map  $\Psi_e(p, w) = \Psi(e, p, w)$ .

By the differentiability of the evaluation map [Abraham and Robbin (1967, 10.3, p. 25)]  $\Psi$  is a  $C^1$  function.

*Lemma 1.* For every  $(e, p, w)$ ,  $D\Psi(e, p, w)$  is surjective.

*Proof.* Let  $\bar{a} = (\bar{h}, \bar{\omega}, \bar{p}, \bar{w}) \in \mathcal{E}' \times S' \times (1/r, r)^N$ ,  $\bar{\Psi} = \Psi(\bar{a})$ . It will be useful to write the variable  $w$  as  $(w', w^N)$ , where  $w' = (w^1, \dots, w^{N-1})$ . Let  $\hat{\Omega} = \{\hat{\omega} \in \mathbf{R}^l: \hat{\omega} = \sum_{i=1}^N \omega^i, \omega \in \Omega\}$ ; as  $\Omega, \hat{\Omega}$  is a  $C^1$  manifold.

By assumption I(iv),  $\mathbf{R}^l = \sum_{i=1}^N L(X^i) + T(\hat{\Omega})$  where  $T$  denotes tangent space. Let  $\{v_1, \dots, v_k\}$  be a set of linearly independent vectors from  $\bigcup_{i=1}^N L(X^i)$  spanning  $\Gamma_1 = \sum_{i=1}^N L(X^i)$ . Let  $\{u_{k+1}, \dots, u_l\}$  be a set of vectors (possibly empty) from  $T(\hat{\Omega})$  such that, letting  $\hat{u}_h = \sum_{i=1}^N u_h^i$ ,  $\{\hat{u}_{k+1}, \dots, \hat{u}_l\}$  spans a space  $\Gamma_2$  complementary to  $\Gamma_1$ , i.e.,  $\Gamma_1 \oplus \Gamma_2 = \mathbf{R}^l$ . For every  $1 \leq j \leq k$  choose an  $i_j$  such that  $v_j \in L(X^{i_j})$ .

We have  $T_{\bar{\Psi}}(S \times \mathbf{R}^l \times \mathbf{R}^{N-1}) = T_{\bar{p}}(S) \times \mathbf{R}^l \times \mathbf{R}^{N-1} = (T_{\bar{p}}(S) \times \Gamma_2 \times \mathbf{R}^{N-1}) \oplus (\{0\} \times \Gamma_1 \times \{0\})$ . Designate by  $\Pi$  the (not necessarily perpendicular) projection map of  $T_{\bar{p}}(S) \times \mathbf{R}^l \times \mathbf{R}^{N-1}$  on  $T_{\bar{p}}(S) \times \Gamma_2 \times \mathbf{R}^{N-1}$  along  $\{0\} \times \Gamma_1 \times \{0\}$ . The lemma will be proved if it is shown:

- (a)  $D\Psi(\bar{a})$  maps onto  $\{0\} \times \Gamma_1 \times \{0\}$ ,
- (b)  $\Pi \circ D\Psi(\bar{a})$  maps onto  $T_{\bar{p}}(S) \times \Gamma_2 \times \mathbf{R}^{N-1}$ ,

since (a) and (b) imply that  $D\Psi(\bar{a})$  maps  $T_{\bar{a}}(\mathcal{E}' \times S' \times (1/r, r)^N)$  onto  $T_{\bar{\Psi}}(S \times \mathbf{R}^l \times \mathbf{R}^{N-1})$ .

Write  $T_{\bar{a}}(\mathcal{E}' \times A')$  as  $\prod_{i=1}^N T(\hat{\mathcal{E}}^i) \times T(\hat{\Omega}) \times T_{\bar{p}}(S) \times \mathbf{R}^{N-1} \times \mathbf{R}$ .

*Proof of (a).* Let  $v \in \bigcap_{i=1}^N X^i$ . For every  $1 \leq j \leq k$  define  $v_j': P \times \mathbf{R}_{++} \rightarrow L(X^{i_j})$  by  $v_j'(p, t) = v_j - (pv_j/pv)v$ . It is clear that  $v_j' \in T(\hat{\mathcal{E}}^i)$ . An immediate computation [see Abraham and Robbin (1967, p. 25)] gives

$$D\Psi(\bar{a})(0, \dots, \overbrace{v_j'}^{i_j \text{ place}}, \dots, 0; 0; 0; 0; 0) = (0, \bar{v}_j, 0) \in \{0\} \times \Gamma_1 \times \{0\},$$

where  $\bar{v}_j = v_j - (\bar{p}v_j/\bar{p}v)v$ . The collection  $\{\bar{v}_1, \dots, \bar{v}_k\}$  has rank  $k-1$ ; moreover,  $\bar{p}\bar{v}_j = 0$  for every  $1 \leq j \leq k$ . On the other hand,  $\bar{p}D_{w^N}h^N(\bar{p}, \bar{w}^N) = 1$  and, therefore,  $D\Psi(\bar{a})(0; 0; 0; 0; 1) = (0, D_{w^N}h^N(\bar{p}, \bar{w}^N), 0)$  provides the missing vector for a basis of  $\{0\} \times \Gamma_1 \times \{0\}$ .

*Proof of (b).* Write  $T_{\bar{p}}(S) \times \Gamma_2 \times \mathbf{R}^{N-1} = T_{\bar{p}}(S) \times \{0\} \times \mathbf{R}^{N-1} \oplus \{0\} \times \Gamma_2 \times \{0\}$  and let  $\eta$  be the projection of  $T_{\bar{p}}(S) \times \Gamma_2 \times \mathbf{R}^{N-1}$  on  $\{0\} \times \Gamma_2 \times \{0\}$  along  $T_{\bar{p}}(S) \times \{0\} \times \mathbf{R}^{N-1}$ . We have that (i)  $\Pi \circ D\Psi(\bar{a})(\{0\} \times \{0\} \times T_{\bar{p}}(S) \times \mathbf{R}^{N-1} \times \{0\}) =$

$T_{\bar{p}}(S) \times \{0\} \times \mathbf{R}^{N-1}$ , and (ii)  $\eta \circ \Pi \circ D\Psi(\bar{a})$  maps onto  $\{0\} \times \Gamma_2 \times \{0\}$  since, for every  $k < j \leq l$ ,  $D\Psi^2(\bar{a})(0; u_j; 0; 0; 0) = -\hat{u}_j$ ; (i) and (ii) together yield (b). ■

By convention, for every  $j \in M$ , we let  $\mathbf{R}^j = \mathbf{R}$  and, for  $Q \subset M$ ,  $\mathbf{R}^Q = \prod_{j \in Q} \mathbf{R}^j$ .

*Lemma 2.* *There is an open, dense set  $\mathcal{B}' \subset \mathcal{B}$  such that, for every  $Q \subset M$ ,  $Q \neq \emptyset$ , and  $f \in \mathcal{B}'$ , (i)  $S(Q) = \{p \in S' : f_j(p) + a_j p = 0 \text{ for every } j \in Q\}$  is an  $l-1-\#Q$  manifold, and (ii) for every  $p \in S(Q)$ ,  $\{Df_j(p) + a_j : j \in Q\}$  are linearly independent vectors.*

*Proof.* Take  $\emptyset \neq Q \subset M$  and for  $f \in \mathcal{B}$  define  $G^f : P \rightarrow \mathbf{R}^Q$  by  $G^f(p) = f_j(p) + pa_j, j \in Q$ . Denote  $J = \{\lambda p : p \in S', \lambda \in \mathbf{R}_{++}\}$ . Since  $G^f$  is homogeneous of degree one, it follows, by the compactness of  $S'$ , that  $\mathcal{B}'_Q = \{f \in \mathcal{B} : 0 \text{ is a regular value of } G^f \mid J\}$  is open. If  $f \in \mathcal{B}'_Q, p \in J$ , and  $G^f(p) = 0$ , then, by definition,  $\{Df_j(p) + a_j : j \in Q\}$  have to be linearly independent. Moreover, if  $f \in \mathcal{B}'_Q, G^{f^{-1}}(0) \cap J$  is a manifold of dimension  $l-\#M$  and therefore (remembering again that  $G^f$  is linearly homogeneous)  $S(Q) = \{(1/\|p\|)p : p \in G^{f^{-1}}(0) \cap J\}$  is a  $l-1-\#Q$  manifold.

Since there is only a finite number of distinct  $Q$  it suffices to prove that  $\mathcal{B}'_Q$  is dense in  $\mathcal{B}$ . For every  $j \in Q$ , pick  $b_j \in E_j, b_j > 0$ . Take  $f \in \mathcal{B}$  and define  $\varphi : P \times \mathbf{R}^Q \rightarrow \mathbf{R}^Q$  by  $\varphi_j(p, t) = f_j(p) + pa_j + t_j pb_j$ . Since, for  $p \in P$  and  $j \in Q, pb_j \neq 0, D_t \varphi(p, t)$  is a linear isomorphism for every  $(p, t) \in P \times \mathbf{R}^M$  and, therefore,  $\varphi$  is a regular map. Hence, by the Transversal Density Theorem [see Abraham and Robbin (1967, 19.1, p. 48) and Abraham (1964, p. 39)] and the hypothesis on  $\mathcal{B}$ , there is an arbitrarily small  $\bar{t} \in \mathbf{R}^M$  such that, defining  $\bar{f}$  by  $\bar{f}_j(p) = f_j(p)$  if  $j \notin Q$  and  $\bar{f}_j(p) = f_j(p) + \bar{t}_j pb_j$  if  $j \in Q$ , we have: (i)  $\bar{f} \in \mathcal{B}$ , (ii)  $G^{\bar{f}} = \varphi(\cdot, \bar{t})$  is a regular map. Hence  $\bar{f} \in \mathcal{B}'_Q$ . Clearly, by taking  $\bar{t}$  sufficiently close to 0,  $\bar{f}$  can be made arbitrarily close to  $f$ . ■

Given a (production) set  $Y \subset \mathbf{R}^l$  define  $\hat{Y} = \{(p, x) \in S \times Y : py \leq px \text{ for every } y \in Y\} \subset S \times \mathbf{R}^l$ . Let  $\theta$  be the zero element of  $\mathbf{R}^{N-1}$ . We can rephrase the definition of equilibrium as follows:

$$(p, w) \in H \text{ is an equilibrium for } (h, w) \in \mathcal{E}' \text{ and } Y \text{ if and only if } \Psi(h, \omega, p, w) \in \hat{Y} \times \{\theta\}.$$

We shall prove now that the set  $\mathcal{B}' \subset \mathcal{B}$  given by Lemma 2 has all the properties required in the proposition.

Let  $f \in \mathcal{B}'$ . This  $f$  is kept fixed for the rest of the proof.

For every  $j \in M$ , let  $C_j = \{x \in \mathbf{R}^l : px \leq f_j(p) + pa_j \text{ for every } p \in S\}$ . Then if  $(p, x) \in \hat{Y}_f, x$  can be written in the form  $x = \sum_{j \in M} t_j x_j, t_j \geq 0, x_j \in C_j$ ; moreover, if  $x \in C_j$  for some  $j \in M$ , then  $x = Df_j(p)$ .

For every  $Q \subset M$  we shall define certain collections  $A^0_Q, \{A^1_Q : j \in Q\}$  of

submanifolds of  $S \times \mathbf{R}^l$  in the following manner: if  $Q = \phi$ , let  $A_Q^0 = S \times \{0\}$ ; if  $Q \neq \phi$ , then, using Lemma 2, we see that the function  $\Phi_Q: S(Q) \times \mathbf{R}^Q \rightarrow S \times \mathbf{R}^l$  given by  $\Phi_Q(p, t) = (p, \sum_{j \in Q} t_j Df_j(p))$  is a diffeomorphism; take  $A_Q^0 = \Phi_Q(S(Q) \times \mathbf{R}^Q)$  and  $A_Q^j = \Phi_Q(S(Q) \times \mathbf{R}_j^Q)$ , where  $\mathbf{R}_j^Q = \{t \in \mathbf{R}^Q: t_j = 0\}$ . Observe that  $A_Q^0$  is a manifold of  $\dim = l-1$  and that, for every  $j \in Q$ ,  $A_Q^j$  is a manifold of  $\dim < l-1$ .

Let  $\mathcal{E}'_f = \{e \in \mathcal{E}': \Psi_e \pitchfork A_Q^h \times \{\theta\} \text{ for every } Q \subset M \text{ and } h = 0 \text{ or } h \in Q\}$ . We prove now that  $\mathcal{E}'_f$  has the desired properties. In the first place, by the Transversal Density Theorem [see Abraham and Robbins (1967, 19.1, p. 48) and Abraham (1964, p. 39)],  $\mathcal{E}'_f$  is a dense subset of  $\mathcal{E}'$  since, by Lemma 1,  $\Psi$  is regular and this, obviously, implies  $\Psi \pitchfork A_Q^h \times \{\theta\}$  for  $Q \subset M$  and  $h = 0$  or  $h \in Q$ .

Let  $e \in \mathcal{E}'_f$ . This  $e$  will be kept fixed for the rest of the proof.

Since  $\dim A_Q^j < l-1$ ,  $\Psi_e \pitchfork A_Q^j \times \{\theta\}$ ,  $j \in Q$ , does necessarily mean  $\Psi_e(S' \times (1/r, r)^N) \cap A_Q^j \times \{\theta\} = \phi$  (note that  $\text{codim } \Psi_e^{-1}(A_Q^j \times \{\theta\}) = \text{codim } A_M^j \times \{\theta\} > l+N-1 = \dim S' \times (1/r, r)^N$ ). Since  $\Psi_e(H)$  is compact and  $A_Q^j \times \{\theta\}$  closed (relative to  $S \times \mathbf{R}^l \times \mathbf{R}^{N-1}$ ) there is, by the continuity of  $\Psi$ , an open neighborhood of  $e$ ,  $\mathcal{O} \subset \mathcal{E}'$  such that if  $e' \in \mathcal{O}$ , then  $\Psi_{e'}(H) \cap A_Q^j \times \{\theta\} = \phi$  for every  $Q \subset M$ ,  $Q \neq \phi$ ,  $j \in Q$ .

Let  $(p, w) \in W^{Y_f}(e) \cap H$ ; we have  $f_j(p) + a_j p \leq 0$  for every  $j \in M$ . Take  $Q = \{j \in M: f_j(p) = 0\}$ .

Suppose first that  $Q \neq \phi$ .  $\Psi_e^2(p, w)$  can be written as  $\Psi_e^2(p, w) = \sum_{j \in Q} t_j x_j$ ,  $t_j \geq 0$ ,  $x_j \in C_j$ ,  $(p, x_j) \in \hat{Y}_f$ ; since, then,  $x_j = Df_j(p)$  we have that  $\Psi_e(p, w) \in A_Q^0 \times \{\theta\}$ . Since  $\Psi_e \pitchfork A_Q^0 \times \{\theta\}$  and  $\dim S \times P = \text{codim } A_Q^0 \times \{\theta\} = l+N-1$ , we can [by the Implicit Function Theorem; see Abraham and Robbin (1967, 20.1, p. 51)] take open neighborhoods of  $e$  and  $(p, w)$ ,  $\mathcal{O}' \subset \mathcal{O}$  and  $U \subset S' \times (1/r, r)^N$ , respectively, and a continuous function  $\varphi: \mathcal{O}' \rightarrow U$  such that  $\Psi(e', p', w') \in A_Q^0 \times \{\theta\}$ ,  $(e', p', w') \in \mathcal{O}' \times U'$  if and only if  $(p', w') \in \varphi(e')$ . From now on, for  $e' \in \mathcal{O}'$ , we denote  $x(e') = \Psi_e^2(\varphi(e'))$ ,  $p(e') = \varphi^1(e')$  (where  $\varphi^1$  is the composition of  $\varphi$  with the projection on  $S$ ).

For  $e' \in \mathcal{O}'$ ,  $x(e')$  can be written uniquely as  $x(e') = \sum_{j \in Q} t_j(e') Df_j(p(e'))$ ; since  $e \notin A_Q^j$ , for  $j \in Q$ ,  $t_j(e) \neq 0$  for every  $j \in Q$  and so (remember that  $t_j(e) \geq 0$ )  $t_j(e) > 0$ ,  $j \in Q$ . By continuity there is a neighborhood of  $e$ ,  $\mathcal{O}'' \subset \mathcal{O}'$  such that for  $e' \in \mathcal{O}''$ ,  $t_j(e') > 0$  for every  $j \in Q$  and  $f_j(p(e')) < 0$  for every  $j \notin Q$ ; hence [remember that  $p(e')x(e') = 0$ ] if  $e \in \mathcal{O}''$ , then  $(p(e'), x(e')) \in \hat{Y}_f$  and, therefore,  $\varphi(e') \in W^{Y_f}(e')$ .

If  $(p', x') \in \hat{Y}_f$  is sufficiently close to  $(p(e), x(e))$  the generators of  $x'$  should belong to  $C_j$ 's with indices in  $Q$ . Hence, by the continuity of  $\Psi$ , there are open sets  $\mathcal{O}''' \subset \mathcal{O}''$ ,  $U' \subset U$  such that if  $(e', p', w') \in \mathcal{O}''' \times U'$  and  $(p', w') \in W^{Y_f}(e')$ , then  $\Psi^2(e', p', w')$  can be written in the form  $\sum_{j \in Q} t_j x_j$ ,  $t_j \geq 0$ ,  $x_j \in C_j$ ; since we have that  $x_j = Df_j(p)$ , it follows that  $\Psi(e', p', w') \in A_Q^0 \times \{\theta\}$  and therefore  $(p', w') \in \varphi(e')$ .

Summing up,  $(p', w') \in W^{Y_f}(e')$ ,  $(e', p', w') \in \mathcal{O}''' \times U'$  if and only if  $(p', w') \in \varphi(e')$ . This concludes the proof for the case  $Q \neq \phi$ .

For  $Q = \phi$  the same arguments apply,  $Q = \phi$  implies  $\Psi^2(e, p, w) = \{0\}$ . Since  $\Psi \neq S \times \{0\} \times \{\theta\} = A_Q^0 \times \{\theta\}$  there are open neighborhoods of  $e, (p, w)$ ,  $\mathcal{O}' \subset \mathcal{O}$  and  $U \subset S' \times (1/r, r)^N$ , respectively, and a continuous function  $\varphi: \mathcal{O}' \rightarrow U$  such that  $\Psi(e', p', w') \in A_Q^0 \times \{\theta\}$ ,  $(e', p', w') \in \mathcal{O}' \times U$ ; if and only if  $(p', w') \in \varphi(e')$ . There are also open neighborhoods  $\mathcal{O}'' \subset \mathcal{O}'$  and  $U' \subset U$  such that (i) if  $e' \in \mathcal{O}''$  and  $(p', w') \in W^{Y_f}(e') \cap U'$ , then  $\Psi_e^2(p', w') = 0$ ; (ii) if  $(p', w') \in U'$ , then  $(p', 0) \in \hat{Y}_f$ . Hence, letting  $\mathcal{O}''' = \mathcal{O}'' \cap \varphi^{-1}(U')$ , if  $e' \in \mathcal{O}'''$  and  $(p', w') \in U'$ , then  $(p', w') \in W^{Y_f}(e')$  if and only if  $(p', w') \in \varphi(e')$ .

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