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REGULAR, NONCONVEX ECONOMIES[1]

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1. INTRODUCTION

IT IS WELL KNOWN that the general competitive model is not an appropriate theoretical account of market situations with large nonconvexities. While this is a basic failure we may nevertheless stress the qualification "large." If individual (consumers or producers) nonconvexities are small relative to the size of the economy, then aggregation effects should be expected to arise and to cancel, on the average, the individual "discontinuous" behavior. This is, of course, a familiar fact and in the set-up of a pure exchange economy—the only market problem that shall concern us from now on—rigorous models have been presented by Starr [**25**] and Aumann [**1**]. Those authors have shown that if there is a large number of traders (in Aumann's case, a continuum), then an appropriately defined competitive equilibrium does exist (approximately, in Starr's case). More precisely, an equilibrium in the Starr and Aumann sense (or, for that matter, in the Arrow-Debreu sense) is defined as a price vector *and* an allocation of goods such that markets are cleared and consumers maximize preferences on their budget sets.

Important as those results are, the notion of equilibrium they deal with has some unattractive features. In particular, knowledge by the consumers of the equilibrium price system (plus the preference maximization hypothesis) does not determine the equilibrium; one needs, in addition, a possibly very careful specification of each consumer's commodity bundles. This makes the equilibrium a "decentralized" one only in some weak sense. Even at the technical level, this problem with the determinateness of equilibrium creates many difficulties. For example, on its account, the limit theorems for the core with nonconvex consumer preferences are weaker and more complex than the corresponding ones with convex preferences (see Hildenbrand [**17**, 3.3]).

It is the purpose of this paper to improve (at some cost, obviously) on the above situation with the help of the concept of a regular economy. This notion was introduced by Debreu [**4**] and the "generic" approach (i.e., "focus on the general case") which underlies it rests on the exploitation of smoothness hypothesis. So, we shall restrict ourselves to a world with smooth preferences; this implies a restriction of substance: indivisible commodities are ruled out.

In [**4**] Debreu dealt with a finite number of traders which were implicitly assumed to have convex preferences. His analysis has been extended by H. Dierker [**11**] to exchange economies with infinitely many traders (formalized according to Hildenbrand [**17**]) having convex preferences. The present study can be viewed as an extension of H. Dierker's work to economies with nonconvex preferences.

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Following Hildenbrand [**17**] we define an economy (more properly a continuum economy) to be a distribution with compact support in a metric space of traders' characteristics. In our case, those are endowments and (smooth) preferences equipped with a C^1-type metric. Two topologies (the α and the β) for economies are considered. In the α one, being close means that the distributions are close with respect to the weak convergence and that mean endowments are close. In the β topology, being close means that the distributions are close in the α sense and, moreover, that their supports are also close. Both topologies have been used by Hildenbrand [**17**] and H. Dierker [**11**].

Starr [**25**] and Hildenbrand, Schmeidler, and Zamir [**18**] have established that, under appropriate conditions, large economies have approximate equilibria. As a complement to their result we show that there exists a β-open and dense set of economies such that if one considers a sequence of finite economies with an increasing number of participants and "limit" in this set then, eventually, an *exact* equilibrium exists.

While the result of the last paragraph is probably good enough for the sake of arguing that the exploitation of differentiability assumptions in the present context is not pointless, it is not yet a very useful one. The reason is that the open-dense set identified is not of general relevance, i.e., it "works" for the particular property contemplated in that paragraph but not for others. What one wants is a notion of regularity that, as in the convex case, is intrinsically defined (i.e., not with relation to some particular property) and implies strong determinateness of equilibrium properties (and we are purposely vague on which those are).

With the motivation given at the beginning of this introduction, we propose to define an economy as regular if, in the first place, for every equilibrium price, all consumers in the economy (more precisely, all consumers with characteristics in the support of the measure describing the economy) have a unique, nondegenerate maximizer in their budget sets. Since this implies that the mean excess demand correspondence of the economy is a C^1 function in a neighborhood of the equilibrium set, we further require that, at every equilibrium price, its derivative map be of maximal possible rank (this last part is nothing but the notion of regularity used by Debreu [**4**] and H. Dierker [**11**]). Observe that, in a regular economy, the patterns of equilibria appear "locally" as in a convex economy.

Let us parenthetically observe that this definition of regularity is quite demanding. It is doubtful, however, that anything weaker would be of any use, at least in the present *distribution approach* to the modeling of the space of large economies. It may be possible that in a more stringent *parametric approach* (see Sondermann [**24**]) to the same problem one could exploit a notion of regularity defined much the same way except that now "all consumers in the economy" would mean "almost all" in the measure theoretic sense, rather than "all which have characteristics in the support of the measure." So far, however, the development of this program of work has been hindered by the difficulties of obtaining suitable smoothness properties for aggregate demand.

Clearly, the set of regular economies is open in the β topology (it is obviously not so in the α one) but, unfortunately, it is not β-dense. We can prove, however (and this is the main result of this paper), that it is dense in the (weaker) α topology. To the extent that the α and β topologies are not terribly dissimilar, the result can probably be read as asserting the "bigness" of the set of regular economies.

2. THE MODEL

A. *Notation*

Let X be a topological space and $A, B \subset X$; $\bar{A}$, bdry A, int A, and $A \backslash B$ stand, respectively, for closure, boundary, interior, and set theoretic subtraction. For x, $y \in R^n$, $x \gg y$ means $x^i > y^i$ for all i, $x \geqslant y$ means $x^i \geqslant y^i$ for all i, and $x > y$ means $x \geqslant y$ but not $y \geqslant x$; $R^n_+ = \{x \in R^n : x \geqslant 0\}$. The Euclidean norm in R^n is $\| \|$; $B_\varepsilon(x) = \{y \in R^n : \|y - x\| \leqslant \varepsilon\}$; $|x| = \Sigma_{i=1}^n |x^i|$. The convex hull of $A \subset R^n$ is co A. If f: $U \to R^n$, $U \subset R^m$, is a C^1 function, $Df(x)$, $\nabla f(x)$ are, respectively, the derivative map and the gradient vector at $x \in U$.

B. *Preferences*

The *consumption set* is $P = \{x \in R^l : x \gg 0\}$.

DEFINITION: A preference relation $\gtrsim \subset P \times P$ is a reflexive, transitive, complete preorder on P. A $\gtrsim$ is monotone if $x \geqslant y$ implies $x > y$.

DEFINITION: A function $u : P \to R$ is a utility function for $\gtrsim$ if "$x \gtrsim y \Leftrightarrow u(x) \geqslant u(y)$."

DEFINITION: A monotone preference relation $\gtrsim$ is $C^r (2 \leqslant r \leqslant \infty)$ if:

(1) for all $x \in P$, the closure in R^l of $\{y \in P : y \gtrsim x\}$ is contained in P;

(2) the set $\{(x, y) \in P \times P : x \gtrsim y, y \gtrsim x\}$ is a C^r manifold.

See Figure 1. Our definitions are as in Debreu [5]. A condition equivalent to (2) is the following:

(2′) $\gtrsim$ has a C^r utility function $u : P \to R$ with no critical point, i.e., $\nabla u(x) \neq 0$ for all $x \in P$.[2]

[2] That (2′) implies (2) is clear. To see that (2) implies (2′), suppose first that for every $y \in P$ there is an open neighborhood $y \in U$ and a C^2 function u: $U \to R$ such that $Du(x) \neq 0$ for $x \in U$ and "$x \gtrsim z \Leftrightarrow u(x) \geqslant u(z)$" for $x, z \in U$; via an easy compactness argument reiterated application of the implicit function theorem yields then that the function $x \to \lambda(x)$ implicitly defined by $u(x) = u(\lambda(x)e)$, where $e = (1, \ldots, 1)$, is C^2 and has no critical point; this function is a utility for $\gtrsim$. To verify that (2) implies for every $y \in P$ the existence of an appropriate local utility, let $y \in U$ and h: $U \times U \to R$ be a C^2 function such that $Dh(y, y) \neq 0$ and "$x \gtrsim z$, $z \gtrsim x \Leftrightarrow h(x, z) = 0$" for $x, z \in U$. Note (this is simply seen) that $Dh(y, y)$ has the form $(f(y), -f(y))$; suppose $f^1(y) \neq 0$. Then the function $x \to \lambda(x)$ implicitly defined by $h(\lambda(x), f^2(y), \ldots, f^l(y), x) = 0$ is C^2 in some $U' \subset U (y \in U')$ and $D\lambda(x) \neq 0$ (in particular, $\partial_1 \lambda(x) = 1$) for $x \in U'$.

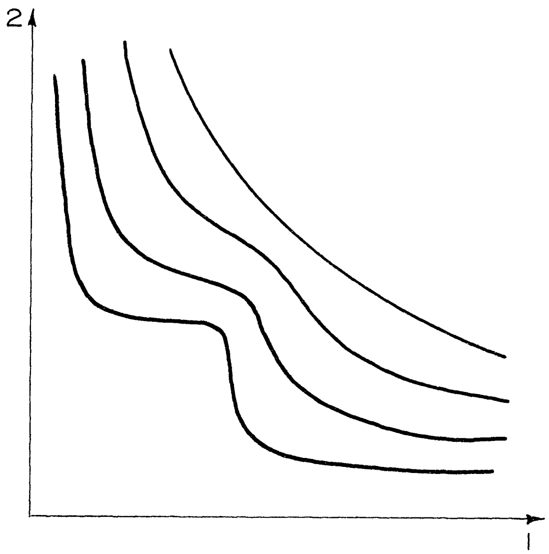

FIGURE 1

The set of C^r monotone preference relations is denoted $\mathcal{P}_s^r$. We are primarily concerned with $r=2$; let $\mathcal{P}_s^2=\mathcal{P}_s$.

A preference relation $\succsim$ is *convex* if $\{y \in P : y \succsim x\}$ is convex for every $x \in P$. It will be convenient to denote by $\mathcal{P}$ the set of continuous (i.e., closed) monotone preference relations on R_+^l (i.e., $\succsim \subset R_+^l \times R_+^l$). In the obvious way, $\mathcal{P}_s$ can be regarded as a subset of $\mathcal{P}$ (extend $\succsim \in \mathcal{P}_s$ to R_+^l by letting "$y \in \partial R_+^l \Rightarrow x \succsim y$ for all x").

Denote by $\mathcal{U}$ the set of C^2 functions $u : P \to R$ with the property that, for all $x \in P$, $\nabla u(x) \gg 0$ and the closure in R_+^l of $\{y \in P : u(y) \geq u(x)\}$ is contained in P. Every $\succsim \in \mathcal{P}_s$ has a utility function in $\mathcal{U}$ and, conversely, every function $u \in \mathcal{U}$ induces a $\succsim_u \in \mathcal{P}_s$ by letting "$x \succsim_u y \Leftrightarrow u(x) \geq u(y)$." From now on, by a utility for a $\succsim \in \mathcal{P}_s$ we always mean a member of $\mathcal{U}$. The Hessian matrix of $u \in \mathcal{U}$ at $x \in P$ is denoted $Hu(x)$.

DEFINITION: A point $x \in P$ is regular for $\succsim \in \mathcal{P}_s$ if, letting u be a utility for $\succsim$, the form $Hu(x)$ restricted to $\operatorname{Ker} Du(x) = \{y \in R^l : Du(x)y = 0\}$ has full rank or, equivalently, if x is a nondegenerate critical point of $u|\operatorname{Ker} Du(x)$.

It is immediately verified that the definition of regular point does not depend on the particular u chosen. At any rate, this is implied by the following well-known fact (see, for example, H. Dierker [**10**, p. 59]). Let $u \in \mathcal{U}$ be a utility for $\succsim \in \mathcal{P}_s$ and, for every $x \in P$, put $x^* = (x^1, \ldots, x^{l-1})$ and take a neighborhood $x^* \subset U \subset R^{l-1}$ small enough to be able to implicitly define a function $\xi[u, u(x)] : U \to R$ by

$u(y, \xi[u, u(x)](y)) = u(x)$. Denoting by $H\xi[u, u(x)](y)$ the Hessian matrix at y, one has:

(3) $x \in P$ is a regular point for $\succsim \in \mathscr{P}_s$ if and only if, letting $u \in \mathscr{U}$ be a utility for $\succsim$, rank $H\xi[u, u(x)](x^*) = l - 1$.

If every $x \in P$ is regular for $\succsim \in \mathscr{P}_s$, we say that $\succsim$ is regular. Observe that *every regular* $\succsim \in \mathscr{P}_s$ *is convex.*

The importance of regular points stems from the following, easily verified, property:

(4) Let $p \in R^l$, $p \gg 0$, $\omega \in P$, and $x \in P$ be regular for $\succsim$. Suppose that, letting u be a utility for $\succsim$, the system $(\nabla u(x) = \lambda p,\ px = p\omega)$ has a solution $\lambda \in R$. Then there are open sets $(p, \omega) \in U_1 \subset R^l \times P$, $x \in U_2$, and a C^1 function $f: U_1 \to U_2$ such that every $(p', \omega', x') \in U_1 \times U_2$ satisfies the above system for some λ', if and only if $x' = f(p', \omega')$.

Let $e = (1, \ldots, 1) \in P$, $\mathscr{U}' = \{u \in \mathscr{U} : u(\lambda e) = \lambda \text{ for all } \lambda > 0\}$. Every $\succsim \in \mathscr{P}_s$ has a unique utility function in $\mathscr{U}'$, denoted $u_{\succsim}$.

For every $\succsim \in \mathscr{P}_s$ define $g_{\succsim} \in P \to S^{l-1}$ by $g_{\succsim}(x) = \nabla u(x)/\|\nabla u(x)\|$ where u is a utility for $\succsim$; this definition is obviously independent of the particular u chosen.

We endow $\mathscr{P}_s$ with the topology induced by the C^1 uniform convergence on compacta of the $g_{\succsim}$'s functions. It has been used by H. Dierker [**11**] and we refer to her for a discussion of its reasonableness. In particular, $\mathscr{P}_s$ becomes a separable, metrizable space.

We remark that, if $\mathscr{U}$ is endowed with the topology of C^r uniform convergence on compacta, the map $u \mapsto \succsim_u$ from $\mathscr{U}$ to $\mathscr{P}_s$ is open and continuous, hence an identification. So, the topology on $\mathscr{P}_s$ is nothing but the identification topology induced by this map (Dugundji [**12**, VI.1]).

C. *Economies and Equilibrium*

The space of *consumer's characteristics* is $\mathscr{A} = \mathscr{P}_s \times P$ (with the product topology); generic elements of $\mathscr{A}$ are $a = (\succsim_a, \omega_a)$.

The price domain is $\Delta = \{p \in R^l : \Sigma_i p^i = 1, p \geq 0\}$; $\Delta^0 = \{p \in \Delta : p \gg 0\}$.

Define the (individual) demand (resp. excess demand) correspondence φ (resp. $\bar{\varphi}$): $\mathscr{A} \times \Delta^0 \to R^l$ by $\varphi(a, p) = \{x \in P : px \leq p\omega_a \text{ and } x \succsim_a y \text{ whenever } py \leq p\omega_a\}$ (resp. $\bar{\varphi}(a, p) = \varphi(a, p) - \omega_a$). It is easily verified that:

(5) $\varphi, \bar{\varphi}$ are compact-valued, upper-hemicontinuous (u.h.c.) correspondences.

As in H. Dierker [**11**] and, in a formally similar context, Delbaen [**8**] and K. Hildenbrand [**16**], we have the following definition:

DEFINITION: An economy is a (Borel) probability measure ν on $\mathscr{A}$ with compact support and such that, denoting by ν_p the marginal distribution of ν on P

and by $i: P \to P$ the identity map, one has $\int i \, d\nu_P \ll \infty$ (bounded mean initial endowments).

As it will become clear in Section 4 on finite economies, the entity we call now an economy would more properly be called "abstract," "infinite," or "atomless" economy. Since we will deal with finite economies only tangentially, no confusion will arise; but it is important for the understanding of the equilibrium notion to be defined to bear in mind that the intended interpretation of an economy ν presupposes a *continuum* of consumers.

Define the *mean demand correspondence* $\varphi_\nu : \Delta^0 \to R^l$ of an economy ν by $\varphi_\nu(p) = \int \text{co}\, \varphi(a, p)\, d\nu(a)$; the mean excess demand $\bar{\varphi}_\nu$ is defined accordingly. We use the same symbols φ, $\bar{\varphi}$ for the corresponding individual and aggregate (mean) concepts; no confusion will arise. For the definition of the integral of a correspondence, see Hildenbrand [**17**, DII].

As in Hildenbrand [**17**, 1.3] one verifies:

(6) The correspondence φ_ν is compact-valued, u.h.c., and satisfies the following boundary condition: $\varphi_\nu(\Delta^0)$ is bounded below and if $p_n \to p \in \partial\Delta$, $p_n \in \Delta^0$, and $x_n \in \varphi_\nu(p_n)$, then $\|x_n\| \to \infty$. The same properties hold for $\bar{\varphi}$.

DEFINITION: $p \in \Delta^0$ is an equilibrium price vector for ν if $0 \in \bar{\varphi}_\nu(p)$.

Denote by $\Pi(\nu)$ the set of equilibrium price vectors for ν. By (6), $\Pi(\nu)$ is a nonempty, compact set (Hildenbrand [**17**, p. 150]).

D. *Space of Economies*

Let $\mathcal{M}$ be the set of economies. We will consider two topologies (the α and β topologies) on $\mathcal{M}$. The α topology is the one mainly used by Hildenbrand [**17**], while both have been utilized by him and by H. Dierker [**11**]. We refer to them for a discussion of their reasonableness. The β topology is stronger, but only "slightly," than the α. For definitions of the mathematical concepts to which we appeal, see Hildenbrand [**17**].

DEFINITION: $\nu^n \underset{\alpha}{\to} \nu$ if ν^n converges to ν weakly and $\int i \, d\nu^n_P \to \int i \, d\nu_P$.

Let $\mathcal{K}$ be the space of nonempty compact subsets of $\mathcal{A}$; endow $\mathcal{K}$ with the topology induced by the Hausdorff distance derived from *any* metric on $\mathcal{A}$ (the topology induced by the Hausdorff distance on the set of nonempty, closed subsets of $\mathcal{A}$ depends on the metric of $\mathcal{A}$, but the relativization to $\mathcal{K}$ depends only on the topology).

DEFINITION: $\nu^n \underset{\beta}{\to} \nu$ if $\nu^n \underset{\alpha}{\to} \nu$ and $\text{supp}\,(\nu^n) \to \text{supp}\,(\nu)$; note that $\text{supp}\,(\nu^n)$, $\text{supp}\,(\nu) \in \mathcal{K}$.

As in Hildenbrand [**17**, 2.2], one shows:

(7) The correspondence $\Pi: \mathcal{M} \to \Delta^0$ is compact-valued and u.h.c. (with the α or β topology on $\mathcal{M}$).

Both the α and β topologies are metrizable and separable.

E. *Finite Economies*

An economy as so far defined is nothing but a model for a *finite, large economy*. As those are not the main focus of the present paper, we shall be somewhat loose in this section.

Let T be a generic symbol for a nonempty, finite indexing set. A *finite economy* is a map $\mathscr{E}: T \to \mathscr{A}$. Two $\mathscr{E}$, $\mathscr{E}'$ are equivalent if $\mathscr{E}$ can be obtained from $\mathscr{E}'$ by an automorphism of T (i.e., the names of the traders do not matter). Every $\mathscr{E}$ induces a distribution $\nu_{\mathscr{E}}$ on $\mathscr{A}$ by $\nu_{\mathscr{E}}(J) = \#\{t : \mathscr{E}(t) \in J\} / \#(T)$. Note that $\mathscr{E}: T \to \mathscr{A}$, $\mathscr{E}': T' \to \mathscr{A}$ are equivalent if and only if $\# T = \# T'$ and $\nu_{\mathscr{E}} = \nu_{\mathscr{E}'}$. A price vector $p \in \Delta^0$ is an equilibrium for $\mathscr{E}: T \to \mathscr{A}$ if $0 \in \Sigma_{t \in T} \bar{\varphi}(\mathscr{E}(t), p)$; the set of equilibrium price vectors is $\Pi(\mathscr{E})$; $\Pi(\mathscr{E})$ is nonempty, compact, and only depends on $\# T$ and $\nu_{\mathscr{E}}$.

Let ν be an economy. In the spirit of Bewley [**2**] we could let a *size n finite representation of* ν be a (random) finite economy $\mathscr{E}_n: T_n \to \mathscr{A}$, with $\# T_n = n$, obtained by (independently) sampling from $\mathscr{A}$ n times according to the distribution ν. By the Glivenko-Cantelli theorem (Hildenbrand [**17**, p. 52]) if $\mathscr{E}_n$ is a sequence of size n representations of ν, then, with probability one, $\nu_{\mathscr{E}_n} \underset{\beta}{\to} \nu$ as $n \to \infty$. Thus, if n is large, $\mathscr{E}_n$ will be almost equal to ν in the three relevant aspects: cardinality (i.e., n) distribution (i.e., $\nu_{\mathscr{E}_n}$), and mean endowments.

F. *Regular Economies*

In this section a certain simple notion of regularity for a ν in $\mathcal{M}$ is proposed. We discuss some of its properties and investigate the relative position of the set of regular economies in $\mathcal{M}$. A more informal discussion of the regularity problem in the present nonconvex preferences setup is deferred until the next section.

DEFINITION: $\bar{\varphi}_\nu$ is C^1 at $p \in \Delta^0$ if $\bar{\varphi}_\nu$ happens to be a C^1 function on a neighborhood of p.

DEFINITION: Let $\bar{\varphi}_\nu: \Delta^0 \to R^l$ be C^1 at every $p \in \Pi(\nu)$. Then $\bar{\varphi}_\nu$ is transversal to 0 if, for every $p \in \Pi(\nu)$, rank $D\bar{\varphi}_\nu(p) = l - 1$.

It is immediately verified that if $\bar{\varphi}_\nu$ is transversal to 0, then $\# \Pi(\nu) < \infty$.

We now state an important condition for a $\nu \in \mathcal{M}$:

(8) For every $a \in \text{supp}(\nu)$ and $p \in \Pi(\nu)$, $\# \varphi(a, p) = 1$ and $\varphi(a, p)$ is a regular point for $\succsim_a$ (see Figure 2).

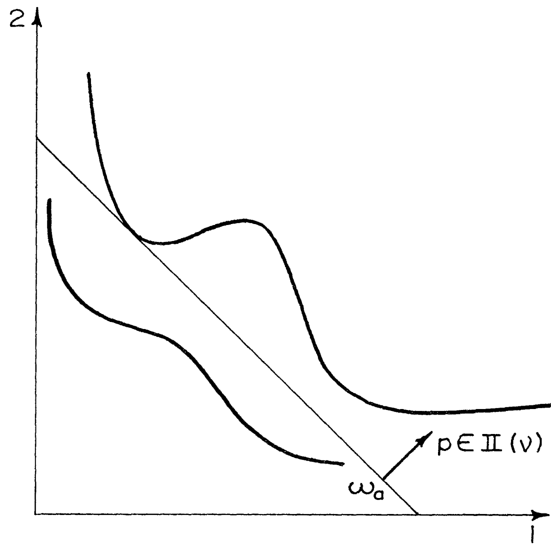

FIGURE 2

If (8) is satisfied for ν, then there is an open set $U \subset \Delta^0$ such that $\Pi(\nu) \subset U$ and for every $a \in \text{supp}(\nu)$, $p \mapsto \bar{\varphi}(a, p)$ is a C^1 function on U (see (4)). Since supp (ν) is compact, $p \mapsto \bar{\varphi}_\nu(p) = \int \text{co}\, \bar{\varphi}(a, p)\, d\nu(a)$ is also C^1 on U; hence, *if* (8) *is satisfied*, $\bar{\varphi}_\nu$ *is* C^1 *at every* $p \in \Pi(\nu)$.

DEFINITION: A $\nu \in \mathcal{M}$ is regular if (8) holds and $\bar{\varphi}_\nu$ is transversal to 0.

Heuristically, we could say that if ν is regular, then in a neighborhood of the (compact) set $J_\nu = \{(a, x, p) \in \text{supp}(\nu) \times P \times \Pi(\nu) : x \in \varphi(a, p)\} \subset \mathcal{A} \times P \times \Delta^0$ everything looks as in a regular economy with convex preferences. Hence, any result for regular, convex economies whose proof uses the convexity property of preferences only on arbitrarily small neighborhoods of J_ν does generalize to regular nonconvex economies. Thus, consider the following two important properties for a $\bar{\nu} \in \mathcal{M}$.

(9) Π is β-continuous at $\bar{\nu}$ and there is a β neighborhood of $\bar{\nu}$, $\bar{\nu} \in U \subset \mathcal{M}$, such that $\# \Pi(\nu)$ is finite and constant on U (see Figure 3).

(10) If $\mathcal{E}_n$ is a sequence of size n representations of $\bar{\nu}$, then with probability 1 there is N such that $\Pi(\mathcal{E}_n) \neq \phi$ for $n > N$.

H. Dierker's contribution [**11**] yields then:

(11) If $\bar{\nu} \in \mathcal{M}$ is regular, then (9) and (10) hold for $\bar{\nu}$.

As another instance, consider the situation in (10). By (11), if $\bar{\nu}$ is regular, then eventually the set of Walras (i.e., competitive) allocations is nonempty and,

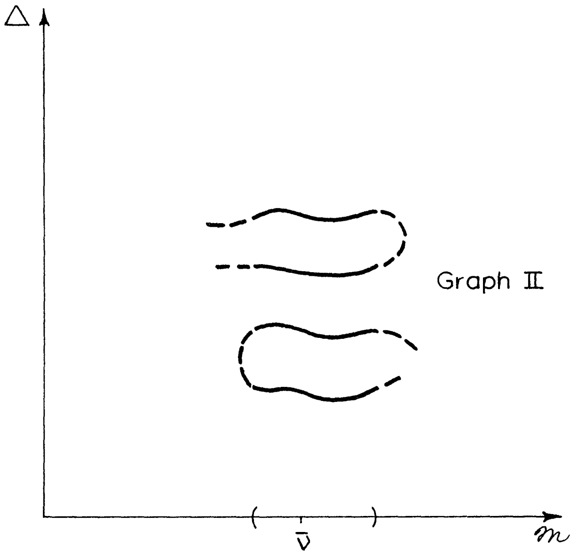

FIGURE 3

therefore, so is the core (see Hildenbrand [**17**, Ch. 2] for definitions of those concepts). As in Debreu [**7**] or Grodal [**14**] one may ask if, as $n \to \infty$, the core "tends" to the set of Walras allocations and, if so, at which rate. The results of Grodal [**14**] generalize to nonconvex economies without modification of the proofs: if $\bar{\nu}$ is regular, then with probability 1, the distance (appropriately defined) between the core and the set of Walras allocations of $\mathscr{E}_n$ (a size n representation of $\bar{\nu}$) goes to zero as the inverse of n.

Let $\mathcal{M}_r \subset \mathcal{M}$ be the set of regular economies. How large is $\mathcal{M}_r$ in $\mathcal{M}$? It is obvious that $\mathcal{M}_r$ is open in the β topology but not in the α one. It is less clear, but it will be substantiated in the Appendix by a simple example, that $\mathcal{M}_r$ is *not* dense in the β topology. Hence, there is no hope of proving that $\mathcal{M}_r$ is open-dense in either the α or β topologies. A somewhat weaker result is true, however:

THEOREM 1: *$\mathcal{M}_r \subset \mathcal{M}$ is α-dense and β-open.*

Let $\mathcal{M}_c = \{\nu \in \mathcal{M}: \succsim_a$ is convex for every $a \in \operatorname{supp} \nu\}$. It has been shown by H. Dierker [**11**] that there is a β-open and dense subset of $\mathcal{M}_c$ such that for every ν in this subset properties (9) and (10) hold; in fact, $\mathcal{M}_r \cap \mathcal{M}_c$ will do. Can this result be generalized to $\mathcal{M}$? With respect to property (9) the answer is no; in the example of the Appendix the set of economies not satisfying (9) contains a β-open set, but even more is true of that example: the β-closure of the set of discontinuity points of Π contains a β-open set. With respect to property (10) the answer is yes. More formally:

THEOREM 2: *There is a β-open and dense set $\mathcal{M}_f \subset \mathcal{M}$ such that for every $\nu \in \mathcal{M}_f$ if $\mathscr{E}_n$ is a finite economy and, as $n \to \infty$, $\nu_{\mathscr{E}_n} \underset{\beta}{\to} \nu$ (which holds with probability* 1 *if $\mathscr{E}_n$ is a size n representation of ν), then there is N such that $\Pi(\mathscr{E}_n) \neq \phi$ for $n > N$.*

It is, incidentally, quite easy to convince oneself that the α-closure of the set of ν which are discontinuity points of Π (hence (9) fails) *and* do not satisfy (10) is the whole $\mathcal{M}$. It is clear that the natural topology on $\mathcal{M}$ from the point of view of the sensitivity analysis of Π is the β.

One should not be misled by Theorem 2. Nonregular economies in $\mathcal{M}_f$ may not be very interesting. In particular, it cannot be presumed that if $\mathscr{E}_n$ is a sequence of finite economies and $\nu_{\mathscr{E}_n} \underset{\beta}{\to} \nu \in \mathcal{M}_f$ (which implies $\Pi(\mathscr{E}_n) \neq \phi$, eventually), then $\Pi(\mathscr{E}_n)$ converges as a set to $\Pi(\nu)$; $\Pi(\mathscr{E}_n)$ may "explode" in the limit, i.e., $Ls\Pi(\mathscr{E}_n) \subset \Pi(\nu)$.

G. *A Remark on the Regularity Problem*

Confronted with the nongeneralizability of property (9) from $\mathcal{M}_c$ to $\mathcal{M}$ there are two possible avenues along which to proceed:

(i) to isolate some interesting and strong notion of regularity, examine its properties and see how far one can go in approaching the open and dense desideratum; this is the program pursued in this paper;

(ii) to conclude that we cannot "make up" with smoothness hypothesis (local by nature) for the nonconvexities of the problem (with its implied abrupt, discontinuous in the large, agents' behavior), drop them and settle for obtaining the strongest determinateness-of-equilibria property one can hope for in a framework of continuity assumptions, namely, the *existence of a dense set of economies such that every economy in the set is a continuity point of the equilibrium correspondence and has a finite number of equilibria.* This continuity-framework analysis of the equilibrium correspondence shall be pursued elsewhere [**23**] in a context where commodities are available only in discrete (i.e., indivisible) amounts. There is gain, and no loss, of conceptual generality in doing so; the hypothesis of perfect divisibility for commodities is of no substance in itself. It is rather a prerequisite for the exploitation of the powerful analytical tools of convexity and differentiability theory, if neither convexity nor smoothness are present, we can do as well without divisibility of commodities.

3. PROOF OF THE THEOREMS

The β-openness of $\mathcal{M}_r$ is clear enough; we shall skip a formal proof.

Let $\mathcal{M}^* = \{\nu \in \mathcal{M}$: if $a \in \text{supp}(\nu)$, then $\omega_a \leq e$ and $\succsim_a$ is regular, convex$\}$. Observe that if $\nu, \nu' \in \mathcal{M}$ and $0 \leq t \leq 1$, then $t\nu + (1-t)\nu' \in \mathcal{M}$. Moreover, if $\nu'_n \in \mathcal{M}^*$, $t_n \to 1$ and we let $\nu_n = t_n\nu + (1-t_n)\nu'_n$, then $\nu_n \underset{\alpha}{\to} \nu$, and if $\bar{\varphi}_\nu$ is C^1 at p, then so is $\bar{\varphi}_{\nu_n} = t_n\bar{\varphi}_\nu + (1-t_n)\bar{\varphi}_{\nu'_n}$.

Denote $\mathscr{A}^\infty = \mathscr{P}_s^\infty \times P \subset \mathscr{A}$. We state some conditions for a $\nu \in \mathscr{M}$.

(12) $\# \operatorname{supp}(\nu) < \infty$ and $\operatorname{supp}(\nu) \subset \mathscr{A}^\infty$;

(13) $\# \operatorname{supp}(\nu) < \infty$; $\# \Pi(\nu) < \infty$; for every $p \in \Pi(\nu)$, $a \in \operatorname{supp}(\nu)$ and $x \in \varphi(a, p)$, x is a regular point for $\succsim_a$;

(14) $\bar{\varphi}_\nu$ is C^1 at every $p \in \Pi(\nu)$; for every $p \in \Pi(\nu)$, $a \in \operatorname{supp}(\nu)$, and $x \in \varphi(a, p)$, x is a regular point for $\succsim_a$;

(15) (14) holds and $\bar{\varphi}_\nu$ is transversal to 0.

The proof of the α-density of $\mathscr{M}_r$ is organized in the following steps:

STEP 1: If $\nu \in \mathscr{M}$, then $\nu_n \underset{\beta}{\to} \nu$ where ν_n satisfies (12).

STEP 2: If ν satisfies (12), then $\nu_n \underset{\beta}{\to} \nu$ where ν_n satisfies (13).

STEP 3: If ν satisfies (13), then $\nu_n \underset{\beta}{\to} \nu$ where ν_n satisfies (14).

STEP 4: If $\bar{\varphi}_\nu$ is C^1 at every $p \in \Pi(\nu)$, then for every $0 < \varepsilon < 1$ there is $\nu_\varepsilon \in \mathscr{M}^*$ such that, putting $\nu'_\varepsilon = \varepsilon \nu_\varepsilon + (1-\varepsilon)\nu$, $\bar{\varphi}_{\nu'_\varepsilon}$ is transversal to 0.

STEP 5: If ν satisfies (15), then $\nu_n \underset{\alpha}{\to} \nu$ where ν_n satisfies (8).

Observe that in Step 4 if ν satisfies (14) and ε is close enough to 0, ν'_ε will satisfy (15) (take into account (7) and the fact $\operatorname{supp}(\nu'_\varepsilon) \backslash \operatorname{supp}(\nu_\varepsilon) \subset \operatorname{supp}(\nu)$). Therefore, following Steps 1 through 5 we end up with a ν satisfying (8); a new application of Step 4 will yield a regular ν.

The approximation embodied in Step 4 is in the α sense, but a more refined argument could replace this step by an equivalent β approximation. The only place where α rather than β is essential is in Step 5.

We skip the proofs of Step 1, which is a well-known fact, and Step 4, which follows immediately applying the perturbation of initial endowments argument of E. and H. Dierker [**10**, p. 872]. The proof of Step 2 is a transversality one (i.e., counting of equations and unknowns). Step 3 is, essentially, a "smoothing by aggregation" (see, for example, Sondermann [**22**]) result restricted to a neighborhood of $\Pi(\nu)$. Step 5 appeals to the recent literature on the characterization of excess demand functions (specifically, to the strengthening in [**22**] of a result of Debreu [**6**]; see also, Mantel [**20**]); the reason why this line of work turns out to be relevant in the present context can be, informally, easily understood. After Step 4 the set of "troublesome" $a \in \operatorname{supp}(\nu)$ is very small; we replace it by another set of consumers having smooth, convex preferences and, as a group, nearly the same excess demand correspondence; clearly the resulting economy approximates in the α sense (but not in the β sense).

From now on, $k(a, x) \neq 0$ is a short-hand for "x is a regular point of $\succsim_a$."

In order to prove Theorem 2, let $\mathscr{M}_f = \{\nu \in \mathscr{M}$: for some $\bar{p} \in \Pi(\nu)$ and all $a \in \operatorname{supp}(\nu)$, $\# \varphi(a, \bar{p}) = 1$ and $k(a, \varphi(a, \bar{p})) \neq 0$. Moreover, rank $D\bar{\varphi}_\nu(\bar{p}) = l - 1\}$. It is clear that $\mathscr{M}_f$ is β-open. In the same manner as H. Dierker [**11**], one verifies that every $\nu \in \mathscr{M}_f$ satisfies the conclusion of Theorem 2. It remains to show that $\mathscr{M}_f$ is β-dense. This will be obtained in due course as an easy consequence of Step 2.

PROOF OF STEP 2: Let ν satisfy (12). Put supp $(\nu)=\{\bar{a}_1, \ldots, \bar{a}_m\}$, $\alpha_j=\nu(\bar{a}_j)$. For this step we change our notation somewhat. The symbol a will denote an element of $(\mathcal{U}^\infty \times P)^m$, i.e., $a=(a_1, \ldots, a_j, \ldots, a_m)$. Every a defines an economy ν_a by the rule $\nu_a(a_j)=\alpha_j$. Without danger of confusion we simply identify a with ν_a and write $\Pi(a)$ for $\Pi(\nu_a)$, etc.

Let $\mathcal{B}(P)$ be the set of compact balls in P with positive radius. For every integer $r_j \geqslant 0$ let $T_{r_j}=\{(x_0, \ldots, x_h, \ldots, x_{r_j}) \in R^{r_j+1}$: $x_h \neq x_{h'}$ if $h \neq h'\}$ and, analogously, $\mathcal{S}_{r_j}=\{(L_{j0}, \ldots, L_{jr_j}) \in (\mathcal{B}(P))^{r_j+1}$: $L_{jh} \cap L_{jh'}=\phi$ if $h \neq h'\}$. For every ordered collection $r=(r_1, \ldots, r_m)$ of integers $r_j \geqslant 0$, let $T_r=\Sigma_{j=1}^m T_{r_j}$, $\mathcal{S}_r=\Pi_{j=1}^m \mathcal{S}_{r_j}$, and $\bar{r}=\Sigma_{j=1}^m r_j$; x (resp. L_j, L) is a generic element of T_r (resp. $\mathcal{S}_{r_j}$, $\mathcal{S}_r$). The n-dimensional simplex in R^{n+1} *is* Δ^n, $\hat{\Delta}^n=\{q \in R^n : \Sigma_{i=1}^n q^i=0\}$.

The space of $(l-1)\times(l-1)$ symmetric matrices, regarded as a submanifold of $R^{(l-1)^2}$, is denoted $\mathcal{G}$. For every $0 \leqslant k \leqslant l-1$, $\mathcal{G}_k$ is the set of elements of $\mathcal{G}$ having rank k; $\mathcal{G}_k$ is a C^∞ submanifold of $\mathcal{G}$ of codimension $\frac{1}{2}(l-1-k)(l-k)$ (Golubitsky and Guillemin [**13**, p. 153]).

For every r, $L \subset \mathcal{S}_r$, and $a \in (\mathcal{U}^\infty \times P)^m$ define a map $\Psi_{L,a}: L \times \Delta^{l-1} \times \Pi_{j=1}^m \Delta^{r_j} \to \Pi_{j=1}^m (R^{r_j} \times (\hat{\Delta}^{l-1} \times R \times \mathcal{G})^{r_j+1}) \times R^{l-1}$ by:[3]

$$(x, p, t) \mapsto$$

$$u_j(x_{j0})-u_j(x_{jh}), \quad 1 \leqslant h \leqslant r_j,$$

$$\left(\frac{1}{|\nabla u_j(x_{jh})|}\right)\nabla u_j(x_{jh})-p,$$

$$p(x_{jh}-\omega_j), \quad H\xi(u_j, u_j(x_{jh}))(x_{jh}^*), \quad 0 \leqslant h \leqslant r_j, \quad 1 \leqslant j \leqslant m;$$

$$\sum_{j=1}^m \alpha_j\left(\sum_{h=0}^{r_j} t_{jh} x_{jh}^i-\omega_j^i\right), \quad 1 \leqslant i \leqslant l-1.$$

Given an r, let κ denote a generic element of $\Pi_{j=1}^m \{0, 1, \ldots, l-1\}^{r_j+1}$ and $J(\kappa)=\Pi_{j=1}^m (\{0\} \times \Pi_{h=0}^{r_j} (\{0\} \times \{0\} \times \mathcal{G}_{k_{jh}})) \times \{0\}$, where k_{jh} is the generic entry of κ.

PROPOSITION: *For every r and $x \in T_r$ there is $L \in \mathcal{S}_r$ with $x \in L$ such that $\mathcal{A}_L=\{a \in (\mathcal{U} \times P)^m$: $\Psi_{L,a} \pitchfork J(\kappa)$ for every $\kappa\}$ is a Baire set.*

Suppose, for the moment, that the proposition is true. Then for every r there is a countable collection $\mathcal{L}_r$ of $L \subset \mathcal{S}_r$ whose interiors cover T_r. The set $\bigcap_r \bigcap_{L \in \mathcal{L}_r} \mathcal{A}_L$ is dense in $(\mathcal{U}^\infty \times P)^m$. We claim that every $a \in \bigcap_r \bigcap_{L \in \mathcal{L}_r} \mathcal{A}_L$ satisfies (13).

Note that, for all r, L, and κ, dimension domain $\Psi_{L,a}=(l-1)+l+\bar{r}(l+1) \leqslant$ codimension of $J(\kappa)$, equality holding if and only if $k_{jh}=l-1$ for all jh. Therefore, $\{(x, p, t)$: $\Psi_{L,a}(x, p, t) \in J(\kappa)\}$ is a discrete (resp. empty) set if $k_{jh}=l-1$ for all jh (resp. $k_{jh}<l-1$ for some jh).

Let $\Pi_1(a)=\{p \in \Pi(a)$: for all j and $z \in \varphi(a_j, p)$, $k(z, a_j) \neq 0\}$. By taking into account (Caratheodory's theorem) that every point of co $\varphi(a_j, p)$ can be written as

[3] We let $a=(u, \omega)$. If $r_j=0$, then $R^{r_j}=\{0\}$ and we convene that the value of the R^{r_j} coordinate of $\Psi_{L,a}$ is automatically defined to be 0. Hence, to give the formal definition of $\Psi_{L,a}$ we proceed as if $r_j \neq 0$; although, of course, this will not in general be the case.

a convex combination of, at most, l points of $\varphi(a_j, p)$, one easily verifies that if $p \in \Pi(a) \backslash \Pi_1(a)$, then $\Psi_{L,a}(x, p, t) \in J(\bar{\kappa})$ for some $r, L \in \mathcal{L}_r, x \in L, t \in \Pi_j \Delta^{r_j+1}$, and $\bar{\kappa}$ with $k_{jh} < l-1$ for some jh. But this contradicts the previous paragraph. Hence, $\Pi_1(a) = \Pi(a)$.

Let $\bar{p} \in \Pi(a)$. Since $\bar{p} \gg 0$ and $k(z, a_j) \neq 0$ for all j and $z \in \varphi(a_j, \bar{p})$, there is an r, a neighborhood of $\bar{p}$, $\bar{p} \in U \subset \Delta^{l-1}$, and a finite collection of C^∞ functions $f_{jh}: U \to P$, $0 \leq h \leq r_j$, $1 \leq j \leq l-1$, such that, for all j and $p \in U$, $\varphi(a_j, p) \subset \bigcup_{h=0}^{r_j} \{f_{jh}(p)\}$ and $f_{jh}(p) \neq f_{jh'}(p)$ if $h \neq h'$. Suppose that $p_n \to \bar{p}$, $p_n \in \Pi(a)$; then, letting $x_{jhn} = f_{jh}(p_n)$ we can find (extracting a subsequence if necessary) $t_n \in \Pi_j \Delta^{r_j+1}$, $t_n \to t$ with $\Sigma_j \alpha_j(\Sigma_{h=0}^{r_j} t_{jhn} x_{jhn} - \omega_j) = 0$; so, if we put $x_{jh} = f_{jh}(\bar{p})$ and let x be in the interior of $L \in \mathcal{L}_r$ we can assume that for all n, $\Psi_{L,a}(x_n, p_n, t_n) \in J(\kappa)$ where $k_{jh} = l-1$ for all jh. Since this equation has a discrete set solution, the sequence p_n is trivial (i.e., $p_n = \bar{p}$ for all but finitely many n); hence, being $\Pi(a)$ compact, we conclude $\# \Pi(a) < \infty$.

We now proceed to prove the proposition. We only show that $\mathcal{A}_L$ is dense; that it is in fact a Baire set follows by noting that every $\mathcal{A}_k$ is σ-compact and $\Psi_{L,a}$ has a compact domain.

Pick an arbitrary $\bar{a} = (\bar{u}, \bar{\omega}) \in (\mathcal{U}^\infty \times P)^m$ and let r, $\bar{x} \in T_r$ be given. Pick any $L \in \mathcal{S}_r$ with $\bar{x} \in L$.

Write any $y \in R^l$ in the form $y = (y^*, y^l)$, where $y^* = (y^1, \ldots, y^{l-1})$. For any jh and $z \equiv (q, A) \in R^{l-1} \times \mathcal{G}$ define $\eta_z^{jh}: P \to R^l$ by $(y^*, y^l) \mapsto (y^*, y^l - (y^* - \bar{x}_{jh}^*)q - (y^* - \bar{x}_{jh}^*)A(y^* - \bar{x}_{jh}^*))$.

For any jh let $U_{jh} \subset R^{l-1} \times \mathcal{G}$ be a neighborhood of zero sufficiently small for all the subsequent definitions to make sense.

For every jh define $F_{jh}: L_{jh} \times \bar{U}_{jh} \times \Delta^{l-1} \times P \to \hat{\Delta}^{l-1} \times R \times \mathcal{G}$ by:

$$(y, z, p, \omega) \mapsto \Big(\Big(\frac{1}{|\nabla \bar{u}_j(\eta_z^{jh}(y))|} \Big) \nabla \bar{u}_j(\eta_z^{jh}(y)) - p, \\ p(y-\omega), H\xi[u_j \circ \eta_z^{jh}, u_j(\eta_z^{jh}(y))](y^*) \Big).$$

LEMMA 1: *For every jh and $p \in \Delta^{l-1}$, $D_{y,z}F_{jh}(\bar{x}_{jh}, 0, p, \bar{\omega}_{jh})$ is surjective.*

PROOF: Drop the subindexes jh and let F_1, F_2, F_3 be the compositions of F with the projections on $\hat{\Delta}^{l-1}$, R, $\mathcal{G}$, respectively. Obviously, $D_z F^2(\bar{x}, 0, p, \bar{\omega}) \equiv 0$ and $D_y F^2(\bar{x}, 0, p, \bar{\omega})$ is surjective. So, it suffices to show that $D_z(F^1, F^3)(\bar{x}, 0, p, \bar{\omega})$ is onto. Putting $z = (q, A)$, easy manipulations and computations yield $F^1(\bar{x}, z, p, \bar{\omega}) = (1/|\nabla \bar{u}(\bar{x}) - (q, 0)\, \partial_l \bar{u}(\bar{x})|)(\nabla \bar{u}(\bar{x}) - (q, 0)\, \partial_l \bar{u}(\bar{x})) - p$ and $F^3(\bar{x}, z, p, \bar{\omega}) = H\xi[\bar{u}, \bar{u}(\bar{x})](\bar{x}^*) + A$. But then $D_q F^1$ is surjective (this is easily checked), $D_A F^1 \equiv 0$, $D_A F^3$ is surjective (in fact, the identity map), and $D_q F^3 \equiv 0$. The desired conclusion follows.

Now define

$$\Phi: L \times \Delta^{l-1} \times \prod_j \Delta^{r_j} \times \prod_{j,h} \bar{U}_{jh} \times \prod_j [-1, 1]^{r_j+1} \times P^m \\ \to \prod_{j=1}^m (R^{r_j} \times (\hat{\Delta}^{l-1} \times R \times \mathcal{G})^{r_j+1}) \times R^{l-1}\,{}^{4}$$

[4] With respect to the factor R^{r_j} the same conventions as in footnote 3 apply.

by

$$(y, p, t, z, v, \omega) \mapsto u_j(\eta^{j0}_{z_{j0}}(y_{j0}) + v_{j0} - u_j(\eta^{jh}_{z_{jh}}(y_{jh})) - v_{jh}, \quad 1 \leq h \leq r_j,$$

$$F_{jh}(y_{jh}, z_{jh}, p, \omega_{jh}), \quad 0 \leq h \leq r_j, \quad 1 \leq j \leq m;$$

$$\sum_{j=1}^{m} \alpha_j \Big(\sum_{h=0}^{r_j} t_{jh} y^i_{jh} - \omega^i_{jh} \Big), \quad 1 \leq i \leq l-1.$$

It is obvious from the lemma that for every $p, t \in \Delta^{l-1} \times \Pi_j \Delta^{r_j}$, $D\Phi(\bar{x}, p, t, 0, 0, \bar{\omega})$ is surjective. Since the domain of p, t is compact, there is $L' \subset L$ such that $\bar{x} \in L'$ and $D\Phi(x, p, t, 0, 0, \bar{\omega})$ is onto for all p, t and $x \in L'$. By the Transversality Theorem (see Guillemin and Pollack [**15**, p. 68]) there is $(\bar{z}, \bar{v}, \bar{\bar{\omega}}) \in \Pi_{jh} U_{jh} \times \Pi_j [-1, 1]^{r_j+1} \times P^m$ arbitrarily close to $(0, 0, \omega)$ and such that $\Phi_{\bar{z},\bar{v},\bar{\bar{\omega}}} \pitchfork J(\kappa)$ for all κ.

For every jh let $u_{jh}: L'_{jh} \to R$ be given by $y \mapsto u_j(\eta^{jh}_{z_{jh}}(y)) + \bar{v}_{jh}$. Take $L'' \subset L'$ with $\bar{x} \in L'' \subset \operatorname{int} L'$ and, for every j, let $\beta_j: P \to [0, 1]$ be C^∞ and equal to 1 on L''_j and to 0 on a neighborhood of $P \backslash L'_j$. Define $u_j: P \to R$ by $u_j(y) = \sum_{h=0}^{r_j} \beta_j(y) u_{jh}(y) + (1 - \beta_j(y)) \bar{u}_j(y)$. Then $a = (u, \bar{\bar{\omega}}) \in (\mathscr{U}^\infty \times P)^m$ is arbitrarily close to $\bar{a}$ and $\Psi_{L'',a} = \Phi_{\bar{z},\bar{v}}$.

Q.E.D.

PROOF OF STEP 3: Let ν satisfy (13). Denote $\operatorname{supp} \nu = \{a_1, \ldots, a_m\}$; without possible confusion we regard a_j as a member of $(\mathscr{U} \times P)^m$, i.e., $a_j = (u_j, \omega_j)$, $\nu = \sum_{j=1}^m \alpha_j \delta_{a_j}$; $\Pi(\nu) = \{p_1, \ldots, p_n\}$.

Because of (4) there is $\mu > 0$ and, for every $1 \leq i \leq n$, an open set $p_i \in U_i \subset \Delta^{l-1}$ such that for any $1 \leq j \leq m$ and $x \in \varphi(a_j, p_i)$ there is a C^1 function $f_{ijx}: \bar{U}_i \to B_\mu(x)$ with the property that $k(a_j, y) \neq 0$ for any $y \in B_\mu(x)$ and "$p \in U_i$, $y \in B_\mu(x)$, $(1/|\nabla u_j(y)|) \nabla u_j(y) = p$, $py = p\omega_j$" if and only if $y = f_{ijx}(p)$.

We assume that, for all j, the sets $\{B_\mu(x): x \in \varphi(a_j, p), p \in \Pi(\nu)\}$ are pairwise disjoint; let $J_{i,j} = \bigcup_{x \in \varphi(a_j, p_i)} B_\mu(x)$, $J_j = \bigcup_{i=1}^n J_{ij}$. We assume that $i \neq i'$ implies $U_i \cap U_{i'} = \phi$ and that, for all $p \in \bar{U}_i$ and j, $\varphi(a_j, p) \subset \operatorname{int} J_j$. Put $U = \bigcup_{i=1}^n U_i$.

We shall proceed by replacing every δ_{a_j} by a ν_j with the property that the compact set $\operatorname{supp}(\nu_j)$ is contained in a neighborhood of a_j as small as we wish and φ_{ν_j} is a C^1 function on $\bar{U}$. This quite suffices since, letting $\nu' = \sum_{j=1}^m \alpha_j \nu_j$, $\varphi_{\nu'}$ will be C^1 on $\bar{U}$ and, being ν' as close as we wish to ν, we can assume that (i) $\Pi(\nu') \subset U$, (ii) $k(a, y) \neq 0$ for all j, $a \in \operatorname{supp} \nu_j$ and $y \in J_j$, and (iii) $\varphi(a, p) \in J_j$ for all j, $a \in \operatorname{supp}(\nu_j)$ and $p \in \Pi(\nu')$.

From now on we consider a single fixed j. For notational economy we drop, when there is no ambiguity, the subscript j.

For every $1 \leq i \leq n$ let $\varphi(a, p_i) = \{x_{i1}, \ldots, x_{is_i}\}$, $B_{i,h} = B_\mu(x_{ih})$, $f_{ih} = f_{ijx_{ih}}$. Of course, $\varphi(a, p) \subset \bigcup_{h=1}^{s_i} \{f_{ih}(p)\}$ for every $p \in \bar{U}_i$.

Let $E_{ih} \subset P$, $1 \leq i \leq n$, $1 \leq h \leq s_i$, be a collection of pairwise disjoint relatively compact neighborhoods of the B_{ih}'s. Let $\xi_{ih}: P \to [0, 1]$ be C^∞ and such that $\xi(x) = 1$ for $x \in B_{ih}$ and $\xi(x) = 0$ for $x \in P \backslash E_{ih}$.

For every $v \in \Pi_{i=1}^{n} R^{s_i}$ define $u_v: P \to R$ to be $u_v = u + \Sigma_{i,h} v_{ih}\xi_{ih}$. For $\gamma > 0$ sufficiently small if $v \in [-\gamma, \gamma]^{\Sigma_i s_i} \equiv T$, then $u_v \in \mathcal{U}$; moreover, the function $v \mapsto u_v$ from T to $\mathcal{U}$ is continuous. For every $v \in T$ we define a_v in the obvious way, i.e., $\omega_{a_v} = \omega_a$ and $\succsim_{a_v}$ is the preference relation represented by u_v. The function $v \mapsto a_v$ from T to $\mathcal{P}_s \times P$, denoted V, is continuous.

If $\gamma > 0$ is small enough, any a_v, $v \in T$, is arbitrarily close to a; in particular, we can assume that for any i, $p \in \bar{U}_i$ and $v \in T$ one has $\varphi(a_v, p) \subset \bigcup_{h=1}^{s_i} B_{ih}$. By noting, then, that $u_v|B_{ih} = u|B_{ih} + v_{ih}$, we conclude $\varphi(a_v, p) \subset \bigcup_{h=1}^{s_i} \{f_{ih}(p)\}$ for any i, $p \in \bar{U}_i$ and $v \in T$.

Let $F: R \to [0, 1]$ be an arbitrary C^∞ cumulative distribution function with $F(-\gamma) = 0$, $F(\gamma) = 1$. Denote by μ the *product measure* induced by F on $R^{\Sigma_{i=1}^{m} s_i}$; of course, supp $(\mu) \subset T$.

Define $\nu_j = \mu \circ V^{-1}$; clearly supp (ν_j) is compact since supp $(\nu_j) \subset V(\text{supp}\, \mu)$ and V is continuous. By taking γ small enough, ν_j is arbitrarily close to δ_{a_j}.

It remains to prove that φ_{ν_j} is C^1 on every U_i.

Let $p \in \bar{U}_i$ and $h \neq h'$; then $\mu\{v \in T: u_v(f_{ih}(p)) = u_v(f_{ih'}(p))\} = 0$ because $\{v \in T: u_v(f_{ih}(p)) = u_v(f_{ih'}(p))\} = \{v \in T: u(f_{ih'}(p)) - u(f_{ih}(p)) = v_{ih} - v_{ih'}\}$ and this last set, which is the intersection of T with an hyperplane, has Lebesgue, hence μ, measure zero. Therefore, given $p \in \bar{U}_i$, $\varphi(a_v, p)$ will be a singleton for μ-a.e. $v \in T$ and so, $\varphi(\nu_j, p) = \int \varphi(a, p)\, d\nu_j(a) = \int \varphi(a_v, p)\, d\mu(v)$ will be a singleton for all $p \in \bar{U}_i$.

For $p \in \bar{U}_i$ and $1 \leq h \leq s_i$ let $A_h(p) = \{v \in T: \varphi(a_v, p) = f_{ih}(p)\}$. Then $\varphi(\nu_j, p) = \Sigma_h^{s_i} \mu(A_h(p)) f_{ih}(p)$ and so, φ_{ν_j} will be C^1 on $\bar{U}_i$ if, for every h, $p \mapsto \mu\ (A_h(p))$ is C^1 on $\bar{U}_j$. Observe that $\mu(A_h(p)) = \mu\{v \in T: u_v(f_{ih}(p)) \geq u_v(f_{ih'}(p))$ for all $h' \neq h\}$. For $p \in \bar{U}_i$, $w \in [-\gamma, \gamma]$ and $h' \neq h$ let $\eta_{h'}(p, w) = u(f_{ih}(p)) - u(f_{ih'}(p)) + w$; $\eta_{h'}$ is C^1 on $U_i \times T$.

An immediate computation yields $\mu(A_h(p)) = \int_{-\gamma}^{\gamma} \Pi_{h' \neq h} F(\eta_{h'}(p, w))\, dw$ and so, $p \mapsto \mu(A_h(p))$ is C^1 on $\bar{U}_1$.

Q.E.D.

PROOF OF STEP 5: Let $\hat{\nu}$ be the given economy satisfying (15). Denote supp $(\hat{\nu}) = K$.

It will be convenient in this section to take as the price domain $S = \{p \in R^l_{++}: \|p\| = 1, p^i > 0\}$; $S_\varepsilon = \{p \in S: p^i \geq \varepsilon\}$.

Define a vector field on S by $g(p) = e - (pe)p$, where $e = (1, \ldots, 1)$.

LEMMA 2: *There are $\xi > 0$ and $\varepsilon > 0$ such that for every $\delta > 0$ and $\nu \in \mathcal{M}$ with supp $(\nu) \subset K$ one has $\bar{\varphi}_\nu(p)g(p) > 0$ for every $p \in S \backslash S_\varepsilon$ and there exists a ν' such that: (i) for every $a \in$ supp (ν'), $\|\omega_a\| < \xi$ and $\succsim_a$ is C^∞, convex, and regular; (ii) $\bar{\varphi}_{\nu'}(p)g(p) > 0$ for every $p \in S \backslash S_\varepsilon$; (iii) graph $(\bar{\varphi}_{\nu'}|S_\varepsilon) \subset B_\delta(\bar{\varphi}_\nu|S_\varepsilon)$.*

PROOF: In **[22]** it is proved, as an extension of results of Debreu **[5]**: *Let $h: S \to R$ be a continuous function and $\varepsilon > 0$ be given; then there is $\xi > 0$ such that for every continuous function $f: S \to R^l$ satisfying (i) $pf(p) = 0$ for all $p \in S$, (ii) $\|f(p)\| < h(p)$ for $p \in S$, (iii) $f(p)g(p) > 0$ for $p \in S \backslash S_\varepsilon$, there is $\{(\succsim_i, \omega_i)\}_{i=1}^{l}$ with the properties*

(i) for all i, $\|\omega_i\| < \xi$ *and* $\succsim_i \subset R^l_+ \times R^l_+$ *is continuous, convex, and monotone,*
(ii) $\sum_{i=1}^l \bar{\varphi}(\succsim_i, \omega_i, p) = f(p)$ *for* $p \in S_\varepsilon$ *and* $(\sum_{i=1}^l \bar{\varphi}(\succsim_i, \omega_i, p))g(p) > 0$ *for* $p \in S \backslash S_\varepsilon$.
The lemma follows by combining this with the next four facts.

Fact 1: Given a compact $H \subset \mathscr{P}_s$ there is $\varepsilon > 0$ such that, for every ν with supp $(\nu) \subset H$, $\bar{\varphi}_\nu(p)g(p) > \varepsilon$ for $p \in S \backslash S_\varepsilon$ (apply Hildenbrand [**17**, Prop. 6, p. 119]; take into account that H is closed convergence compact and that the set of $\nu \in \mathscr{M}$ with supp $(\nu) \subset H$ is compact).

Fact 2: For every ε, $\mu > 0$ and ν with supp $(\nu) \subset K$ there is a continuous function $f: S \to R$ such that $pf(p) = 0$ for all p, graph $(f|S_\varepsilon) \subset B_\mu$(graph $(\bar{\varphi}_\nu|S_\varepsilon)$) and $f(p)g(p) > 0$ for $p \in S \backslash S_\varepsilon$ (combine Fact 1 with the well-known approximability on their graphs of u.h.c. convex-valued correspondences by continuous functions; see Cellina [**3**]).

Fact 3: Any continuous, convex, monotone preference relation $\succsim \subset R^l_+ \times R^l_+$ can be approximated in closed convergence by a $\succsim^* \in \mathscr{P}_s$ which is convex and regular (see Mas-Colell [**21**, Theorem 2 and Remark 5]).

Fact 4: The mean-excess demand correspondence $\bar{\varphi}_\nu(p)$ is u.h.c. in the domain $\mathscr{M} \times S$ (Hildenbrand [**17**, p. 117]).

Since $k(a, x) \neq 0$ for every $p \in \Pi(\hat{\nu})$, $a \in K$ and $x \in \varphi(a, p)$, we have that $A_p = \{a \in K;\ \#\varphi(a, p) > 1\}$ is compact (there is $\delta > 0$ such that if $x, x' \in \varphi(a, p)$ and $x \neq x'$, then $\|x - x'\| > \delta$). Being $\Pi(\hat{\nu})$ finite and $\varphi_{\hat{\nu}}(p)$ a singleton at every $p \in \Pi(\hat{\nu})$ the set $A = \bigcup_{p \in \Pi(\hat{\nu})} A_p$ is compact and $\hat{\nu}(A) = 0$.

Let $U_n \subset U_{n+1} \subset \ldots \subset \mathscr{P}_s \times P$ be a sequence of open sets with $A = \bigcap_n U_n$. Then $\hat{\nu}(U_n) \to 0$ and $K \backslash U_n$ is compact. If $\hat{\nu}(U_n) = 0$ for some n we are done. From now on we assume $\hat{\nu}(U_n) > 0$ for all n. For every n, there is an open set $\mathscr{O}_n \subset S$ such that $\Pi(\hat{\nu}) \subset \mathscr{O}_n$ and if $p \in \mathscr{O}_n$, $a \in K \backslash U_n$, then (a, p) is a singleton and $k(a, \varphi(a, p)) \neq 0$.

Let $\varepsilon > 0$ and $\xi > 0$ be as in Lemma 2. For a $\delta_n > 0$ to be specified later let ν' be the economy given by the lemma with respect to $\nu \equiv (1/\hat{\nu}(U_n))\hat{\nu}|U_n$. Let $\hat{\hat{\nu}} = (1/\hat{\nu}(K \backslash U_n))\hat{\nu}|K \backslash U_n$. For $p \in S \backslash S_\varepsilon$, $\hat{\nu}(U_n)\bar{\varphi}_{\nu'}(p) + \hat{\nu}(K \backslash U_n)\bar{\varphi}_{\hat{\hat{\nu}}}(p) \neq 0$, since $\bar{\varphi}_{\nu'}(p)g(p) > 0$ and $\bar{\varphi}_{\hat{\hat{\nu}}}(p)g(p) > 0$. But then if δ_n is sufficiently small, $\{p \in S: \hat{\nu}(U_n)\bar{\varphi}_{\nu'}(p) + \hat{\nu}(K \backslash U_n)\bar{\varphi}_{\hat{\hat{\nu}}}(p) = 0\} \subset \mathscr{O}_n$. With this Step 5 is in fact finished; let $\nu^n = \hat{\nu}(U_n)\nu' + \hat{\nu}(K \backslash U_n)\hat{\hat{\nu}}$; then $\Pi(\nu^n) \subset \mathscr{O}$ and if $a \in$ supp (ν^n), either $a \in K \backslash U_n$ in which case $\#\varphi(a, p) = 1$ and $k(a, \varphi(a, p)) \neq 0$ for all $p \in \mathscr{O}$, or $a \in$ supp (ν') in which case $\#\varphi(a, p) = 1$ and $k(a, \varphi(a, p)) \neq 0$ for *all* $p \in S$.

We have that $\nu^n \to \hat{\nu}$ weakly because $\nu(U_n) \to 0$. We have $\int i\, d\nu^n_P \to \int i\, d\hat{\nu}$ because the bound ξ on initial endowments is independent of n. Hence, $\nu^n \to_\alpha \hat{\nu}$.

Q.E.D.

PROOF OF THEOREM 2: We can assume ν satisfies (13). We want $\nu^n \underset{\beta}{\to} \nu$ with $\nu_n \in \mathscr{M}_f$.

As in Step 3 let supp $\nu = \{a_1, \ldots, a_m\}$, $a_j \in (\mathscr{U} \times P)^m$, $a_j = (u_j, \omega_j)$, $\nu = \sum_{j=1}^m \alpha_j \delta_{a_j}$. Pick $p \in \Pi(\nu)$ arbitrarily. Then $0 = \sum_{j=1}^m \alpha_j y_j$ for some $y_j \in \text{co}\, \bar{\varphi}(a_j, p)$. Let $\varphi(a_j, p) = \{x_{j1}, \ldots, x_{js_j}\}$. By (4) there exists an open $p \in U \subset \Delta$ and $\mu > 0$ such that for every jh, $(\omega', p') \mapsto \varphi(u_j, \omega', p') \cap B_\mu(x_{jh})$ is C^1 on $B_\mu(\omega_j) \times U$. Moreover, we can assume $B_\mu(x_{jh}) \cap B_\mu(x_{jh'}) = \varnothing$ if $h \neq h'$.

We shall proceed by replacing every δ_{a_j} by a ν_j with the properties: (i) supp (ν_j) is finite and concentrated around a_j as closely as we wish, (ii) $y_j \in \text{co}\ \bar{\varphi}_{\nu_j}(p)$, (iii) for every $a \in \text{supp}\ (\nu_j)$, $\# \varphi(a, p) = 1$ and $k(a, \varphi(a, p)) \neq 0$. Letting then $\nu' = \Sigma_j \alpha_j \nu_j$ we have $p \in \Pi(\nu')$ and so, with respect to p, ν' satisfies all the conditions for belonging to $\mathcal{M}_f$ except, possibly, rank $D\bar{\varphi}_{\nu'}(p) = l - 1$. But this can be easily fixed up by perturbing the initial endowments of any $a \in \text{supp}\ (\nu')$ in the manner of E. and H. Dierker [**10**].

From now on we consider a single fixed j. For notational economy we drop when there is no ambiguity the subscript j. What will be done is illustrated in Figure 4.

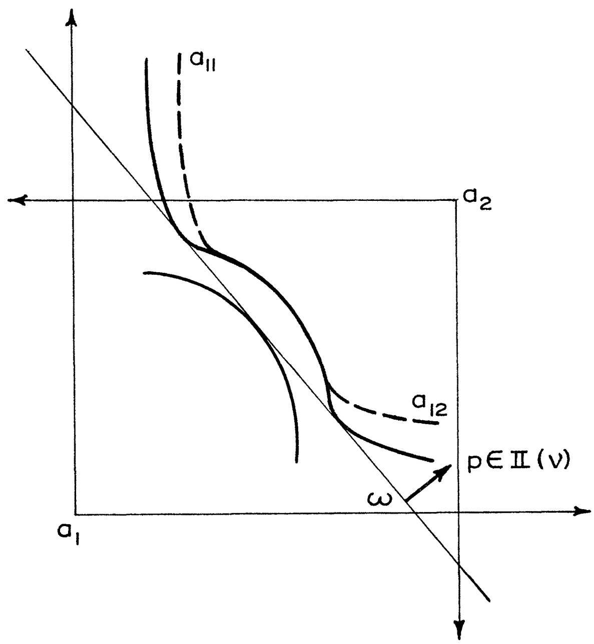

FIGURE 4

Let $F_h \subset P$, $1 \leqslant h \leqslant s$, be a collection of pairwise disjoint relatively compact neighborhoods of the $B_\mu(x_h)$. For every h, let $\xi_h \colon P \to [0, 1]$ be C^∞ and such that $\xi(x) = 1$ for $x \in B_\mu(x_h)$ and $\xi(x) = 0$ for $x \in P \backslash F_h$.

Take an arbitrarily small constant $\eta > 0$. Put $y = \Sigma_{h=1}^s t_h(x_h - \omega)$, where $t_h \geqslant 0$, $\Sigma_{h=1}^s t_h = 1$. For every h define $a_h \in (\mathcal{U} \times P)^m$ by $\omega_{a_h} = \omega_j$ and $u_{a_h}(x) = u_j(x) + \eta \xi_h(x)$. Then a_h is as close as we wish to a, $\varphi(a_h, p) = x_h$ and, of course, $k(a_h, x_h) \neq 0$. Now define $\nu_j = \Sigma_{h=1}^{s_j} t_h\, \delta_{a_h}$.

Q.E.D.

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APPENDIX

In this appendix we describe the example announced in Section 2.

There are two commodities; taking $p_2 = 1$, p will be the relative price of good 1. We start with an economy $\bar{\nu} = \frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$ which is best described by the Edgeworth's box in Figure 5. The excess demand correspondence of a_1 for commodity 1 in a neighborhood of $p = 1$ is given in Figure 6. Figure 7 describes excess demand correspondence (always for good 1) of $\bar{\nu}$. In particular, $p = 1$ is an equilibrium. The dotted lines in Figure 5 indicate "restricted" demand functions; i.e., we assume that there is an open set of prices $U \subset R$ containing $p = 1$ such that for $i = 1, 2$: "if $x \in \varphi(a_i, 1)$, then there is a C^1 function f: $\bar{U} \to R^2$ with $f(1) = x$ and the property that, in a neighborhood V of x and all $p \in U$, $f(p)$ is the unique element of $V \succsim_{a_i}$-maximal on the p-budget set. *Moreover, $\partial_1 f(p) > 0$ for all $p \in U$.*" We can let $U = (s, t)$ and assume $\bar{\varphi}_{\bar{\nu}}(s) > 0$, $\bar{\varphi}_{\bar{\nu}}(t) < 0$ (see Figure 7).

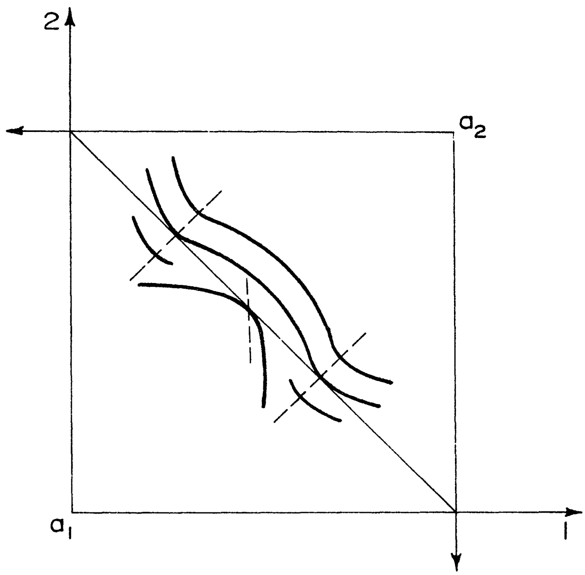

FIGURE 5

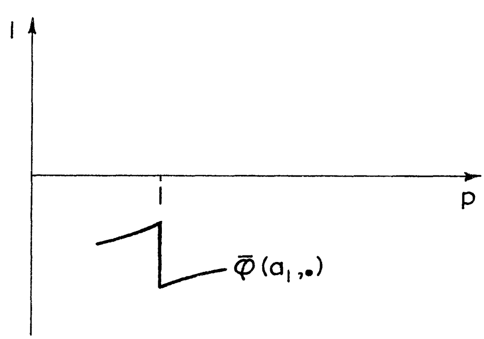

FIGURE 6

Let $\mathcal{O} \subset \mathcal{M}$ be a β-open set of economies containing $\bar{\nu}$ and such that for every $\nu \in \mathcal{O}$: (i) the bracketed statement of the last paragraph holds true, with respect to U, for every $a \in \operatorname{supp}(\nu)$; (ii) $\bar{\varphi}_\nu(s) > 0$, $\bar{\varphi}_\nu(t) < 0$. Clearly, the existence of such an open set follows by continuity arguments.

We will now argue that the set of economies at which the correspondence Π is not stable (as $\bar{\nu}$ in Figure 8) lies dense in $\mathcal{O}$ (and, therefore, includes the whole of $\mathcal{O}$). Similar arguments would yield the density in $\mathcal{O}$ of the discontinuity points of Π.

Let $\mathcal{O}'$ be the set of $\nu \in \mathcal{O}$ satisfying: (i) $\# \operatorname{supp}(\nu) < \infty$, (ii) for every $a \in \operatorname{supp}(\nu)$, $\#\{p \in U: \# \varphi(a, p) > 1\} < \infty$. It is a simple argument to verify that $\mathcal{O}'$ is dense in $\mathcal{O}$. It is clear that: *for every* $\nu \in \mathcal{O}'$ *there is* $\bar{p} \in U$ *such that* $\bar{p} \in \Pi(\nu)$ *and* $\# \varphi(a, \bar{p}) > 1$ *for some* $a \in \operatorname{supp}(\nu)$. Indeed, in going from s to t, the graph of $\bar{\varphi}_\nu$ has to cross the horizontal axis at some $\bar{p}$ "from above" but if $\# \varphi(a, \bar{p}) = 1$ for all $a \in \operatorname{supp}(\nu)$, then the conditions defining $\mathcal{O}$ and $\mathcal{O}'$ imply that $\bar{\varphi}_\nu$ *is positively sloped at* $\bar{p}$. Hence the conclusion (see Figure 9).

Let $\nu \in \mathcal{O}'$ and $\bar{p} \in \Pi(\nu) \cap U$ be such that $\# \varphi(a, \bar{p}) > 1$ for some $a \in \operatorname{supp}(\nu)$. In an *arbitrarily small* neighborhood of $\bar{p}$, $\bar{\varphi}_\nu$ looks as in Figure 7 between s and t. Put $0 = \Sigma_{a \in \operatorname{supp}(\nu)} \nu(a)(x_a - \omega_a)$, $x_a \in \operatorname{co} \varphi(a, \bar{p})$. Suppose that $\# \varphi(\bar{a}, \bar{p}) > 0$, i.e., $x_{\bar{a}} = q x_1 + (1-q) x_2$, $0 \leqslant q \leqslant 1$, $x_1, x_2 \in \varphi(\bar{a}, \bar{p})$. Define from $\bar{a}$ two new a_1, a_2 in the manner indicated in Figure 10. The reader may verify that there is nothing special in the figure, the essential point is that for $\bar{a}_i$, $i = 1, 2$, $\varphi(\bar{a}_i, \bar{p}) = x_i$ and the indifference map remains unaltered in a neighborhood of x_i. Define a new $\bar{\nu}$ by replacing $\bar{a}_1$, with weight $q\nu(\bar{a})$, and $\bar{a}_2$, with weight $(1-q)\nu(\bar{a})$, for $\bar{a}$. Perform this same replacement operation for every $a \in \operatorname{supp}(\nu)$ with $\# \varphi(a, \bar{p}) > 1$. At the end we get $\bar{\nu}$ such that: (i) $\bar{\nu}$ is arbitrarily close to ν; (ii) $\bar{p} \in \Pi(\bar{\nu})$; (iii) for every $a \in \operatorname{supp}(\bar{\nu})$, $\# \varphi(a, \bar{p}) = 1$ and, therefore, $\bar{\varphi}_{\bar{\nu}}$ *is* C^1 *and positively sloped at* $\bar{p}$. But then for an arbitrarily small neighborhood of $\bar{p}$ (take one in which $\bar{\varphi}_\nu$ looks as in Figure 7 between s and t) if $\bar{\nu}$ is sufficiently close to ν, the graph of $\bar{\varphi}_{\bar{\nu}}$ has to cross the horizontal axis several times in this neighborhood (see Figure 11). Hence, Π is not stable at ν and our claim is established.

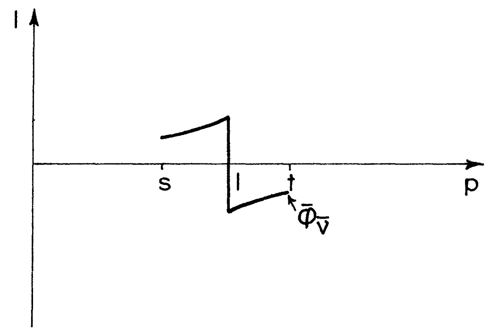

FIGURE 7

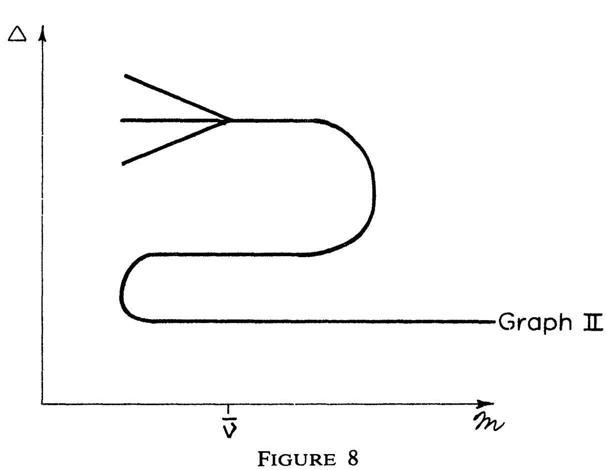

FIGURE 8

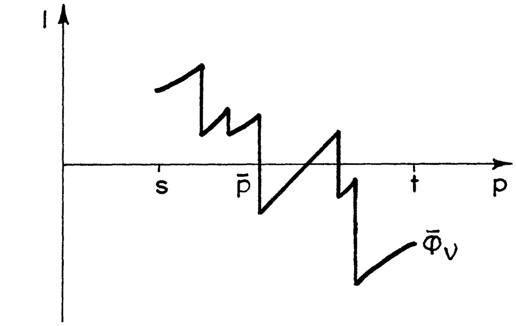

FIGURE 9

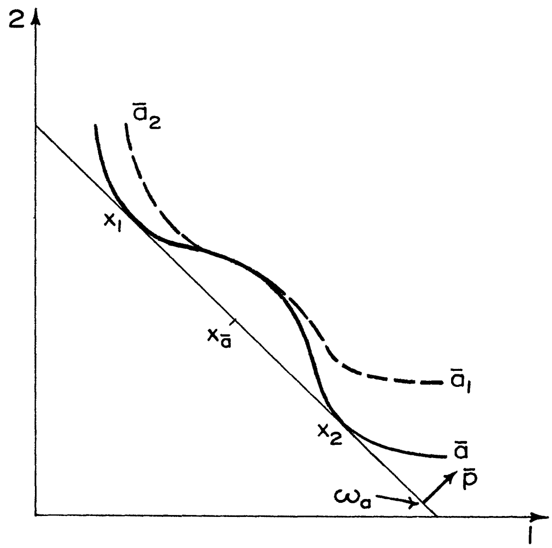

FIGURE 10

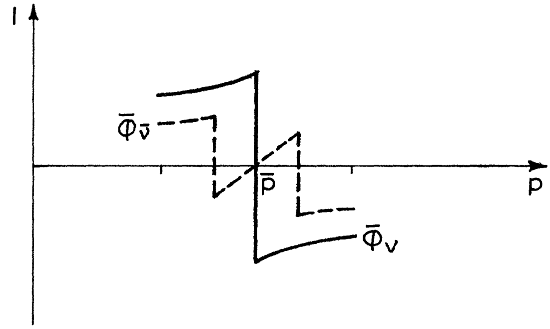

FIGURE 11

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