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THE RECOVERABILITY OF CONSUMERS' PREFERENCES FROM MARKET DEMAND BEHAVIOR¹

BY ANDREU MAS-COLELL

1. INTRODUCTION

A BASIC INGREDIENT of the theory of competitive analysis is the concept of a continuous, monotone, convex preference relation (see Debreu [4], Arrow-Hahn [1]). It is assumed in the theory that economic agents (to be specific, consumers) are endowed with one such and that they behave in competitive markets situations, i.e., when facing parametric prices, by choosing preferred points in budget sets. The demand function or, more generally, correspondence thus generated contains all the relevant information about the behavior of the consumer (setting aside how its income is determined).

Suppose now that the demand function is known, say, by means of market observations, and that we want either to predict the behavior of the consumer in some noncompetitive situation, or to make some definite welfare statement. Is this possible? For the question to be meaningful we should assume that underlying the given demand function there is a preference relation; to get to the heart of the matter, we shall postulate that there is a continuous, monotone, convex one. The problem can then be formulated thusly: does the given demand function identify uniquely a monotone, convex, continuous preference relation? It turns out that, in spite of its intuitive plausibility, such a contention is false. There are two distinct continuous, monotone, strictly convex preference relations, defined in the non-negative orthant of R^l ($l \geq 2$), giving rise to identical demand functions. Needless to say, they are very pathological.

This negative result poses the problem of determining a wide enough class of preferences having what could be denominated the *recoverability property*. It leads us to define, in a certain natural manner, a class of "lipschitzian" preference relations. It can be characterized (Theorem 1) as the class of preferences representable by utility functions which are lipschitzian and satisfy a further, quite weak, regularly condition (akin to the "nonvanishing of the gradient vector"). Once this representability result is available it is proved (Theorem 2) that the recoverability property holds for lipschitzian preference relations.

From the work of Houthakker [12] and Uzawa [18] it is known that continuous preferences yielding income lipschitzian demand functions (this concept will be defined precisely later on) are recoverable. It is proved (Theorem 3) that those preferences are lipschitzian, in our sense. The converse is not true; in particular, smooth (C^r , $r \geq 1$; see Debreu [5]) and concavifiable (i.e., representable by concave utility functions) preferences are lipschitzian but they do not necessarily give rise to income lipschitzian demand functions. It would appear that the class of

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lipschitzian preferences is quite broad and includes as proper subsets most of the classes of preferences useful in applications. Thus, while avoiding pathology, virtually no conceptual generality will be lost by confining oneself to a lipschitzian preferences setup.

2. RESULTS

A. Some Definitions and a Fact

The *consumption set* will be $\Omega = \{x \in R^l : x \geq 0\}$.² Every proof, and therefore every result, in this paper remains valid if Ω is interpreted to be the strictly positive orthant R^l_{++} .

A complete preorder \geq on Ω^3 will be called a *preference relation*. For every y and $x \in \Omega$ let $V(\geq, x) = \{y \in \Omega : y \geq x\}$; $x > y$ means $\neg(y \geq x)$.

A preference relation \geq is: (i) *upper semicontinuous* (u.s.c.) if $V(\geq, x)$ is closed for every x ; (ii) *continuous* on $A \subset \Omega$ if its graph is closed in $A \times A$; (iii) *continuous* if it is continuous on Ω ; (iv) *convex* if $x \geq y$ implies $tx + (1-t)y \geq y$ for every $t \in [0, 1]$; (v) *strictly convex* if it is convex and $x > y$ implies $tx + (1-t)y > y$ for every $t \in (0, 1]$; (vi) *monotone*, if $x \gg y$ implies $x > y$; (vii) *strictly monotone*, if $x > y$ implies $x \gg y$.

The set of continuous, convex, monotone preference relations will be denoted \mathcal{P} ; \mathcal{P}_{sc} (resp. \mathcal{P}_{sm}) is the subset of strictly convex (resp. strictly monotone) ones.

Given an u.s.c. preference relation \geq the *demand correspondence* $h^{\geq} : R^{l+1} \rightarrow \Omega$ for \geq is defined as usual, i.e., $h^{\geq}(p, w) = \{x : px \leq w, \text{ and } py \leq w \text{ implies } x \geq y\}$; if \geq is strictly convex, then h^{\geq} is a function.

The following fact shows the inadequacy of the continuity assumption for extracting welfare conclusions out of the analysis of consumers' behavior in competitive markets. It should be regarded as the main motivation for this paper.

FACT: *There are two continuous, monotone, strictly convex preference relations \geq, \geq' such that $\geq \neq \geq'$ and $h^{\geq} = h^{\geq'}$, i.e., there is a continuous, monotone, strictly convex preference relation which is not recoverable from its demand function.*

REMARK 1: Although the prevailing opinion in the literature seems to be the contrary (see, for example, Gorman [9, (iv), p. 104]) the possibility described by the just stated fact has been previously recognized by T. Rader [16, p. 243], L. Shapley and, no doubt, others.

² Subscripts denote vectors, superscripts components of vectors; $x \geq 0$ means $x^i \geq 0$ for every i , $x > 0$ means $x \geq 0$ and $x \neq 0$, $x \gg 0$ means $x^i > 0$ for every i . Intervals and segments in R^n are denoted in the customary manner. The euclidean norm is $\|\cdot\|$; for $x \in R^n$, $\varepsilon > 0$, let $B_\varepsilon(x) = \{y \in R^n : \|y - x\| < \varepsilon\}$, $B_\varepsilon = B_\varepsilon(0)$; the appropriate dimension will be clear by the context; $S^{l-1} = \{x \in R^l : \|x\| = 1\}$; $S^{l+1}_+ = \{x \in S^{l-1} : x \gg 0\}$. Given $A \subset R^n$, \bar{A} , $\text{bdry } A$, $\text{int } A$ designate, respectively, the closure, boundary, and interior of A relative to R^n ; the interior of A relative to C , $A \subset C \subset R^n$, is $\text{int}_C A$; $\text{co } A$ is the convex hull of $A \subset R^n$. $R^n_+ = \{x \in R^n : x \geq 0\}$; $R^n_{++} = \{x \in R^n : x \gg 0\}$.

³ A complete preorder \geq on Ω is a subset of $\Omega \times \Omega$ which is *reflexive* (i.e., $x \geq x$, for every $x \in \Omega$), *transitive* (i.e., $x \geq y, y \geq z$ implies $x \geq z$), and *complete* (i.e., for every $x, y \in \Omega$, either $x \geq y$ or $y \geq x$).

REMARK 2: If so desired, the preferences \succeq, \succeq' in the fact can be taken to have C^1 indifference hypersurfaces.

B. Lipschitzian Preferences: Definition and a Representation Theorem

Let (\mathcal{C}, d) be the metric space formed by the set of nonempty, compact, convex subsets of R^l with the Hausdorff metric d derived from the euclidean norm in R^l .⁴ Denote by e the vector in R^l all of whose components equal 1; for every nonnegative real r let $K_r = \{x \in \Omega : (1/(1+r))e \leq x \leq (1+r)e\}$.

For every u.s.c., monotone \succeq and $0 < r \leq \infty$ define a function $V_{\succeq, r}: K_r \rightarrow \mathcal{C}$ by $V_{\succeq, r}(x) = V(\succeq, x) \cap K_r$. Then the following can be seen:

A convex, monotone, u.s.c. \succeq is continuous on R^l_{++} if and only if, for every $0 \leq r < \infty$, $V_{\succeq, r}: K \rightarrow \mathcal{C}$ is continuous.

This suggests the following definition:

DEFINITION: *An u.s.c., convex, monotone preference relation \succeq is lipschitzian if, for every $0 < r < \infty$, $V_{\succeq, r}: K \rightarrow \mathcal{C}$ is a lipschitzian function.*⁵

This concept of lipschitzian preferences is devised to capture the idea of indifference curves fitting together not too wildly.

A utility function for a preference relation \succeq is a function $u: \Omega \rightarrow R$ such that $x \succeq y$ if and only if $u(x) \geq u(y)$.

DEFINITION: *A utility function u is lipschitzian if, for every $r > 0$, $u|_{K_r}$ is lipschitzian. A utility function u for a strictly monotone \succeq is regular if, for any $r > 0$, there is $\delta > 0$ such that if $x \geq y$, $x, y \in K_r$, then $u(x) - u(y) \geq \delta \|x - y\|$.*

The term "regular" is suggested by the fact that if u were continuously differentiable, then the definition is equivalent to the gradient of u being nonvanishing on R^l_{++} .

We have the following representation result:

THEOREM 1: *A continuous, monotone, strictly convex \succeq is lipschitzian if and only if there is a lipschitzian and regular utility function for \succeq .*

Let $\mathcal{P}_l \subset \mathcal{P}_{sc}$ be the subset of lipschitzian preferences in \mathcal{P}_{sc} .

Two important and interesting subsets of \mathcal{P}_{sc} are:

$$\mathcal{P}_{co} = \{\succeq \in \mathcal{P}_{sc} : \text{there is a concave utility function for } \succeq\};$$

$$\mathcal{P}_d = \{\succeq \in \mathcal{P}_{sc} : \succeq \text{ is } C^1\};$$

for a \succeq being C^r , $r \geq 1$, we mean, following Debreu [5], that there is a C^r utility for

⁴ The Hausdorff metric is defined as follows (see Hausdorff [10, p. 168]): given two nonempty, compact sets $A, A' \subset R^n$, let $\hat{d}(A, A') = \max_{x \in A} \min_{y \in A'} \|x - y\|$; then $d(A, A') = \max\{\hat{d}(A, A'), \hat{d}(A', A)\}$.

⁵ A function f from a metric, compact space (A_1, ρ_1) to a metric space (A_2, ρ_2) is lipschitzian if there is a real number H such that for every $v, v' \in A_1$, $\rho_2(f(v), f(v')) \leq H\rho_1(v, v')$.

\succeq with nonvanishing gradient (for an analysis of the case $r = 1$, not covered by Debreu, see Moulin [14]).

An obvious corollary of Theorem 1 is:

COROLLARY: $\mathcal{P}_{co} \subset \mathcal{P}_1$ and $\mathcal{P}_d \subset \mathcal{P}_1$, i.e., *concavifiable and smooth (monotone and strictly convex) preferences are lipschitzian.*

A systematic analysis of the lipschitzian preferences concept shall not be attempted here since the focus of this paper is the recoverability property.

REMARK 3: The careful reader of the proof will verify that Theorem 1 remains valid if \succeq is required to be upper semicontinuous rather than continuous.

REMARK 4: The definition of lipschitzian preferences can be stated in several equivalent forms. Perhaps the following is the more transparent one; for $x \in R^n$, $A \subset R^n$ let $\delta(x, A) = \inf_{y \in A} \|x - y\|$.

A continuous, convex, monotone \succeq is lipschitzian if and only if, for every $r > 0$, there are reals $H > 0$ and $\varepsilon > 0$ such that if $x, y, z \in K$, $x \succeq y$, $y \succeq x$, and $\|x - z\| < \varepsilon$, then $\delta(x, V(\succeq, z)) \leq H\delta(y, V(\succeq, z))$.

In this form it is clear that the concept of lipschitzian preferences is not new; it has been introduced before by Rader under the name of *uniformly sensitive* preferences [16, p. 171];⁶ it is also closely related (but, naturally enough, weaker) to a condition isolated by Moulin [14] for the purpose of characterizing preferences representable by C^1 utility functions with no critical point.

REMARK 5: Preferences representable by lipschitzian (not necessarily regular) utility functions need not be lipschitzian. Examples can be constructed along the lines of the ones exhibited by di Finetti in [8] with a different purpose.

REMARK 6: The definition of lipschitzian preferences is independent of the norm used in R^l ; i.e., any of them gives rise to the same class of preferences (note that any two norms $\|\cdot\|_1, \|\cdot\|_2$ in R^l are topologically equivalent, hence—see Dieudonné [6, p. 106]—there are $a > 0, b > 0$ such that $a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2$ for all $x \in R^l$).

REMARK 7: The definition given for regularity of utility functions requires that the function be strictly increasing; indeed, Theorem 1 is obviously false if strict convexity is replaced by convexity in its statement. Let \succeq be a monotone (not necessarily strictly) preference relation and u a utility function for \succeq ; then an appropriate definition of regularity is: u is regular if for every $r > 0$ there is $\delta > 0$ such that if $x \in K$, and $\lambda > 0$, then $u(x + \lambda e) - u(x) \geq \varepsilon \lambda$. With this definition, and no further complication in the proof, Theorem 1 remains true for convex (not necessarily strictly) preferences.

⁶ Theorem 1 is also related to a theorem of his [16, p. 172].

REMARK 8: Let \mathcal{U} be the set of utility functions for preferences in \mathcal{P} and denote by $\Phi: \mathcal{U} \rightarrow \mathcal{P}$ the map which assigns to every u the preference relation it represents. Say that a set $\mathcal{P}' \subset \mathcal{P}$ (resp. $\mathcal{U}' \subset \mathcal{U}$) is *uniformly lipschitzian* if, for every $r > 0$ and $\succeq \in \mathcal{P}'$ (resp. $u \in \mathcal{U}'$), the maps $V_{\succeq, r}$ (resp. $u|K_r$) admit a common lipschitz constant (so, for example, the set formed by a single Cobb-Douglas utility function is *not* uniformly lipschitzian).

Define the *uniform regularity* of $\mathcal{U}' \subset \mathcal{U}$ analogously, but with respect to the notion of regularity of Remark 7. The counterpart of Theorem 1 (which is proved in exactly the same manner) is the following:

THEOREM 1': *A set $\mathcal{P}' \subset \mathcal{P}$ is uniformly lipschitzian if and only if there is a uniformly lipschitzian and regular set $\mathcal{U}' \subset \mathcal{U}$ such that $\Phi(\mathcal{U}') = \mathcal{P}'$.*

Endow \mathcal{P} with the closed convergence topology considered by Hildenbrand [11]. It can be seen that this topology is equivalent to the one induced by the uniform convergence of the function $V_{\succeq, r}$ (i.e., $\succeq_n \rightarrow \succeq$ if, for all r , $V_{\succeq_n, r} \rightarrow V_{\succeq, r}$ uniformly). Theorem 1' has then an interesting implication: *if $\mathcal{P}' \subset \mathcal{P}$ is uniformly lipschitzian, then it is relatively compact.* Indeed, let $\mathcal{U}' \subset \mathcal{U}$ be as in Theorem 1'; with the compact open topology on \mathcal{U} , Φ is continuous (see Mas-Colell [13, 1.18]); because \mathcal{U}' is a uniformly lipschitzian and regular set, it has a limit point u in \mathcal{U} and, therefore, $\Phi(u)$ is a limit point of \mathcal{P}' in \mathcal{P} . Since any subset of \mathcal{P}' is again uniformly lipschitzian this proves the relative compactness of \mathcal{P}' .

Compact sets of continuous preferences play an important role in the theoretical study of large economies (see Hildenbrand [11], Bewley [3]).

REMARK 9: All the definitions have been given in terms of a specific collection of compact cubes in R^l ; Theorem 1 remains true if we had used the collection of compact, convex subsets of R^l_{++} with nonempty interior.

C. Lipschitzian Preferences and the Recoverability Property

The relevance of lipschitzian preferences for the problem of this paper stems from the following theorem:

THEOREM 2: *Let \succeq, \succeq' be continuous, monotone, strictly convex preference relations. If $h^{\succeq} = h^{\succeq'}$ and \succeq is lipschitzian, then $\succeq = \succeq'$.*

REMARK 10: Instead of Theorem 2 we will prove with no additional complication the more general theorem:

THEOREM 2': *Let \succeq, \succeq' be upper semicontinuous, monotone, convex preference relations. If $h^{\succeq} = h^{\succeq'}$ (these may be correspondences) and \succeq is representable by a lipschitzian utility function, then $\succeq = \succeq'$.*

REMARK 11: A recoverability condition has been given by Rader in [16, p. 234]. It requires the existence of the indirect demand function, an assumption we want to avoid. It is unknown to us if his condition implies that preferences are lipschitzian.

REMARK 12: Let \succeq be a preference relation. It is said that $x \in \Omega$ is revealed preferred (r.p.) to $y \in \Omega$ if for some $(p_i, w_i) \in R_{++}^{l+1}$, $i = 1, \dots, N$, one has $x \in h^{\succeq}(p_1, w_1)$, $p_N y \leq w_N$, $y \notin h^{\succeq}(p_N, w_N)$ and, for all $i < N$, $p_i x_{i+1} \leq w_i$ for some $x_{i+1} \in h(p_{i+1}, w_{i+1})$. The following is a corollary of Theorem 2:

If \succeq is continuous, lipschitzian, monotone, and convex, then x is revealed preferred to y if and only if $x \succ y$.

PROOF: The “only if” part is trivial. Let $y \in \Omega$; denote $T_y = \{z \in \Omega : y \text{ is not r.p. to } z\}$; it is easily checked that T_y is closed and convex. Define a new preference relation \succeq' by:

$$u \succeq' v \quad \text{if} \quad \begin{cases} \text{“}u \in \text{co}(T_y \cup V(\succeq, v)) \text{ and } v \notin T_y, \text{” or} \\ \text{“}u \in T_y \cap V(\succeq, v) \text{ and } v \in T_x. \text{”} \end{cases}$$

Then \succeq' is upper semicontinuous, monotone, convex, and $h^{\succeq} = h^{\succeq'}$. By Theorem 2', $\succeq' = \succeq$, but this is possible only if $T_y = V(\succeq, y)$. Q.E.D.

From the revealed preference literature (Houthakker [12], Uzawa [18]) it is known that continuous preferences giving rise to demand functions satisfying a certain income lipschitz condition are recoverable. We will prove that those preferences are lipschitzian in our sense.

DEFINITION: *Let \succeq be continuous, monotone, and strictly convex; h^{\succeq} is income lipschitzian if, for every compact $L \subset R_{++}^{l+1}$, there is a real H such that if $(p, w), (p, w') \in L$, then $\|h^{\succeq}(p, w) - h^{\succeq}(p, w')\| \leq H|w - w'|$.*

THEOREM 3: *Let \succeq be a continuous, monotone, strictly convex preference; if h^{\succeq} is income lipschitzian, then \succeq is lipschitzian.*

Let $\mathcal{P}_{il} = \{\succeq \in \mathcal{P}_{sc} : h^{\succeq} \text{ is income lipschitzian}\}$. Then Theorem 3 says $\mathcal{P}_{il} \subset \mathcal{P}_l$; the converse inclusion does not hold. We already pointed out that $\mathcal{P}_{co} \subset \mathcal{P}_l$ and $\mathcal{P}_d \subset \mathcal{P}_l$. In fact, none of the sets \mathcal{P}_{il} , \mathcal{P}_{co} , \mathcal{P}_d is contained in the union of the remaining two; this is well known for \mathcal{P}_d and Examples 2 and 3 will show, respectively, that $\mathcal{P}_{il} \not\subset \mathcal{P}_{co}$, $\mathcal{P}_{co} \not\subset \mathcal{P}_{il}$. So, the set of lipschitzian preferences contains as proper subsets the classes of preferences defined by the regularity properties stronger than continuity which have proved most fruitful. We may remark that the intersection $\mathcal{P}_{il} \cap \mathcal{P}_{co} \cap \mathcal{P}_d$ contains the set of C^2 preferences having indifference hypersurfaces with everywhere nonzero curvature (see Debreu [5], Fenchel [7, Ch. 8], Aumann [2]) and is therefore dense in \mathcal{P} with respect to the closed convergence topology (see Mas-Colell [13]).

REMARK 13: Theorems 1 and 3 have obvious implications in revealed preference theory. To keep the length of this paper within bounds they shall be developed elsewhere.

3. PROOF OF THE THEOREMS

For any monotone, convex preference relation \succeq define a correspondence $\sigma(\succeq, \cdot): \Omega \rightarrow S^{l-1}$ by $\sigma(\succeq, x) = \{p: \text{if } y \in V(\succeq, x), \text{ then } py \geq px\}$.

LEMMA 1: *If \succeq is monotone, convex, and u.s.c., then $\sigma(\succeq, \cdot)$ is an upper hemicontinuous correspondence.*

PROOF: Let $x_n \rightarrow x, p_n \in \sigma(\succeq, x_n), p_n \rightarrow p$. If $\neg(x \succeq y)$, then $\neg(x_n \succeq y)$ for n larger than some N , hence $p_n y \geq p_n x_n$ and so $py \geq px$; monotonicity implies then $py \geq px$ for any $y \in V(\succeq, x)$. Q.E.D.

A. Proof of Theorem 1

Pick $\succeq \in \mathcal{P}_{sc}$ and $\bar{r} > 1$. This \succeq and \bar{r} will remain fixed for the rest of the proof. If $x \succeq y$ and $y \succeq x$, we write $x \sim y$. We denote $V_{K_{\bar{r}}}$ by $V_{\bar{r}}$.

For every $x \in \Omega$, let $v(x)$ be the unique point such that $x \sim v(x)$ and $v(x) = \lambda(x)e$ for some real $\lambda(x) \geq 0$.

LEMMA 2: $d(V_{\bar{r}}(v(x)), V_{\bar{r}}(v(y))) \geq |\lambda(x) - \lambda(y)|$ for all $x, y \in \Omega$.

PROOF: If $|\lambda(x) - \lambda(y)| \neq 0$ and $x \succeq y$, then, by monotonicity, $B_{|\lambda(x) - \lambda(y)|}(v(y)) \cap V_{\bar{r}}(v(y)) = \emptyset$. Q.E.D.

LEMMA 3: *If $0 < \varepsilon < 1/\bar{r}, x, y \in K_{\bar{r}}, x \succeq y$, and $d(V_{\bar{r}}(x), V_{\bar{r}}(y)) \leq \varepsilon/2$, then there is $z \in \Omega$ such that $x \sim y$ and $\|x - z\| \leq l^{\frac{1}{2}} \bar{r}^2 \varepsilon$.*

PROOF: Let $(1 - \alpha) = \varepsilon \bar{r} < 1$; then, $\|x - \alpha x\| \leq \varepsilon \bar{r}^2 l^{\frac{1}{2}}$ and $x' \ll x$ for every $x' \in B_{\varepsilon}(\alpha x)$. Therefore $y \succeq \alpha x$ and so $z \sim y$ for some $z \in [x, \alpha x]$. Q.E.D.

LEMMA 4: *If $x_n, y_n \in K_{2\bar{r}}, x_n \sim y_n$, and $x_n \neq y_n$ for all n , then $\{(x_n - y_n)/\|x_n - y_n\|\}_{n=1}^{\infty}$ does not have any accumulation point in Ω .*

PROOF: Follows immediately from the strict convexity and monotonicity of \succeq . Q.E.D.

LEMMA 5: *There is $\delta > 0$ such that if $x > y, x, y \in K_{\bar{r}}$, then $d(V_{\bar{r}}(x), V_{\bar{r}}(y)) \geq \delta \|x - y\|$.*

PROOF: Assume the contrary. Then, using Lemma 2, we find $x_n, y_n, z_n \in K_{2\bar{r}}$ such that $x_n > y_n, z_n \sim y_n, z_n \neq y_n$ and $\|z_n - x_n\|/\|x_n - y_n\| \rightarrow 0$. By this and the

triangle inequality, one has $\|z_n - y_n\|/\|x_n - y_n\| \rightarrow 1$, hence

$$\lim_n \left(\frac{z_n - y_n}{\|z_n - y_n\|} - \frac{x_n - y_n}{\|x_n - y_n\|} \right) = \lim_n \frac{z_n - x_n}{\|x_n - y_n\|} = 0;$$

therefore,

$$\left\{ \frac{z_n - y_n}{\|z_n - y_n\|} \right\}_{n=1}^\infty$$

has some accumulation point in Ω which contradicts Lemma 4.

Q.E.D.

LEMMA 6: *If $x, y \in K_{\bar{f}}$, $x \succeq y$ and $d(V_{\bar{f}}(x), V_{\bar{f}}(y)) \geq \delta$, then there are $x', y' \in K_{\bar{f}}$ such that $x \sim x'$, $y \sim y'$, $x' \succ y'$ and $\|x' - y'\| \geq \delta$.*

PROOF: Since $V_{\bar{f}}(y) \notin V_{\bar{f}}(x) + B_\delta$ we can pick $y' \in K_{\bar{f}}$ such that $y \sim y'$ and $y' \notin V_{\bar{f}}(x) + B_\delta$. Let x' be the point where the boundary (rel. to $K_{\bar{f}}$) of $V_{\bar{f}}(x)$ intersects the closed segment connecting y' and $\bar{r}e$.

Q.E.D.

We are ready for the proof of the theorem.

PROOF OF THEOREM 1: (i) Let \succeq be lipschitzian; then defining $u: \Omega \rightarrow R$ by $u(x) = \lambda(x)$, u is a utility function for \succeq and we have:

(a) u is lipschitzian on $K_{\bar{f}}$. Indeed, let H be a lipschitz constant for $V_{\bar{f}}: K_{\bar{f}} \rightarrow \Omega$. Then H is a lipschitz constant for $u|_{K_{\bar{f}}}$ since (using Lemma 2) for every $x, y \in K_{\bar{f}}$, $|u(x) - u(y)| = |\lambda(x) - \lambda(y)| \leq d(V_{\bar{f}}(v(x)), V_{\bar{f}}(v(y))) = d(V_{\bar{f}}(x), V_{\bar{f}}(y)) \leq H\|x - y\|$.

(b) u is regular on $K_{\bar{f}}$. Indeed, suppose this is not so, i.e., there are $x_n, y_n \in K_{\bar{f}}$ such that $x_n \succ y_n$ and $(u(x_n) - u(y_n))/\|x_n - y_n\| \rightarrow 0$. Since $\|v(x_n) - v(y_n)\| = l^{\frac{1}{2}}|u(x_n) - u(y_n)|$, one has $\|x_n - y_n\|/\|v(x_n) - v(y_n)\| \rightarrow \infty$. By Lemma 4 $d(V_{\bar{f}}(x_n), V_{\bar{f}}(y_n))/\|v(x_n) - v(y_n)\| \rightarrow \infty$ or (remember that $x_n \sim v(x_n)$, etc.) $d(V_{\bar{f}}(v(x_n)), V_{\bar{f}}(v(y_n)))/\|v(x_n) - v(y_n)\| \rightarrow \infty$ contradicting the hypothesis that $V_{\bar{f}}$ is lipschitzian.

(ii) Let \succeq be representable by a lipschitzian, regular utility function $u: \Omega \rightarrow R$ and suppose that $V_{\bar{f}}$ is not lipschitzian, i.e., there are $x_n, y_n \in K_{\bar{f}}$ such that $x_n \neq y_n$, $x_n \succeq y_n$, and $d(V_{\bar{f}}(x_n), V_{\bar{f}}(y_n))/\|x_n - y_n\| \rightarrow \infty$. By Lemma 5 we can pick $x'_n, y'_n \in K_{\bar{f}}$ such that: $\|x'_n - y'_n\| \geq \frac{1}{2}d(V_{\bar{f}}(x_n), V_{\bar{f}}(y_n))$, $x'_n \succ y'_n$, $x'_n \sim y_n$, $y'_n \sim y_n$. One has $\|x'_n - y'_n\|/\|x_n - y_n\| \rightarrow \infty$. Letting H be a lipschitz constant for $u|_{K_{\bar{f}}}$, $u(x'_n) - u(y'_n) = u(x_n) - u(y_n) \leq H\|x_n - y_n\|$. Therefore $\|x'_n - y'_n\|/(u(x'_n) - u(y'_n)) \rightarrow \infty$ or $(u(x'_n) - u(y'_n))/\|x'_n - y'_n\| \rightarrow 0$ which contradicts the regularity of u . Hence, $V_{\bar{f}}$ is lipschitzian.

Q.E.D.

B. Proof of Theorem 2'

LEMMA 7: *If \succeq_1, \succeq_2 are u.s.c., convex, monotone preference relations and $h^{\succeq_1} = h^{\succeq_2}$, then $\sigma(\succeq_1, \cdot) = \sigma(\succeq_2, \cdot)$.*

PROOF: Let $p \in \sigma(\succeq_1, x)$ and suppose that $y \succeq_2 x, py < px$. By monotonicity we can assume $\neg(x \succeq_2 y)$. Let $\bar{p} \gg 0$ be such that $\bar{p}z > \bar{p}x$ for every $z \in V(\succeq_2, y)$; it is clear that such a \bar{p} exists. Pick a $t \in (0, 1)$ such that, letting $p_t = t\bar{p} + (1-t)p, p_t y < p_t x$; note that $p_t \gg 0$. Take $v \in h^{\succeq_2}(p_t, p_t x)$; then $v \in V(\succeq_2, y)$ and so $\bar{p}v > \bar{p}x$. Since $p_t v = p_t x$, we have $p v < p x$ which implies $v \notin V(\succeq_1, x)$ and, therefore, $v \notin h^{\succeq_1}(p, p x)$. Since this is impossible, we conclude $p \in \sigma(\succeq_2, x)$. Q.E.D.

Let \succeq_1, \succeq_2 be two u.s.c., convex, monotone preference relations and suppose that \succeq_1 is representable by a lipschitzian utility function $u: \Omega \rightarrow R$.

For every $x \in \Omega$ define $I_x = \{y: y \in V(\succeq_2, x) \text{ and if } y' < y, \text{ then } \neg(y' \succeq_2 x)\}$. Because of monotonicity (and u.s.c.) in order to establish $\succeq_1 = \succeq_2$, it suffices to prove that, for every $x \in \text{int } \Omega, V(\succeq_2, x) \subset V(\succeq_1, x)$, or, simply, $I_x \subset V(\succeq_1, x)$. Hence, let $x \in \text{int } \Omega, y \in I_x$; we will show $u(y) \geq u(x)$.

Since I_x is contained in the boundary of the convex set $V(\succeq_2, x)$, there is a lipschitzian function $f: [0, 1] \rightarrow \Omega$ with $f(0) = x, f(1) = y, f((0, 1)) \subset I_x \cap \text{int } \Omega$. The function $g = u \circ f: [0, 1] \rightarrow R$ is also lipschitzian and so (see Royden [17, p. 108]) both g and f are differentiable a.e. and $\int_0^1 g'(s) ds = g(1) - g(0) = u(y) - u(x)$. Let $f'(\bar{t}), g'(\bar{t})$ exist at $0 < \bar{t} < 1$ and let $t_n \rightarrow \bar{t}, \bar{t} < t_n < 1$.

Suppose that $g'(\bar{t}) < 0$, i.e., $0 < \lim_n (g(t_n) - g(\bar{t})) / (\bar{t} - t_n) < \infty$. Let $z_n = f(\bar{t}) + Df(\bar{t})(\bar{t} - t_n)$; by definition of the derivative map $\lim_n \|f(t_n) - z_n\| / (\bar{t} - t_n) = 0$. Therefore,

$$\lim_n \frac{g(t_n) - g(\bar{t})}{\|f(t_n) - z_n\|} = \lim_n \frac{g(t_n) - g(\bar{t})}{\bar{t} - t_n} \frac{\bar{t} - t_n}{\|f(t_n) - z_n\|} = \infty.$$

We can assume that $z_n \in \text{int } \Omega$ for every n . Since $f([0, 1]) \subset I_x$ and $V(\succeq_2, x)$ is convex, $Df(\bar{t})(R) \cap \text{int } V(\succeq_2, x) = \emptyset$; therefore, there is $p \in \sigma(\succeq_2, f(\bar{t}))$ such that $p z_n \leq p f(\bar{t})$. By Lemma 7, $p \in \sigma(\succeq_1, f(\bar{t}))$; hence, for every n (remember $z_n, f(\bar{t}) \in \text{int } \Omega$), $u(z_n) \leq u(f(\bar{t}))$, i.e., $u(z_n) \leq g(\bar{t})$. Therefore,

$$\lim_n \frac{u(f(t_n)) - u(z_n)}{\|f(t_n) - z_n\|} = \lim_n \frac{g(t_n) - u(z_n)}{\|f(t_n) - z_n\|} \geq \frac{g(t_n) - g(\bar{t})}{\|f(t_n) - z_n\|} = \infty,$$

which contradicts the fact that u is lipschitzian (since $z_n \rightarrow f(\bar{t}), f(t_n) \rightarrow f(\bar{t})$, the sequences $f(t_n), z_n$ can eventually be enclosed in a compact subset of $\text{int } \Omega$). We conclude that $g'(t) \geq 0$ a.e. and so $u(y) - u(x) = \int_0^1 g'(s) ds \geq 0$.

C. Proof of Theorem 3

Let $\succeq \in \mathcal{P}_{sc}$ be such that h^{\succeq} is income lipschitzian.

Define $M: S_{++}^{l-1} \times \Omega \rightarrow [0, \infty)$ by $M(p, x) = \min\{py: y \in U(R, x)\}$; let $\rho(p, x) = M(p, x), \rho(p, x) \in V(R, x)$; $\rho(p, x)$ is uniquely defined by those conditions. The function M has the following well-known properties (see, for example, Nikaido [15, p. 298]): (i) M is continuous; (ii) for fixed $x, M(\cdot, x)$ is continuously differentiable and $D_p M(p, x) = \rho(p, x)$; (iii) for fixed $p, M(p, \cdot)$ is a utility function for R ; (iv) for every $(p, x) \in S_{++}^{l-1} \times \Omega, h(p, M(p, x)) = \rho(p, x)$.

We shall show, and this will end the proof, that for every $p \in S^{l-1}_{++}$ the utility function $M_p = M(p, \cdot)$ is lipschitzian and regular.

Pick an arbitrary $\bar{p} \in S^{l-1}_{++}$ and let $K \subset \text{int } \Omega$ be any compact set. Then the set $J = \text{co}(\sigma(\{\succeq\} \times K) \cup \{\bar{p}\})$ is compact; moreover, $J \subset \text{int } \Omega$. Therefore there is $s > 0$ such that $J \times M(J \times K) \subset J \times [1/s, s]$.

LEMMA 8: *There is $H > 0$ such that if $x, y \in K, p, p' \in \sigma(\{\succeq\} \times K) \cup \{\bar{p}\}$, then $|M(p, x) - M(p, y)| \leq H|M(p', x) - M(p', y)|$.*

PROOF: Suppose the contrary; then there are sequences $x_n, y_n \in K, p_n, p'_n \in \sigma(\{R\} \times K) \cup \{\bar{p}\}$, such that $\neg(y_n \succeq x_n)$ and $(M(p'_n, x_n) - M(p'_n, y_n))/(M(p_n, x_n) - M(p_n, y_n)) \rightarrow \infty$. For every n and $t \in [0, 1]$ let $p_n(t) = tp'_n + (1-t)p_n \in J$ and define $g_n: [0, 1] \rightarrow (0, \infty)$ by $g_n(t) = M(p_n(t), x_n) - M(p_n(t), y_n)$. Then $g_n(1)/g_n(0) \rightarrow \infty$ and, by (ii), the functions g_n are C^1 . Let $c_n = \sup_t g'_n(t)/g_n(t)$. Since $\log g_n(1) - \log g_n(0) = \int_0^1 (g'_n(t)/g_n(t)) dt \leq c_n$ and $\log(g_n(1)/g_n(0)) \rightarrow \infty$, one has $c_n \rightarrow \infty$. Therefore there is $t_n \in [0, 1]$, such that $g'_n(t_n)/g_n(t_n) \rightarrow \infty$.

Denote $\bar{p}_n = p_n(t_n) \in J, v'_n = \rho(\bar{p}_n, x_n), v_n = \rho(\bar{p}_n, y_n), w'_n = M(\bar{p}_n, x_n), w_n = M(\bar{p}_n, y_n)$. Note that $v_n \neq v'_n$ and $1/s \leq w_n < w'_n \leq s$. By (ii), $g'_n(t_n) = (p'_n - p_n)(v'_n - v_n), g_n(t_n) = \bar{p}_n(v'_n - v_n)$. Therefore $(p'_n - p_n)(v'_n - v_n)/\bar{p}_n(v'_n - v_n) \rightarrow \infty$ which implies $(\|v'_n - v_n\|/\bar{p}_n(v'_n - v_n))((p'_n - p_n)(v'_n - v_n)/\|v'_n - v_n\|) \rightarrow \infty$. Since, for every $n, (p'_n - p_n)(v'_n - v_n)/\|v'_n - v_n\| \leq 2$, we have $\|v'_n - v_n\|/\bar{p}_n(v'_n - v_n) \rightarrow \infty$, which can be rewritten because of (iv), as $\|h^{\succeq}(\bar{p}_n, w'_n) - h^{\succeq}(\bar{p}_n, w_n)\|/(w'_n - w_n) \rightarrow \infty$, contradicting the hypothesis that h^{\succeq} is income lipschitzian. Q.E.D.

We are ready now to prove the theorem.

(i) $M_{\bar{p}}$ is lipschitzian on K . Suppose not; then there are $x_n, y_n \in K$ such that $\neg(y_n \succeq x_n)$ and $(M(\bar{p}, x_n) - M(\bar{p}, y_n))/\|x_n - y_n\| \rightarrow \infty$. For every n , pick $p_n \in \sigma(\succeq, y_n)$; since $p_n y_n = M(p_n, y_n) < M(p_n, x_n) \leq p_n x_n$, one has $0 < M(p_n, x_n) - M(p_n, y_n) \leq p_n(x_n - y_n) \leq \|p_n\| \|x_n - y_n\| = \|x_n - y_n\|$. Therefore, $(M(\bar{p}, x_n) - M(\bar{p}, y_n))/(M(p_n, x_n) - M(p_n, y_n)) \rightarrow \infty$, which contradicts Lemma 8.

(ii) $M_{\bar{p}}$ is regular on K . Suppose not, then there are $x_n, y_n \in K$ such that $x_n > y_n$ and $(M(\bar{p}, x_n) - M(\bar{p}, y_n))/\|x_n - y_n\| \rightarrow 0$. Let $\varepsilon > 0$ be such that $p > \varepsilon e$ for every $p \in J$. For every n , pick $p_n \in \sigma(R, x_n)$; since $M(p_n, y_n) \leq p_n y_n \leq p_n x_n = M(p_n, x_n)$, one has $M(p_n, x_n) - M(p_n, y_n) \geq p_n(x_n - y_n) \geq \varepsilon \|x_n - y_n\|$. Therefore, $(M(p_n, x_n) - M(p_n, y_n))/(M(\bar{p}, x_n) - M(\bar{p}, y_n)) \rightarrow \infty$ which, again, contradicts Lemma 8.

4. EXAMPLES

A. Example 1

We will exhibit two distinct $\succeq, \succeq' \in \mathcal{P}_{sc}$ with $h^{\succeq} = h^{\succeq'}$. Since the details are messy, it will be helpful to give first an intuitive explanation of the construction. The main idea (also to be found in Rader [16, p. 244] and attributed to L. Shapley) goes as follows: suppose one has in the plane two distinct finite families of (nonintersecting) indifference curves such that any two curves from different

families either do not intersect or intersect in a tangential manner (see Figure 1); moreover, some couple of lines do in fact intersect. It is possible to convince oneself by inspection that there is no difficulty in adding to any of the families a new line passing through an arbitrary point and preserving the tangential intersection property. Iterating, one ends up with two families *dense* in the plane and it is reasonably clear that they define two distinct "nice" preference relations having the same demand function.

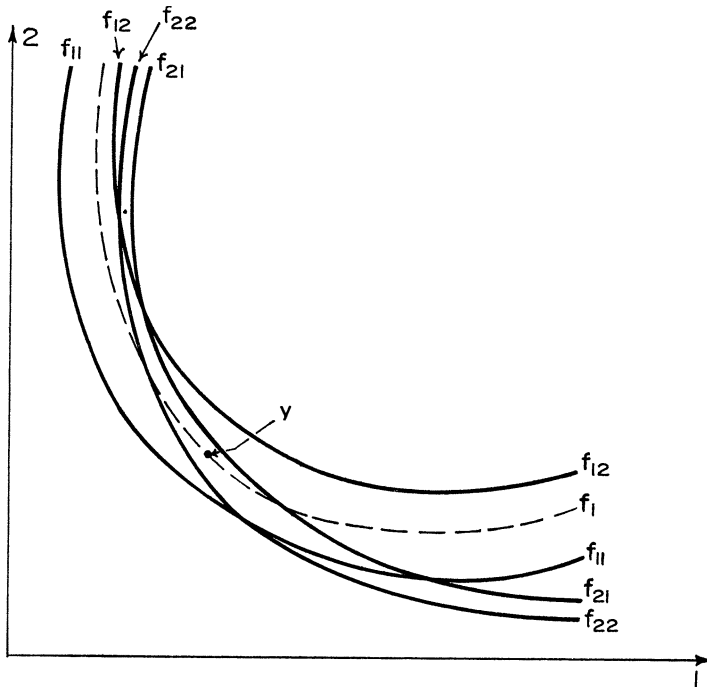


FIGURE 1

The construction is organized in three sections. The first carries out the iterative construction of the two families of lines; the second deals with the limiting operations; the third puts the results together. Not to be exceedingly long, many small and easy proofs shall be left out.

SECTION 1: Define $\varphi: [0, 1] \rightarrow \mathbb{R}$ by $\varphi(t) = 3 - (t/(1+t))$; φ is C^1 , decreasing and strictly convex; let Fe be the space of C^1 , decreasing, strictly convex functions $f: [0, 1] \rightarrow \mathbb{R}$ such that $0 \ll f \ll \varphi$.⁷ The set of nonempty, finite subsets of Fe is \mathcal{F} . A

⁷ $f \ll_{[a,a']} f^*, f, f^*: A \rightarrow \mathbb{R}, [a, a'] \subset A \subset \mathbb{R}$, shall mean $f(t) < f^*(t)$ for every $t \in [a, a']$ (or $[a, a')$, etc.); $f \ll f^*$ stands for $f \ll_A f^*$; analogously, $f \leq f^*$ will mean $f(t) \leq f^*(t)$ for every $t \in A$; without possible confusion we shall, at times, let 0 stand for the constant function with value 0. The derivative of $f: A \rightarrow \mathbb{R}, A \subset \mathbb{R}$, at $t \in A$, if it exists, is $f'(t)$; in general (i.e., $f: A \rightarrow \mathbb{R}^n, A \subset \mathbb{R}^m$) the derivative map at $x \in A$ is denoted $Df(x)$.

Given $f, f^*: A \rightarrow \mathbb{R}, A \subset \mathbb{R}$, let $f \vee f^*, f \wedge f^*: A \rightarrow \mathbb{R}$ be given by $(f \vee f^*)(t) = \max \{f(t), f^*(t)\}$, $(f \wedge f^*)(t) = \min \{f(t), f^*(t)\}$.

generic element of $\mathcal{F} \times \mathcal{F}$ shall be denoted $(\{f_{1,i}\}_1^{N_1}, \{f_{2,i}\}_1^{N_2})$. We state the following condition (see Figure 1):

$(\{f_{1,i}\}_1^{N_1}, \{f_{2,i}\}_1^{N_2}) \in \mathcal{F} \times \mathcal{F}$ satisfies:

- (1a) for every $j = 1, 2$ and $1 \leq i, i' \leq N_j$, either $f_{j,i} \ll f_{j,i'}$ or $f_{j,i'} \ll f_{j,i}$ or $f_{j,i} = f_{j,i'}$;
- (1b) for every $1 \leq i \leq N_1, 1 \leq i' \leq N_2$ the equation $f_{1,i}(t) = f_{2,i'}(t)$ has at most one solution;
- (1c) for every $1 \leq i \leq N_1, 1 \leq i' \leq N_2, 0 \leq \bar{i} \leq 1, f_{1,i}(\bar{i}) = f_{2,i'}(\bar{i})$ implies $f'_{1,i}(\bar{i}) = f'_{2,i'}(\bar{i})$.

Note that, for every $1 \leq i \leq N_1, 1 \leq i' \leq N_2, f_{1,i} \vee f_{2,i'} \in Fe$ and $f_{1,i} \wedge f_{2,i'} \in Fe$. We will need the following fact.

With every eight-tuple $(a_1, a_2, b_1, b_2, c_1, c_2, g_1, g_2) \in R^6 \times Fe^2$ satisfying:

- (2a) $0 \leq a_1 < a_2 \leq 1, b_1 < b_2, g_1 \ll g_2, g_1(a_1) \leq b_1 \leq g_2(a_1), g_1(a_2) \leq b_2 \leq g_2(a_2)$;
- (2b) for every $a_1 < t < a_2, g_2(t) > \max\{b_1 + c_1(t - a_1), b_2 + c_2(t - a_2)\}$;
- (2c) $b_2 > b_1 + c_1(a_2 - a_1), b_1 > b_2 + c_2(a_1 - a_2)$,

one can associate a C^1 , strictly concave function $\beta[a_1, a_2, b_1, b_2, c_1, c_2, g_1, g_2]: [a_1, a_2] \rightarrow R$ such that (dropping, by notational economy, the bracketed terms):

- (3) $g_1 \ll_{(a_1, a_2)} \beta \ll_{(a_1, a_2)} g_2, \beta(a_1) = b_1, \beta(a_2) = b_2, \beta'(a_1) = c_1, \beta'(a_2) = c_2$.

See Figure 2. We skip the proof of this assertion (which, details aside, is obvious).

We now show the following: For $j = 1, 2$ there are functions $\Psi_j: \mathcal{F} \times \mathcal{F} \times E \rightarrow Fe$ such that if $(\{f_{1,i}\}_1^{N_1}, \{f_{2,i}\}_1^{N_2}) \in \mathcal{F} \times \mathcal{F}$ satisfies (1) and $y \in E$, then letting $f_j = \Psi_j(\{f_{1,i}\}_1^{N_1}, \{f_{2,i}\}_1^{N_2}, y)$,

- (4a) $f_j(y^1) = y^2$;
- (4b) if $j = 1$ (resp. $j = 2$), $(\{f_{1,i}\}_1^{N_1} \cup \{f_1\}, \{f_{2,i}\}_1^{N_2}$ (resp. $(\{f_{1,i}\}_1^{N_1}, \{f_{2,i}\}_1^{N_2} \cup \{f_2\}))$ satisfies (1).

See Figure 1.

PROOF: By symmetry, it suffices to prove that Ψ_1 exists. The problem reduces to exhibiting a set of rules such that for any $(\{f_{1,i}\}_1^{N_1}, \{f_{2,i}\}_1^{N_2}) \in \mathcal{F} \times \mathcal{F}$ satisfying (1) a unique $f \in Fe$ is obtained such that $f(y^1) = y^2$ and $(\{f_{1,i}\}_1^{N_1} \cup \{f\}, \{f_{2,i}\}_1^{N_2})$ satisfies (1).

Take an arbitrary $y \in E$ and $(\{f_{1,i}\}_1^{N_1}, \{f_{2,i}\}_1^{N_2}) \in \mathcal{F} \times \mathcal{F}$ for which (1) holds. Let $f_{1,0} = f_{2,0} = 0, f_{1, N_1+1} = f_{2, N_2+1} = \varphi$; we can assume $f_{1,0} \leq f_{1,1} \leq \dots \leq f_{1, N_1+1}, f_{2,0} \leq f_{2,1} \leq \dots \leq f_{2, N_2+1}$. For some $1 \leq j_1 \leq N_1, f_{1, j_1}(y^1) \leq y^2 \leq f_{1, j_1+1}(y^1)$; if $y^2 = f_{1, j_1}(y^1)$ (resp. $y^2 = f_{1, j_1+1}(y^1)$) take $f = f_{1, j_1}$ (resp. $f = f_{1, j_1+1}$). From now on suppose that $f_{1, j_1}(y^1) < y^2 < f_{1, j_1+1}(y^1)$. Let $f_{2, j_2}(y^1) \leq y^2 \leq f_{2, j_2+1}(y^1), 1 \leq j_2 \leq N_2$, and denote $g_i = f_{2,i}$ if $i \leq j_2, g_{i+1} = f_{2,i}$ if $i \geq j_2 + 1$ (hence, g_{j_2+2} is not defined). Take $m_1 = \max\{i: g_i \ll f_{1, j_1+1} \text{ and } g_i(y^1) < y^2\}, m_2 = j_2 + 1, m_3 = \min\{i: g_i \gg f_{1, j_1} \text{ and}$

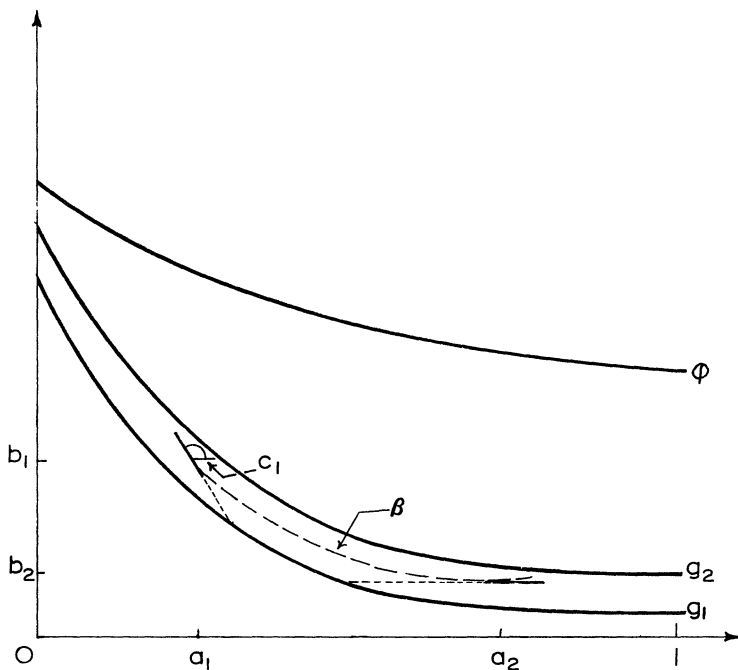


FIGURE 2

$g_i(y^1) > y^2$ }; clearly, $m_1 \leq m_2 - 1$, $m_3 \geq m_2 + 1$. Let $\bar{f} = f_{1,j_1} \vee g_{m_1}$, $\bar{f} = f_{1,j_1+1} \wedge g_{m_3}$; then $\bar{f} \ll \bar{f}$ and, of course, $\bar{f}, \bar{f} \in Fe$. Define the symbols:

$$g_{m_2}(y^1) = y^2, g'_{m_2}(y^1) = \begin{cases} (\bar{f} \wedge g_{m_2+1})'(x^1) & \text{if } g_{m_2-1}(y^1) < y^2 \leq g_{m_2+1}(y^1); \\ g'_{m_2-1}(y^1) & \text{if } g_{m_2-1}(y^1) = y^2. \end{cases}$$

By the hypothesis and the definition of \bar{f}, \bar{f} the following two cases are exhaustive and mutually exclusive (see Figure 3): (i) $g_i(0) \geq \bar{f}(0)$ for every $m_1 \leq i \leq m_2 - 1$ and $g_i(1) \leq \bar{f}(1)$ for every $m_2 + 1 \leq i \leq m_3$; (ii) $m_3 > m_1 + 2$, $g_i(1) \geq \bar{f}(1)$ for every $m_1 \leq i \leq m_2 - 1$ and $g_i(0) \leq \bar{f}(0)$ for every $m_2 + 1 \leq i \leq m_3$.

We construct now the function f for case (i); case (ii) is entirely parallel.

Let $a_{m_2} = y_1$ and, for $i = m_2$, define numbers $\bar{a}_i, \bar{a}_i, \bar{a}_i, a'_i, a_i$ by the following conditions:

If $m_1 < i \leq m_2 - 1$ let $g_i(\bar{a}_i) = \bar{f}(\bar{a}_i)$, $g_{i+1}(\bar{a}_{i+1}) + g'_{i+1}(\bar{a}_{i+1})(\bar{a}_i - \bar{a}_{i+1}) = g_i(\bar{a}_i)$, $\bar{a}_i = \sup \{t \in [0, 1]: g_i(t) > \bar{f}(t)\}$,

$$a'_i = \frac{1}{2}\bar{a}_i + \frac{1}{2} \min \{\bar{a}_i, \bar{a}_i\}, a_i = \begin{cases} a'_i & \text{if } i < m_2 - 1, \text{ or } i = m_2 - 1 \text{ and } y^1 > \bar{a}_i; \\ y^1 & \text{otherwise.} \end{cases}$$

If $m_2 + 1 \leq i < m_3$, let $g_i(\bar{a}_i) = \bar{f}(\bar{a}_i)$, $g_i(\bar{a}_i) + g'_i(\bar{a}_i)(\bar{a}_{i-1} - \bar{a}_i) = g_{i-1}(\bar{a}_{i-1})$, $\bar{a}_i = \inf \{t \in [0, 1]: g_i(t) < \bar{f}(t)\}$, $a'_i = \frac{1}{2}\bar{a}_i + \frac{1}{2} \max \{\bar{a}_i, \bar{a}_i\}$,

$$a_i = \begin{cases} a'_i & \text{if } i > m_2 + 1, \text{ or } i = m_2 + 1 \text{ and } y^1 < \bar{a}_i; \\ y^1 & \text{otherwise.} \end{cases}$$

One has $0 < a_{m_1+1} < \dots < a_{m_2-1} \leq a_{m_2} \leq a_{m_2+1} < \dots < a_{m_3-1} < 1$.

Define the following symbols:

$$\bar{g}_{m_1}(0) = \frac{1}{2} \min \{\bar{f}(0), g_{m_2+1}(0)\} + \frac{1}{2} \max \{\bar{f}(0), g_{m_1+1}(a_{m_1+1}) - g'_{m_1+1}(a_{m_1+1})a_{m_1+1}\};$$

$$\bar{g}'_{m_1}(0) = 2g'_{m_1+1}(a_{m_1+1});$$

$$\bar{g}_{m_2}(y_1) = g_{m_2}(y_1); \bar{g}'_{m_2}(y_1) = g'_{m_2}(y_1);$$

$$\bar{g}_{m_3}(1) = \frac{1}{2} \max \{\bar{f}(1), g_{m_3-1}(a_{m_3-1}) + g'_{m_3-1}(a_{m_3-1})(1 - a_{m_3-1}), g_{m_2-1}(1)\} + \frac{1}{2} \min \{\bar{f}(1), g_{m_3-1}(a_{m_3-1})\};$$

$$\bar{g}'_{m_3}(1) = \frac{1}{2}g'_{m_3-1}(a_{m_3-1}); \bar{f} \vee \bar{g}_{m_1} = \bar{f} \vee g_{m_1}; \bar{f} \wedge \bar{g}_{m_3} = \bar{f} \wedge g_{m_3};$$

and, finally, for $m_1 < i < m_3$, $i \neq m_2$, let $\bar{g}_i = g_i$.

Let, then,

$$f(t) = \begin{cases} g_{m_1}(0) & \text{if } t = 0; \\ \beta[a_i, a_{i+1}, \bar{g}_i(a_i), \bar{g}_{i+1}(a_{i+1}), \bar{g}'_i(a_i), \bar{g}'_{i+1}(a_{i+1}), \bar{f} \vee \bar{g}_i, \bar{f} \wedge \bar{g}_{i+1}](t) & \text{if } a_i < t \leq a_{i+1}, m_1 \leq i \leq m_2 - 2, \text{ or } a_i \leq t < a_{i+1}, m_2 + 1 \leq i \leq m_3 - 1; \\ \beta[a_{m_2-1}, a_{m_2}, \bar{g}_{m_2-1}(a_{m_2-1}), \bar{g}_{m_2}(a_{m_2}), \bar{g}'_{m_2-1}(a_{m_2-1}), \bar{g}'_{m_2}(a_{m_2}), \bar{f} \vee \bar{g}_{m_2-1}, \bar{f} \wedge \bar{g}_{m_2+1}](t) & \text{if } a_{m_2-1} < t \leq a_{m_2}; \\ \beta[a_{m_2}, a_{m_2+1}, \bar{g}_{m_2}(a_{m_2}), \bar{g}_{m_2+1}(a_{m_2+1}), \bar{g}'_{m_2}(a_{m_2}), \bar{g}'_{m_2+1}(a_{m_2+1}), \bar{f} \vee \bar{g}_{m_2-1}, \bar{f} \wedge \bar{g}_{m_2+1}](t) & \text{if } a_{m_2} \leq t < a_{m_2+1}; \\ \bar{g}_{m_3}(1) & \text{if } t = 1. \end{cases} \quad Q.E.D.$$

Let $\{y_n\}_1^\infty$ be a countable dense subset of $E = \{y \in \mathbb{R}^2: 0 < y^1 < 1, 0 < y^2 < \varphi(y^1)\}$. For the rest of this and the next two sections the set $\{y_n\}_n^\infty$ shall remain fixed; we assume $y_1 = (\frac{1}{2}, 2)$.

We make the following claim: *There are two sequences $f_{1,n}, f_{2,n} \in Fe$, $1 \leq n < \infty$, such that*

- (5a) for every n, n' and $j = 1, 2$, either $f_{j,n} \ll f_{j,n'}$ or $f_{j,n'} \ll f_{j,n}$ or $f_{j,n'} = f_{j,n}$;
- (5b) for every n , $f_{1,n}(y_n^1) = f_{2,n}(y_n^2)$ and $f'_{1,n}(y_n^1) = f'_{2,n}(y_n^2)$;
- (5c) $f_{1,1} \neq f_{2,1}$.

PROOF: This is an obvious consequence of (4). In two initial steps choose for $f_{1,1}, f_{2,1}$, respectively, two arbitrary distinct functions $g_1, g_2 \in Fe$ such that $g_1(\frac{1}{2}) =$

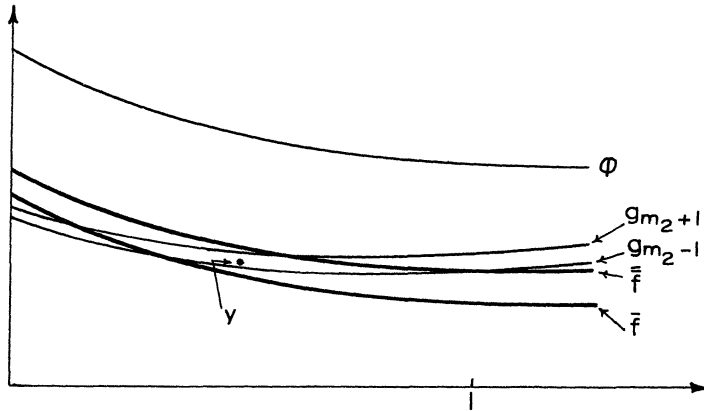


FIGURE 3—Case (i).

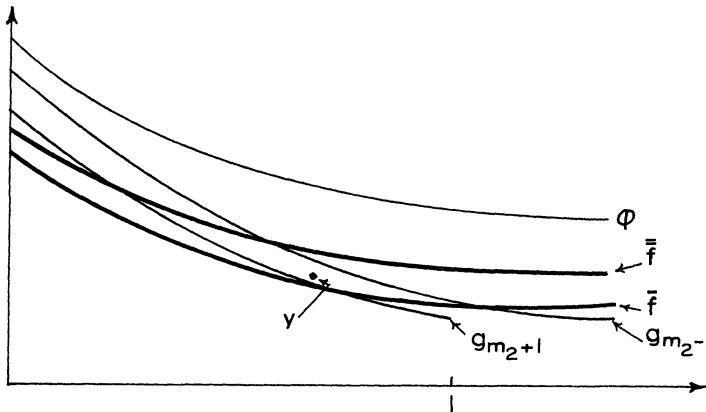


FIGURE 3—Case (ii).

$g_2(\frac{1}{2}) = 2$ and $g'_1(\frac{1}{2}) = g'_2(\frac{1}{2})$. Then, in a recursive manner, apply alternatively the rules Ψ_1, Ψ_2 to generate the full sequences.

SECTION 2: Let $\pi_l: \Omega \rightarrow R$ denote the projection map on the l th coordinate space. The next step allows one to pass from a countable family of curves to the whole space.

Let $A \subset \Omega$ be a dense set and $\{U_x; x \in A\}$ a collection of nonempty, closed convex sets such that:

- (6a) $x \in U_x$ and $U_x + \Omega \subset U_x$ for every $x \in A$;
- (6b) if $x \in \text{int}_\Omega U_{x'}$, then $U_x \subset \text{int}_\Omega U_{x'}$, for every $x, x' \in A$;
- (6c) for every $x, x' \in A$, either $U_x \subset U_{x'}$ or $U_{x'} \subset U_x$;
- (6d) for every $x \in A$, $\pi_l(\Omega \sim U_x)$ is bounded;

then the relation \succeq defined by $x \succeq v$ if and only if, for every $z \in A$, " $v \in U_z$ implies $x \in U_z$," satisfies:

- (6e) $\succeq \in \mathcal{P}$, i.e., \succeq is a continuous, monotone, convex preference relation;
- (6f) $V(\succeq, x) = U_x$ for every $x \in A$;
- (6g) $\pi_1(\Omega \sim U(\succeq, x))$ is bounded for every $x \in \Omega$.

PROOF: By definition \succeq is obviously reflexive and transitive.

(i) \succeq is complete: Suppose that $\neg(x \succeq v)$, $x, v \in \Omega$; then there is $z \in A$ such that $v \in U_z$ and $x \notin U_z$. Therefore, if $x \in U_{v'}$, $v' \in A$, we cannot have $U_{v'} \subset U_z$, hence, by (6c), $U_z \subset U_{v'}$, implying $v \in U_{v'}$. So, $v \succeq x$.

(ii) \succeq is continuous: Let $x_n \rightarrow x$, $v_n \rightarrow v$, $x_n \succeq v_n$. Suppose that for some $z \in A$, $v \in U_z$, $x \notin U_z$. Take $z', z'' \in A$ such that $x \ll z' \ll z''$ and $z'' \notin U_z$, then $U_z \subset U_{z'} \subset \text{int}_\Omega U_{z'}$ (by (6a-c)). Therefore, for some \bar{n} , $v_{\bar{n}} \in U_{z'}$ and $x_{\bar{n}} \notin U_{z'}$, which contradicts $x_{\bar{n}} \succeq v_{\bar{n}}$. Hence $x \succeq v$.

(iii) \succeq is monotone: Let $z \gg x$, $x \in \Omega$. Pick $v \in A$ such that $x \ll v \ll z$. Then $z \in U_v$, but $x \notin U_v$ (if $x \in U_v$, then $v \in \text{int}_\Omega U_v$, which is impossible by (6b)), hence $z > x$.

(iv) (6f) holds: Let $z \in U_x$, $x \in A$; if $x \in U_v$, $v \in A$, then $z \in U_v$ (by (6a-c)); hence $z \in V(\succeq, x)$. Let $z \notin U_x$, then $x \in U_x$ and $z \notin U_x$, hence $z \notin V(\succeq, x)$.

(v) R is a convex preference relation: By continuity, monotonicity, and transitivity it suffices to prove that for every $x \in \Omega$, $\text{int}_\Omega V(\succeq, x)$ is convex. Let $v, z \in \text{int}_\Omega V(\succeq, x)$ and take $v_n \rightarrow v$, $z_n \rightarrow z$, $v_n, z_n \in A \cap \text{int}_\Omega V(\succeq, x)$. By (6f), already proved, $[v_n, z_n] \subset V(\succeq, x)$ and so, $[v, z] \subset V(\succeq, x)$.

(vi) (6g) holds: Let $x \in \Omega$, pick $v \gg x$, $v \in A$. Then $\pi_1(\Omega \sim U(\succeq, x)) \subset \pi_1(\Omega \sim U(\succeq, v)) = \pi_1(\Omega \sim U_v)$, which, by (6d), is bounded. Q.E.D.

Let $C = \{p \in S^{l-1} : p^l > 0\}$. Note that if a $\succeq \in \mathcal{P}$ satisfies (6g), $x \in \text{int } \Omega$ and $p \in \sigma(\succeq, x)$, then $p \in C$.

Define a strictly convex function $\lambda : R_+^{l-1} \rightarrow [0, -1]$ by $\lambda(x) = \sum_{i=1}^{l-1} (1/l)(-x^i/(1+x^i))$ and a continuous one-to-one function $\Phi : \Omega \rightarrow \Omega$ by $\Phi(x^1, \dots, x^l) = (x^1, \dots, x^{l-1}, x^l - \lambda(x^1, \dots, x^{l-1}))$.

Given any $\succeq \in \mathcal{P}$ define \succeq^* by $x \succeq^* v$ if and only if $\Phi x \succeq \Phi v$ (see Figure 4). It is immediate that \succeq^* is a continuous, monotone, preference relation; moreover:

- (7) If \succeq satisfies (6g), then \succeq^* is strictly convex, i.e., $\succeq^* \in \mathcal{P}_{sc}$.

PROOF: Pick $\bar{x} \in \Omega$ and let $\bar{x} = \Phi(\bar{x})$. Since (6g) holds, there is a convex $g : R_+^{l-1} \rightarrow R$ such that $v \succeq \bar{x}$ and $\bar{x} \succeq v$ if and only if $v^l = g(v^1, \dots, v^{l-1})$. Therefore $v' \succeq^* \bar{x}$ and $\bar{x} \succeq^* v'$ if and only if $v'^l = g(v'^1, \dots, v'^{l-1}) + \lambda(v'^1, \dots, v'^{l-1})$. Since $g + \lambda$ is a strictly convex function and \bar{x} is arbitrary, \succeq^* is strictly convex (see Figure 4). Q.E.D.

Define $\rho : C \times \text{int } \Omega \rightarrow C$ by $\rho(p, x) = a(p, x) / \|a(p, x)\|$, where $a(p, x) = ((p^1/p^l) - D_x^1 \lambda(x^1, \dots, x^{l-1}), \dots, (p^{l-1}/p^l) - D_x^{l-1} \lambda(x^1, \dots, x^{l-1}), 1)$; let $\Omega_1 =$

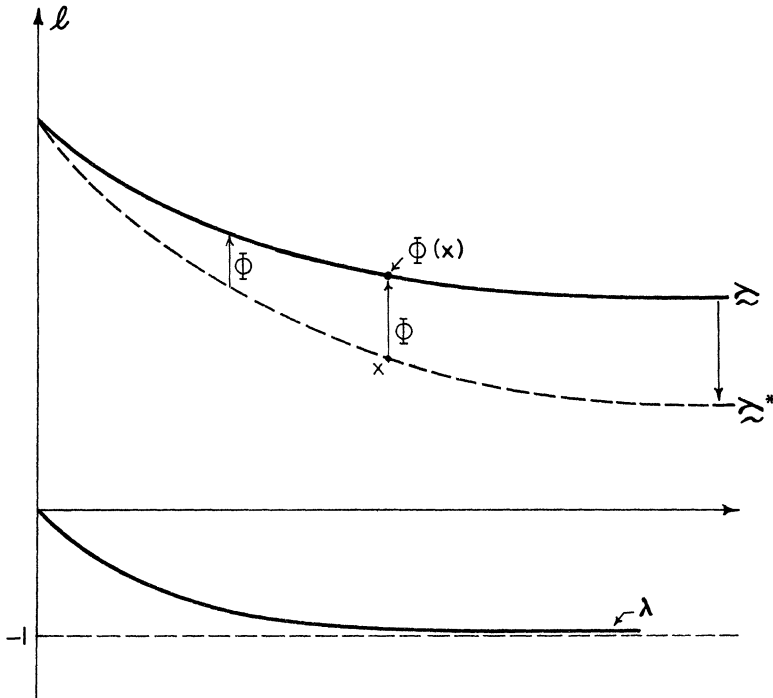


FIGURE 4

$\{x \in \Omega: x^l \geq 1\}$. The following two facts are easy to check:

- (8) If $\succeq_1 \cap \Omega_1 \times \Omega_1 \neq \succeq_2 \cap \Omega_1 \times \Omega_1, \succeq_1, \succeq_2 \in \mathcal{P}$, then $\succeq_1^* \neq \succeq_2^*$;
- (9) If $\succeq \in \mathcal{P}$ satisfies (6g) and $x \in \text{int } \Omega$, then $\sigma(\succeq^*, x) = \rho(\sigma(\succeq, \Phi x) \times \{\Phi x\})$.

We show now:

- (10) Let $A \subset \Omega$ be dense and $\succeq_1, \succeq_2 \in \mathcal{P}$. If $\sigma(\succeq_1, x) = \sigma(\succeq_2, x)$ for every $x \in A$, then $\sigma(\succeq_1, x) = \sigma(\succeq_2, x)$ for every $x \in \Omega$.

PROOF: It is well known (see, for example, Mas-Colell [13, (1.14)]) that for every $\succeq \in \mathcal{P}, \sigma(\succeq, \cdot)$ as a correspondence on Ω is upper hemicontinuous. Therefore, for every $x \in \Omega, \sigma(\succeq_1, x) \cap \sigma(\succeq_2, x) \neq \emptyset$ and, as a consequence, $V(\succeq_1, x) \cup V(\succeq_2, x')$ is convex, for every $x, x' \in \Omega'$.

Let $p \in \sigma(\succeq_1, x), x \in \Omega$. Take any $z \in \Omega$ such that $pz < px$. Then, $[z, x] \subset V(\succeq_1, x) \cup V(\succeq_2, z)$; by hypothesis $[z, x] \cap V(\succeq_1, x) = \emptyset$; therefore, $[z, x] \subset V(\succeq_2, y)$ and so $x \in V(\succeq_2, z)$; by monotonicity $z \notin V(\succeq_2, x)$. Hence $p \in \sigma(\succeq_2, x)$. Q.E.D.

SECTION 3: All the pieces for the construction of the example are now available. Let $f_{1,n}, f_{2,n}$ be two sequences of functions as the ones whose existence is asserted in (5). Denote (see Figure 5) $\hat{E} = \{x \in \Omega : (x^1, x^1) \in E\}$, $H = \bigcup_n \{x \in \Omega : (x^1, x^1) = y_n\}$, $A = \text{int}(\Omega \sim E) \cup H$, $U = \{x \in \Omega : "x^1 \leq 1 \text{ and } x^1 \geq \varphi(x^1)" \text{ or } "x^1 \geq 1 \text{ and } x^1 \geq 1"\}$.

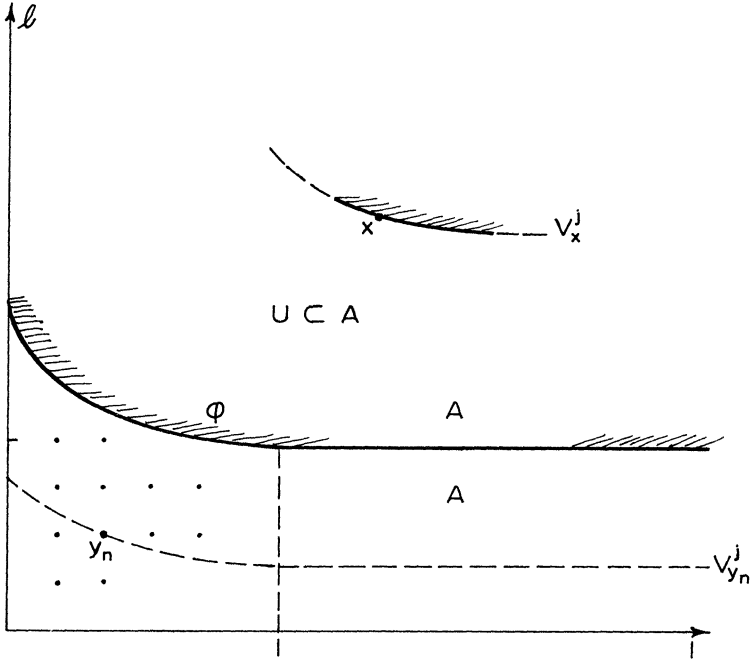


FIGURE 5

Define two families $\{U_x^j : x \in A\}, j = 1, 2$, of closed, convex sets as follows: (i) If $x \in U$, then U_x^j is obtained by homothetic expansion of U subject to the condition $x \in \text{bdry } U_x^j$. (ii) If $x \in \hat{E}$, then $(x^1, x^1) = y_n$ for some n ; let $U_x^j = \{v \in \Omega : "v^1 \leq 1 \text{ and } v^1 \geq f_{j,n}(v^1)" \text{ or } "v^1 \geq 1 \text{ and } v^1 \geq f_{j,n}(1)"\}$. (iii) If $x \notin U$ and $x^1 \geq 0$, let $U_x^j = \bigcap_{v \in H} U_v^j$.

The families $\{U_x^j : x \in A\}, j = 1, 2$, satisfy every one of the conditions in (6). Therefore, if we define $\succeq_j, j = 1, 2$ by $x \succeq_j v$ if, for every $z \in A, v \in U_z^j$ implies $x \in U_z^j$, we have $\succeq_j \in \mathcal{P}, j = 1, 2$, and, consequently, by (7), $\succeq_j^* \in \mathcal{P}_{sc}, j = 1, 2$. We claim that $\succeq_1^* \neq \succeq_2^*$ and $h^{\succeq_1^*} = h^{\succeq_2^*}$.

By the hypothesis on $f_{j,n}, j = 1, 2$, and (6), $\sigma(\succeq_1, x) = \sigma(\succeq_2, x)$ for every $x \in A$ and $\succeq_1 \cap \Omega_1 \times \Omega_1 \neq \succeq_2 \cap \Omega_1 \times \Omega_1$. Therefore, by (8), $\succeq_1^* \neq \succeq_2^*$. Since Φ is a homeomorphism and A is dense in $\Omega, \Omega^{-1}(A)$ is dense in Ω . Let $v \in \Phi^{-1}(A)$; then, by (9), $\sigma(\succeq_j^*, v) = \rho(\sigma(\succeq_j, \Phi v) \times \{\Phi v\}), j = 1, 2$; since $\sigma(\succeq_1, \Phi v) = \sigma(\succeq_2, \Phi v)$ we

have $\sigma(\succeq_1^*, v) = \sigma(\succeq_2^*, v)$. Therefore, by (10), $\sigma(\succeq_1^*, x) = \sigma(\succeq_2^*, x)$ for every $x \in \Omega$ or, equivalently, $h^{\succeq_1^*} = h^{\succeq_2^*}$.

This completes the construction of the example. As described the preference relations obtained may not have smooth hypersurfaces but it should be clear that with some further transformations they could be taken to be so.

B. Example 2

This is an example of a monotone, continuous, convex preference relation generating income lipschitzian demand functions and not possessing any concave utility representation. In fact, in the example there is a point such that no utility function can be concave in a neighborhood of that point. For simplicity the example is given for $l=2$ and the preference relation involved is not strictly concave; with some additional work it can be made strictly convex.

Let $\varphi: [0, \infty) \rightarrow [0, \infty)$ be the function obtained by linear extension given the values $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(1 + (1/n)) = 1 + (1/n)$ for $n \geq 1$, $\varphi(\frac{3}{4}(1 + (1/(n+1))) + \frac{1}{4}(1 + (1/n))) = \frac{1}{2}(1 + (1/(n+1))) + \frac{1}{2}(1 + (1/n))$ for $n \geq 1$, $\varphi(t) = t$ for $t \geq 2$ (see Figure 6). On R_+^2 , \succeq is the preference relation represented by

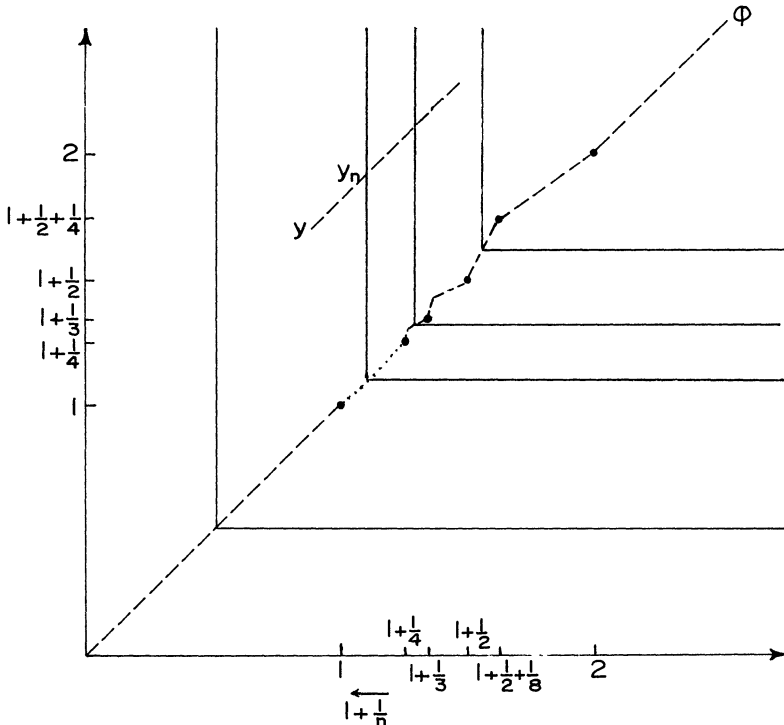


FIGURE 6

the utility function $v(x) = \min \{\varphi(x^1), x^2\}$ (which, incidentally, is lipschitzian and, with the qualification of Remark 7, regular).

Suppose there was a concave utility function $u: \Omega \rightarrow R$ representing \succeq . Let $y = (1, 2)$, $x = (1, 1)$, $y_n = (1 + (1/n), 2 + (1/n))$, $x_n = (1 + (1/n), 1 + (1/n))$. Then, for every $n \geq 2$, $u(\frac{3}{4}y_n + \frac{1}{4}y_{n-1}) = u(\frac{1}{2}x_n + \frac{1}{2}x_{n-1}) \geq \frac{1}{2}u(x_n) + \frac{1}{2}u(x_{n-1}) = \frac{1}{2}u(y_n) + \frac{1}{2}u(y_{n-1})$. Hence, denoting by $\mu(z)$ the directional derivative of u at $z \in \Omega$ in the direction $(1, 1)$,

$$\begin{aligned} \mu(y_n) &\geq 4n(n-1)(u(\frac{3}{4}y_n + \frac{1}{4}y_{n-1}) - u(y_n)) \\ &\geq 4n(n-1)(\frac{1}{2}u(y_n) + \frac{1}{2}u(y_{n-1}) - u(y_n)) \\ &= 2n(n-1)(u(y_{n-1}) - u(y_n)) \geq 2\mu(y_{n-1}). \end{aligned}$$

Since $\mu(y_2)$ has to be positive (monotonicity) $\mu(y_n) \rightarrow \infty$, which is impossible, given the concavity of u .

The same argument shows that no concave function exists representing \succeq in a neighborhood of $(1, 1)$.

Clearly, h^\succeq is income lipschitzian; in fact, the example can be easily modified (smooth out the corners of φ) to yield a differentiable demand function; possibly, a more complicated modification would yield a continuously differentiable one.

C. Example 3

This is an example of a continuous, monotone \succeq representable by a concave utility function and generating a demand function which is not income lipschitzian. Again, for ease of description, $l = 2$ and \succeq will not be strictly convex. It is easy to modify the example to get strict convexity.

Let $\alpha(m): [0, m] \rightarrow R^2$ be the curve defined by (see Figure 7):

$$\alpha(m) = \begin{cases} \left(\frac{m}{2} - 1 - \left(\frac{m}{2} - 1 \right)^{\frac{1}{3}}, \frac{m}{2} - 1 + \left(\frac{m}{2} - 1 \right)^{\frac{1}{3}} \right) & \text{if } \frac{3}{2} \leq m \leq \frac{5}{2}; \\ m^{\frac{2}{3}} \alpha\left(\frac{3}{2}\right) & \text{if } 0 \leq m \leq \frac{3}{2}; \\ m^{\frac{2}{5}} \alpha\left(\frac{5}{2}\right) & \text{if } m \geq \frac{5}{2}. \end{cases}$$

Let $T \subset R^3$ be $T = \{\alpha^1(m), \alpha^2(m), (\alpha^1(m) + \alpha^2(m))^{\frac{1}{2}} : 0 \leq m < \infty\}$. Define a concave, continuous, increasing function $u: \Omega \rightarrow R$ by $u(x) = \sup \{v \in R : (x, v) \in \text{co}(T + (\Omega \times \{0\}))\}$. Then the preference relation \succeq represented by u is such that for $p^1 = p^2$, $h^\succeq(p^1, p^2, \cdot) = \alpha(\cdot)$ and, therefore, h^\succeq is not income lipschitzian at $p^1 = p^2 = 1$, $w = 2$ (see Figure 7).

University of California at Berkeley

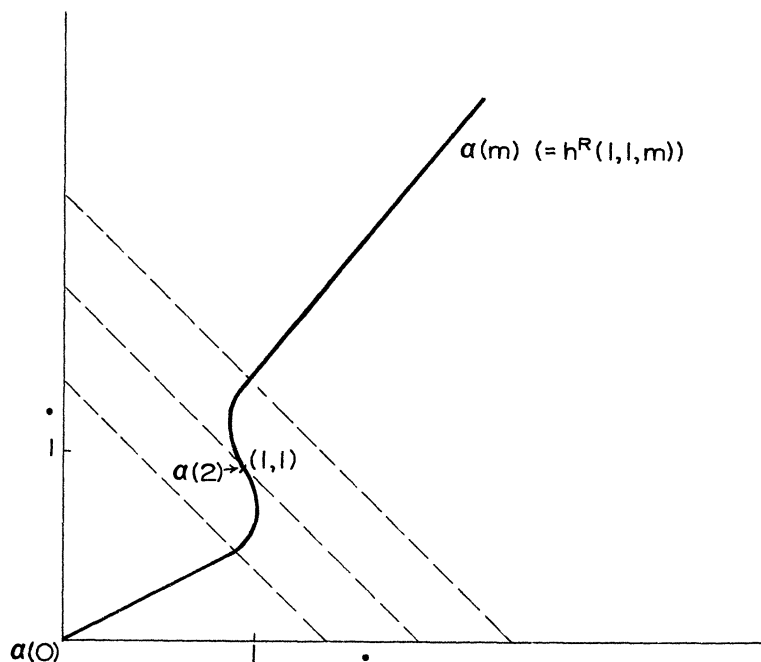


FIGURE 7

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