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A REMARK ON A SMOOTHNESS PROPERTY OF CONVEX, COMPLETE PREORDERS*

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Let $\gtrsim \subset A \times A$ be a complete preorder on A, a closed, convex subset of $\mathbf{R}^n$. Assume (i) $\gtrsim$ is closed (continuity), and (ii) if $x \gtrsim y$, then $\alpha x + (1-\alpha)y \gtrsim y$ for all $0 \leqq \alpha \leqq 1$ (convexity).

Denote

$$B = \{q \in \mathbf{R}^n\colon \|q\| \leqq 1\},$$

and define a ('gradient') mapping $g_{\gtrsim}: A \to B$ by

$$g_{\gtrsim}(x) = \{q \in B\colon \text{ if } \; y \gtrsim x, \; \text{ then } \; qy \geqq qx\};$$

$g_{\gtrsim}(x)$ is a convex set and it has a well-defined dimension. For every $m \leqq n$, let

$$C_m = \{x \in A\colon \; \dim \; g_{\gtrsim}(x) \geqq m\},$$

and denote by μ_m the m-dimensional Hausdorff measure in $\mathbf{R}^n$ [for the definition and some facts about the Hausdorff measure, see Federer (1969, 2.10)]; μ_n coincides with the usual Lebesgue measure. We prove

Theorem. For every $m \leqq n$, $\mu_{n-m+2}(C_m) = 0$; in particular, $\mu_n(C_2) = 0$, i.e., $g_{\gtrsim}$ is a (possibly degenerate) ray a.e. (in the sense of Lebesgue measure).

The theorem gives an analog for convex, complete preorders of the a.e. differentiability properties of concave functions; convex, complete preorders play in 'ordinal' maximization problems, i.e., those where the value of the maximization criterion is inessential and only the maximizers matter, the same

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role that concave functions play in ordinary programming problems. We may remark that, in contrast to concave functions, convex, continuous, complete preorders can be quite pathological; see Mas-Colell (1973) for an example of two distinct $\succsim$, $\succsim'$ having $g_{\succsim} = g_{\succsim}'$.

The method of proof and the results obtained are related to those of Anderson and Klee (1952) for concave functions.[1]

As an application (in economics), let $\succsim$ be a continuous, monotone (not necessarily convex) preference relation (i.e., a complete preorder) in $\mathbf{R}^n_+$. The demand correspondence $h: R^l_{++} \to \mathbf{R}^l_+$ is given by

$$h(p) = \{x \in \mathbf{R}^l_+ : px \leqq 1, \quad \text{and} \quad py \leqq 1 \quad \text{implies} \quad x \succsim y\}.$$

Then:

Corollary. *h is single-valued a.e. (in the sense of Lebesgue measure).*

Proof. Define $\succsim^*$ in $\mathbf{R}^l_{++}$ by $p \succsim^* q$ if and only if $h(q) \succsim h(p)$. Then $\succsim^*$ is continuous and $\{p : p \succsim^* q\} = \varphi^{-1}([1, \infty))$ where φ is the negative of the support function of $\{-y : y \succsim h(q)\}$. Hence $\succsim^*$ is convex and $h(p)$ is single-valued if and only if $g_{\succsim}{}^*(p)$ is a ray. ∎

The corollary has an interesting implication: if budgets are 'sufficiently' random (in prices and income) then, even with non-convex preferences, expected maximizers will be unique.

Proof of the theorem. It follows simply from the definition of the Hausdorff measure that if $E \subset \mathbf{R}^n$ is an F_σ set, $H \subset \mathbf{R}^n$ is an m-dimensional subspace and $\#(E \cap H + \{a\}) \leqq 1$ for every $a \in \mathbf{R}^n$, then $\mu_{n-m+1}(E) = 0$.

Fix $1 \leqq m \leqq n$ and let $\mathscr{H}$ be the (countable) collection of $(m-1)$-dimensional subspaces of $\mathbf{R}^n$ having a base with rational components. For every $x \in A$ define

$$L_x = \{z \in \mathbf{R}^n : pz = 0 \quad \text{for every} \quad p \in g_{\succsim}(x)\},$$

and let $\operatorname{Int} g_{\succsim}(x)$ denote the interior of $g_{\succsim}(x)$ relative to its linearity space. Note that

$$\dim(\operatorname{Int} g_{\succsim}(x)) + \dim L_x = n.$$

For every $H \in \mathscr{H}$ let

$$C_m(H) = \{x \in C_m : \dim(H + L_x) = \dim H + \dim L_x \quad \text{and, for some} \quad p \in \operatorname{Int} g_{\succsim}(x), \quad p \neq 0 \quad \text{is perpendicular to} \quad H\}.$$

[1] For those familiar with the concept of upper semicontinuous collections of sets [see Berge (1959, V, 7)], we point out that $\{\{x\} - g_{\succsim}(x) : x \in A\}$ is one such.

It is easily checked that $C_m(H)$ is F_σ. If H' is a translate of H, then $\#(C_m(H) \cap H') \leqq 1$ for, otherwise, if $x, y \in C_m(H) \cap H'$ and $x \neq y$, then $px < py$ for some $p \in g_{\succsim}(y)$ and $qy < qx$ for some $q \in g_{\succsim}(x)$ which contradicts the completeness hypothesis. Therefore $\mu_{n-m+2}(C_m(H)) = 0$ for every $H \in \mathscr{H}$. On the other hand, let $x \in C_m$ and $p \in \operatorname{Int} g_{\succsim}(x)$ be perpendicular to $I \subset \mathbf{R}^n$, some $(m-1)$-dimensional subspace independent of L_x; if I' is an $(m-1)$ subspace with a rational components base which is sufficiently close to I and we take p' oriented as p and perpendicular to I' and L_x, then $p' \in g_{\succsim}(x)$. Hence

$$C_m = \bigcup_{H \in \mathscr{H}} C_m(H)$$

and therefore $\mu_{n-m+2}(C_m) = 0$. For $m \leqq 1$ the theorem is obvious. ∎

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