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Indivisible Commodities and General Equilibrium Theory

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In this paper we study (pure exchange) economies where some of the commodities traded are available only in indivisible units which are large from the standpoint of the individual consumer but small (in fact, negligible) relative to the size of the economy. The objective is to formulate models and give conditions guaranteeing that the main results of the price equilibrium theory with divisible commodities and convex preferences remain valid. Previous references are Henry [10], Broome [2], and E. Dierker [5].

The original motivation for this work comes from Mas-Colell [13], where it turned out that, for the particular aim there pursued, an indivisible commodities model was a more natural and conveneient starting point than the familiar divisible commodities one.

There are close relations between the situation here and the divisible commodities, nonconvex preferences theory developed by Starr [15], Aumann [1], Hildenbrand [11], and others. The indivisible commodities case, however, is not subsumed by the latter; one can no longer assume, for example, that consumption sets are convex.

As with the previous references we concern ourselves only with economies having a large number of traders. In fact, we adopt the Aumann idealization of a continuum of traders which we model following Hildenbrand [11]. In the assumptions referring to consumption sets and preferences we stay close to Henry [4] and Broome [2]. In particular, we postulate that some commodities are perfectly divisible (for simplicity, just one) and we argue that this hypothesis, besides being reasonable, is a sine qua non; see Mc Kenzie [12].

The two specific problems considered are the existence and the determinateness of equilibrium prices and allocations. The notion of equilibrium we use is the (distribution form) one recently introduced in Hart *et al* [9] and Hildenbrand [11].

For sequences of increasingly large finite numbers of traders economies the problem of existence of "approximate" equilibria has been studied by Broome [2] and E. Dierker [5]. Our line of attack is slightly different: we assume from the beginning that there is a continuum of traders and in order to overcome the (continuity) problems peculiar to the indivisibilities case and

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obtain an exact result we make use of an hypothesis postulating that the distribution of consumers according to the amount of divisible commodity they own does not give a positive weight to any particular amount (i.e., the distribution is "spread"). This is a natural enough assumption but its need had not previously been felt in existence of equilibrium analysis.

The set of economies can be made into a topological space in a by now familiar manner (Hildenbrand [11]). Associating to every economy the set of its equilibrium prices and distributions a correspondence is defined. Ideally, one would like that the correspondence assigns to every economy a finite set (or even a singleton!) which, moreover, does not abruptly vary with perturbations of the economy. It is clear that this is too much to wish for, but in the divisible commodities case with convex, smooth preferences (and appropriate C^1 topologies and boundary conditions) it can be shown that the above determinateness of equilibrium property does, indeed, hold in an open, dense set of economies (for the finite number of consumers case see the seminal paper of Debreu [4], for the continuum see H. Dierker [6]); with a set of weakening qualifications the validity of a similar result can be extended to the whole space of economies with smooth (not necessarily convex) preferences (see Mas-Colell [14]).

In the present indivisible commodities context there is no room for the exploitation of smoothess hypothesis and so we will have to settle for obtaining (Theorem 3) the strongest determinateness of equilibrium property one can hope for in a framework of continuity hypothesis, i.e., *the existence of a dense set of economies having a finite number of equilibria each one of which is "stable" (i.e., not very sensitive) under perturbations of the economy* (for the divisible commodities, convex preferences case see H. Dierker [7]); some relevant "open and dense" statements can be derived from this. We may mention that proving the indivisibility model has this property turns out to be a somewhat delicate matter. Precise definitions, discussion, and more details will be given in Part I.

Part I contains the model and statement of theorems; Part II, the proofs.

I

1. *The Model*

Let Z be the nonnegative integers. The *consumption set* is $\Omega = Z^{l-1} \times [0, \infty)$; a commodity bundle will be denoted $a = (x, s)$. Let $b = (0,..., 0, 1)$.

A *preference relation* $\succsim \subset \Omega \times \Omega$ is a complete, transitive presorder; $\succ$ is the partial order induced from $\succsim$ in the usual manner. We will always assume that preference relations are continuous (i.e., closed) and satisfy the following assumptions:

all the adjustments have taken place on the weights ν_j; preferences, or endowments, have not been altered. From now on the weights remain fixed and only preferences will be adjusted.

Let u_j, $1 \leqslant j \leqslant m$, be utility functions for $\succsim_j$, $1 \leqslant j \leqslant m$. It is easily seen that if a sequence of utility functions approximate u_j uniformly on compacta, then the induced preferences approximate $\succsim_j$ in closed converge. Hence, we can assume that u_j is of class C^∞ and from now on by a perturbation of the u_j's we always mean a C^∞ approximation uniform on compacta.

For any $1 \leqslant j \leqslant m$, $p \in R_{++}^{l-1}$ and $x \in Z^{l-1}$ let $x(p, j) \in Z^{l-1} \times R$ be given by $x(p, j) = (x, \sum_{i=1}^{l-1} p^i \omega_j^i + \omega_j^l - px)$. Let $\Omega^o = Z_+^{l-1} \times (0, \infty)$ and for every L define $\Lambda(L) \subset R^{l-1}$ by $\Lambda(L) = \{p \in R^{l-1}$ for all j, if $x, x' \in L_j$, then $x(p, j), x'(p, j) \in \Omega^o$ and $u_j(x(p, j)) = u_j(x'(p, j))\}$. Note that if $L' \subset L$, then $\Lambda(L) \subset \Lambda(L')$.

Lemma 2. *We can assume that for every L, $\Lambda(L)$ is a C^∞ manifold of dimension $(l - 1) - r$*, where $r = \sum_{j=1}^m r_j$ and $r_j = \# L_j - 1$.

Proof. Let $L_j = \{x_{j0}, \ldots, x_{jr_j}\}$.

The set $Q = \{p \in R_{++}^{l-1}: x(p, j)\ e\ \Omega^o$ for all j and $x \in L_j\}$ is open. For every $p \in Q$, j, and $0 \leqslant h \leqslant r_j$ denote $u_{jh}(p) = u_j(x_{jh}(p, j))$ and, for every j and $1 \leqslant h \leqslant r_j$, define $\eta_{jh}: \Omega \to R$ by $\eta_{jh}(x, s) = s$ if $x = x_{jh}$ and $\eta_{jh}(x, s) = 0$, otherwise.

If $r = 0$, then $\Lambda(L) = Q$ and the lemma is proved. Let $I = \{j : r_j \neq 0\} \neq \varnothing$ and define $\Psi: Q \times \prod_{j \in I} R^{r_j} \to \prod_{j \in I} R^{r_j}$ by $\Psi^{jh}(p, t) = u_{j0}(p) - u_{jh}(p) - t^{jh} \eta_{jh}(x_{jh}(p, j))$. Observe that, for every $p, t \in Q \times \prod_{j \in I} R^{r_j}$, $D_t \Psi(p, t)$ has full rank (since $\eta_{jh}(x_{jh}(p, j)) > 0$). Hence for a $\bar{t} \gg 0$ arbitrarily close to 0 we have rank $D\Psi_{\bar{t}}(p) = r$ whenever $\Psi_{\bar{t}}(p) \equiv \Psi(\bar{t}, p) = 0$ (we are here applying the Transversality Theorem; see, for example, Guillemin and Pollack [8, p. 68]). By redefining the utility functions for $j \in I$ to be $u_j \times \sum_{h=1}^{r_j} \bar{t}^{jh} \eta_{jh}$ we can assume rank $D\Psi_0(p) = r$ whenever $\Psi_0(p) = 0$. Then $\Lambda(L) = \Psi_0^{-1}(0)$, and $\Lambda(L)$ is a C^∞ manifold of dimension $= \dim Q - r = l - 1 - r$. ∎

Lemma 2 implies that ν satisfies (i) and (iii) in the definition of regular economy. Since $\Pi(\omega_j') \subset K$ the collection $\mathscr{L} = \{L : 0 \in F_\nu(p) = \sum_{j=1}^m \nu_j(\text{co } L_j - \omega_j'),\ \ p \in \Pi(\nu)\}$ is finite. If $L \in \mathscr{L}$, i.e., $0 \in F_\nu(p) = \sum_{j=1}^m \nu_j(\text{co } L_j - \omega_j)$ for some p, then $\Lambda(L) = \varnothing$, because $p \in \Lambda(L)$, and, by (4), $r = \sum_{j=1}^m (\# L_j - 1) \geqslant l - 1$; by Lemma 2, $r > l - 1$ implies $\Lambda(L) = \Phi$, hence $L \in \mathscr{L}$ implies $r = l - 1$ and so, $\Lambda(L)$ is a discrete set. Since one has $\Pi(\nu) \subset \bigcup_{L \in \mathscr{L}} \Lambda(L) \cap K$, $\# \Pi(\nu)$ is finite. Also, $0 \in \text{Int} \sum_{j=1}^m \nu_j(\text{co } L_j - \omega_j')$ and $r = l - 1$ imply that L_j are all simplices and that 0 can be written in a unique way in the form $0 = \sum_{j=1}^m \nu_j(x_j - \omega_j')$, $x_j \in \text{co } L_j$. In turn, since L_j is a simplex, x_j can be written in a unique way as a convex combination of the points in L_j; so, if $p \in \Pi(\nu)$ there is a unique τ such that $(p, \tau) \in W(\nu)$.

For every $p \in R_{++}^{l-1}$ let $L(p)$ be the maximal m-tuple of finite subsets of

Remark 1. We refer to Hildenbrand [11] for the ideas underlying the definitions of economies and equilibrium; to identify an economy with a distribution is tantamount to postulate from the beginning an infinity of traders.

Remark 2. The discrete commodity model we use has already appeared in Mc Kenzie [12], Henry [10] and Broome [2]. In contrast to E. Dierker [5], they assume the existence of one (or several) perfectly divisible commodity; it seems to us that this is indeed the minimal requirement for generalizing the usual (perfectly divisible commodities) equilibrium theory to a discrete commodity context without losing any of its substantial results. It goes without saying that "perfectly divisible" is an idealization for almost divisible.

Assumption (1) is the standard (and obviously very strong) strict monotonicity assumption. It is convenient and can be weakened in many familiar direction. Assumption (2) is peculiar to the discrete commodities model (it can be found in Broome [2]); it is not very strong, any continuous preference relation can be approximated in closed convergence by one satisfying the condition.

The assumption defining $\mathscr{P}'$ (i.e., that some amount of the divisible commodity is indispensable) is very strong and shall not be postulated in our main existence theorem; it can be found in Henry [10]; it is, however, a very convenient assumption and once the point of its dispensability is made we will use it freely.

Remark 3. The hypothesis that there is only one divisible commodity is made only for notational simplicity.

2. *The Existence Problem*

Given an economy ν let ν_l be the marginal distribution on $[0, \infty)$, i.e., the distribution of divisible commodity.

THEOREM 1. *Given an economy ν there is an equilibrium (p, τ) for ν if either one of the following two conditions holds*:

(i) *ν_l is absolutely continuous with respect to Lebesgue measure*;

(ii) $\nu(\mathscr{P}' \times \Omega) = 1$.

The main result is existence under condition (i). The role of this condition can be gauged from the following example: let the economy give weight 1 to the endowments–preferences combination depicted in Fig. 2; it is then easy to check that the economy has no equilibrium (mean excess demand is bounded away from zero). The problem is that the mean excess demand correspondence does not have a closed graph, if $p^1 > p^2$ the demand of commodity 1 is zero but as soon as $p^1 = p^2$, every consumer switches to a demand of 2 units of that commodity. Condition (i) avoids this problem because if it holds, consumers do not switch their demands *all at once* (i.e., at the same price) but rather only a negligible fraction of them switches at every single price; the net effect is that mean excess demand behaves in a continuous fashion. That diversification of economic agents characteristics will induce regularity in aggregate behavior is a straightforward and old idea;[1] it is, in fact, rather surprising that (thanks to "convexifying" results such as the Shapley–Folkman and Lyapunov theorems) equilibrium theory has managed to do without. It would appear, however, that if discrete commodities are brought into the picture, then diversification is an essential component of a reasonable theory; it is at least the most natural way to deal with the existence problem.

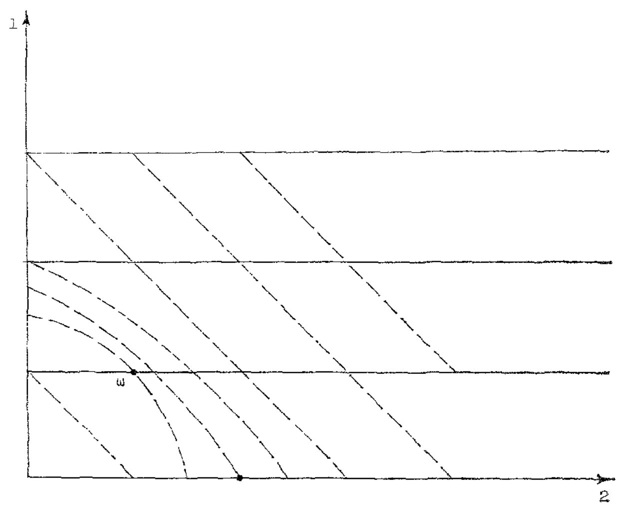

FIGURE 2

3. *The Equilibrium Correspondence*; *Determinateness of Equilibria*

In this section we shall go beyond the existence question and ask how determinate equilibria are. For the sake of simplicity we shall only deal with economies whose consumers have characteristics in $\mathscr{P}' \times \Omega$.

Let $\mathscr{E}$ be the space of economies ν such that $\nu(\mathscr{P}' \times \Omega) = 1$; $\mathscr{E}$ will be topologized in the usual manner, i.e., $\nu^n \to \nu$ if ν^n converges to ν weakly *and* $\int i \, d\nu_\Omega{}^n \to \int i \, d\nu_\Omega$ (see Hildenbrand [11]).

[1] See, for example, Walras [16, p. 58].

Let $\mathscr{M}(\mathscr{P}' \times \Omega \times \Omega)$ be the set of probability measures on $\mathscr{P}' \times \Omega \times \Omega$ with the weak convergence topology. Then $\nu \mapsto W(\nu)$ and $\nu \mapsto \Pi(\nu)$ define correspondences $W: \mathscr{E} \to \mathscr{M}(\mathscr{P}' \times \Omega \times \Omega)$, $\Pi: \mathscr{E} \to R_+^{l-1}$, respectively.

The following theorem is of a familiar variety and it is routinely proved using the techniques in Hildenbrand's book [11].

THEOREM 2. *The correspondences W and Π are compact valued and upper hemicontinuous. Moreover, $\Pi(\nu) \subset R_{++}^{l-1}$ for every $\nu \in \mathscr{E}$.*

Consider an economy ν with a finite support (i.e., the weight of the distribution is concentrated in a finite number of points) and such that for some $p \in \Pi(\nu)$ and $a \in \operatorname{supp}(\nu)$ there is a unique $\succsim_a$ maximizer in the budget set determined by p. It is easily seen (i) every p' in a neighborhood of p is an equilibrium price; (ii) such economies lie dense in $\mathscr{E}$. Hence, there is a dense set of economies having a continuum of price equilibria. Observe, incidentally, that the argument leading to this conclusion remains valid under any "regularity" condition on preferences one may wish to impose.

Is there a dense set of economies having a finite set of equilibria? We shall see the answer is yes, but this is not by itself a very interesting property; what one wants (for, say, estimation or prediction purposes) is that those equilibria be "essential," i.e., that they do not disappear by performing an arbitrarily small perturbation of the economy. Also, if the existence of a dense set of economies having a finite number of equilibria each one of them "essential" can be established, then some reasonable "open and dense" properties can be stated.

In order to arrive at a concept of essential price equilibrium we need to gather some mathematical definitions and results.

Let $S, B \subset R^n$ be, respectively, the unit sphere and the unit ball; in general, the ϵ sphere or ball centered at $x \in R^n$ will be denoted $S_{x\epsilon}$, $B_{x\epsilon}$ (S_ϵ, B_ϵ for $x = 0$). A subset of a sphere in R^n will be called convex if it is the intersection of the sphere with a *pointed* convex cone in R^n. We concern ourselves with the space $\mathscr{F}$ of convex, compact-valued, upper hemicontinuous (u.h.c.) correspondences $f: S \to S$.

Two $f, g \in \mathscr{F}$ are *homotopic* if there is a convex, compact-valued u.h.c. correspondence $H: S \times [0, 1] \to S$ such that $H(\cdot, 0) = f$, $H(\cdot, 1) = g$. Homotopy is an equivalence relation.

A semidistance in $\mathscr{F}$ is defined by letting $d(f, g) = \inf\{\epsilon > 0: \operatorname{Graph}(g) \subset \operatorname{Graph}(f) + B_\epsilon\}$. It is well known that for every $f \in \mathscr{F}$ and $\epsilon > 0$ there is a continuous (or smooth) function $g \in \mathscr{F}$ such that $d(f, g) < \epsilon$ (see Cellina [3]); it is also immediately verified if $f \in \mathscr{F}$ and $\epsilon > 0$ is small enough, then f is homotopic to any $g \in \mathscr{F}$ with $d(f, g) < \epsilon$. Since the *degree* of a continuous function $g: S \to S$ is a homotopy invariant (for the definition of the *degree* and its basic properties see, for example, Guillemin and Pollack [8, Chap. 3]) we can define the degree of $f \in \mathscr{F}$, denoted deg f, to be the degree of any function $g \in \mathscr{F}$ homotopic to f. The basic fact, straightforwardly derived from the corresponding result for continuous (or smooth) functions, is: *$f \in \mathscr{F}$ has degree zero if and only if it can be extended to a convex, compact-valued, u.h.c. correspondence $\hat{f}$: $B \to S$* (see Guillemin and Pollack [8, p. 145]).

Let F: $V \to R^n$, $V \subset R^n$ open, be a convex, compact-valued, u.h.c. correspondence. Suppose $0 \in F(x)$; we say that x is an *essential solution* to the equation $0 \in F(x)$ if for every $\epsilon > 0$ there is $0 < \delta < \epsilon$ such that $0 \notin F(S_{x\delta})$ and the convex, compact-valued, u.h.c. correspondence $\hat{F}$: $S_{x\delta} \to S$ defined by $\hat{F}(y) = \{z/\|z\|: z \in F(y)\}$ has degree different from zero.

Denote $E = \{x \in V \mid 0 \in F(x)\}$ and suppose that every $x \in E$ is essential. Let F_n: $V \to R^n$ be a sequence of compact, convex-valued, u.h.c. maps such that Ls Graph $(F_n) \subset$ Graph $(F)^2$ and $E_n \subset K \subset V$, where $E_n = \{x \in V$: $0 \in F_n(x)\}$ and K is an a priori given compact set. Then E_n converges to E in the Hausdorff metric for the nonempty, compact subsets of R^n, i.e., if n is large E_n and E appear alike. It is obvious that $Ls(E_n) \subset E$; to show $E \in \text{Li}(E_n)$, let $x \in E$, $\epsilon > 0$ and take $\delta > 0$ as in the definition of essential solution. Then $\hat{F}$: $S_{x\delta} \to S$ has degree $\neq 0$ and if n is large, $0 \notin F_n(S_{x\delta})$; in fact, giving to $\hat{F}_n$: $S_{x\delta} \to S$ the obvious meaning, $d(\hat{F}, \hat{F}_n) \to 0$. Therefore, if n is sufficiently large, $\hat{F}$ and $\hat{F}_n$ are homotopic and so, $\deg(\hat{F}_n) = 0$. If $0 \notin F_n(B_{x\delta})$, then $\hat{F}_n$ could be extended to $B_{x\delta}$ which is not the case; hence $E_n \cap B_{x\epsilon} = \varnothing$.

We see then that the definition of essential solution corresponds to the idea of the solution being stable under perturbations of the correspondence F. The converse is, incidentally, also true: if a solution is not essential, then it can be eliminated by an arbitrarily small perturbation of F. The formulation via the degree has the advantage that it is given in terms of F only and does not need, in consequence, the explicit consideration of an ambient space of correspondences.

Now we apply the ideas and concepts of the last six paragraphs to our economic problem.

Define the mean excess demand correspondence F: $\mathscr{E} \times R^{l-1}_{++} \to R^{l-1}$ by $F(\nu, p) = \int \varphi(\succsim, \omega, p)\, d\nu$ where $\varphi(\succsim, \omega, p) = \{x \in R^{l-1}$: for some $s \in R_+$, $p^*(x, s) \leqslant p^*\omega$ and $(x, s) \succsim a$ whenever $p^*a \leqslant p^*\omega\} - \{(\omega^1, ..., \omega^{l-1})\}$. As in Hildenbrand [11, 1.2, 1.3, Chap. 1] one verifies that F is well defined, nonempty, compact, convex-valued, and u.h.c. Moreover, $p \in \Pi(\nu)$ if and only if $0 \in F(\nu, p)$. The correspondence $F(\nu, \cdot)$: $R^{l-1}_{++} \to R^{l-1}$ is denoted F_ν.

In the present context, we define an economy $\nu \in \mathscr{E}$ to be *regular* if the following three conditions are satisfied:

(i) $\Pi(p)$ is a finite set.

[2] For definitions see Hildenbrand [11, p. 15]. One writes $u \in Ls(A_n)$ if every neighborhood of u intersects infinitely many of the A_n.

(ii) Every $p \in \Pi(\nu)$ is an essential solution of $0 \in F_\nu(p)$.

(iii) For every $p \in \Pi(\nu)$ there is a unique τ such that $(p, \tau) \in W(\nu)$.

Given Theorem 2 and the continuity properties of F it is immediate that every regular ν is a continuity point of W.

THEOREM 3. *There is a dense set $\mathscr{E}^* \subset \mathscr{E}$ of regular economies.*

Let ρ be a metric for the weak convergence on $\mathscr{M}(\mathscr{P}' \times \Omega \times \Omega)$. We have a corollary to Theorem 3 (which, however, is weaker than the theorem and could be proved directly somewhat more easily).

COROLLARY. *Let $\epsilon > 0$; then there is an open, dense set $\mathcal{O} \subset \mathscr{E}$ such that if $\nu \in \mathcal{O}$ there is a neighborhood $V \subset \mathcal{O}$ of ν and a finite number of pairwise disjoint open sets $B_i \subset R_{++}^{l-1} \times \mathscr{M}(\mathscr{P}' \times \Omega \times \Omega)$* of radius $< \epsilon$ *with the property that, for every $\nu' \in V$, $W(\nu') \subset \bigcup_i B_i$ and $W(\nu') \cap B_i \neq \varnothing$ for all i.*

Remark 4. Via approximation arguments there is no difficulty in using Theorem 3 to prove an analogous result for a model with perfectly divisible commodities.

II

Proof of Theorem 1 Let ν be given.

The proof follows familiar paths; in particular, it is convenient to formulate the problem in Aumann's [1] representation form. Let $I = [0, 1]$ and λ denote Lebesgue measure. Take a (Borel) measurable map $e: I \to \mathscr{P} \times \Omega$ such that $\nu = \lambda \circ e^{-1}$ (such a map exists, Hildenbrand [11, (37), p. 50]); denote $e(t) = (\succsim_t, \omega_t)$. In this section it will be convenient to let p be a generic element of Δ, the closed unit simplex in R^l; Δ^o will be the open simplex.

For every $(\succsim, \omega) \in \mathscr{P} \times \Omega$ and $p \in \Delta$ let $\varphi(\succsim, \omega, p) = \{a \in \Delta: pa \leqslant p\omega$ and $a \succsim a'$ whenever $pa' \leqslant p\omega\} - \{\omega\}$. Hildenbrand's proof of Proposition 2 [11, p. 102] applies *verbatim* to establish the measurability of Graph φ. Hence, for every $p \in \Delta$, the correspondence $t \mapsto \varphi(\succsim_t, \omega_t, p)$ has, likewise, a measurable graph (Hildenbrand [11, p. 54]) and so, we can define the mean excess demand correspondence $\Phi: \Delta^o \to R^l$ by $\Phi(p) = \int \varphi(\succsim_t, \omega(t), p)\, dt$; Φ is compact, convex-valued (Hildenbrand [11, Theorem 3, p. 62]). We will show that, under any of the two conditions in the statement of Theorem 1, Φ is upper hemicontinuous and $p_n \to p \in \partial\Delta$ implies $\inf\{\| a \| : a \in \Phi(p_n)\} \to \infty$. This will yield the theorem; indeed, it is well known that the above properties, together with the uniform boundedness below of Φ and $p\Phi(p) = 0$ for all

$p \in \Delta^o$, imply the existence of $p \in \Delta^o$ such that $0 \in \Phi(p)$ (via a fixed point argument; see, for example, Hildenbrand [11, p. 150]), i.e., there is a measurable $\mathbf{a}: I \to \Omega$ such that $\int \mathbf{a} \leqslant \int \omega$ and $\mathbf{a}(t) \in \varphi(\succsim_t, \omega(t), p)$ for a.e. $t \in I$, hence $(p, \lambda \circ (e, \mathbf{a})^{-1})$ is an equilibrium. The boundary condition on Φ does also yield the compactness of $W(\nu)$.

In order to prove the claimed properties of Φ, let $p \in \Delta^o$, $p \to p \in \Delta$, $z_n \in \Phi(p_n)$, $z_n = \int \mathbf{a}_n$, $\mathbf{a}_n(t) \in \varphi(\succsim_t, \omega_t, p_n)$; if $p \in \Delta^o$ we can let $z_n \to z$, $z \in \int \mathbf{a}$, $\mathbf{a}(t) \in Ls(\{a_n(t)\})$ (we are applying Fatou's lemma in l dimensions; see Hildenbrand [10, p. 69]). We need to show: (i) if $p \in \Delta^o$, then $\mathbf{a}(t) \in \varphi(\succsim_t, \omega_t, p)$ for a.e. $t \in I$; (ii) if $p \in \partial\Delta$ there is $I' \subset I$ such that $\lambda(I') > 0$ and $\| \mathbf{a}_n(t) \| \to \infty$ for $t \in I'$ (this is easily seen to imply $\| z_n \| \to \infty$).

Suppose first that $p^l = 0$. Since $\lim p_n \int \omega > 0$ there is $I' \subset I$ and $\epsilon > 0$ such that $\lambda(I') > 0$ and $\lim p_n \omega_t > \epsilon$ for $t \in I'$. Let $t \in I'$ and $a \in Ls(\{\mathbf{a}_n(t)\})$. Then $\xi b \succ_t a$ for some real $\xi > 0$, but $\xi pb = 0 < \epsilon < p_n \mathbf{a}_n(t)$ for n sufficiently large, so, by continuity, $a \succsim_t \xi b$, a contradiction. Hence no such a exists, i.e., $\| \mathbf{a}_n(t) \| \to \infty$ for all $t \in I'$.

From now on we assume $p^l > 0$. Pick $I' \subset I$ as follows: if hypothesis (ii) in the statement of Theorem 1 holds, put $I' = I$; if hypothesis (i) holds, put $I' = I \setminus \{t : \text{for some } x \in Z_+^{l-1}, p(x, 0) = p\omega(t)\}$. Then $\lambda(I') = 1$; this is obviously true in the first case, to see it in the second case note that Z^{l-1} is a countable set and hypothesis (i) implies, letting $p' = (p^1, ..., p^{l-1})$, $\lambda\{t : \omega^l(t) = (1/p^l)\, p(x - y)\} = 0$ for all $x, y \in Z^{l-1}$.

Let $t \in I'$ and $a \in \mathrm{Ls}(\{\mathbf{a}_n(t)\})$; if $pa' < p\omega(t)$, then $a \succsim_t a'$ by the usual argument; if $pa' = p\omega(t)$, $a'^l = 0$, and hypothesis (ii) holds, then $a \succsim_t a'$; if hypothesis (i) holds, then $a'^l = 0$ is impossible because $t \in I'$; let $pa' = p\omega(t)$ and $a'^l > 0$, then there is $a_n' \to a'$ such that $p_n a_n' \leqslant p_n \mathbf{a}_n(t)$ and so, $a \succsim_t a'$. In conclusion, if $t \in I'$ and $a \in Ls(\{a_n(t)\})$, then $a \in \varphi(\succsim_t, \omega(t), p)$. If $p^j = 0$, $j \neq l$, this is compatible with the monotonicity of $\succsim_t$ only if $Ls(\{a_n(t)\}) = \varnothing$, i.e., $\| \mathbf{a}_n(t) \| \to \infty$. If $p \in \Delta^o$, then we have $\mathbf{a}(t) \in \varphi(\succsim_t, \omega(t), p)$ for a.e. $t \in I$ and the proof is finished.

Proof of Theorem 2

The proof of this theorem is routine and we will skip it; everything is analogous to the situation with consumption set R_+^l and the proof of Hildenbrand [11, Theorem 3, p. 159] applies almost verbatim.

Proof of Theorem 3

Let $\nu \in \mathscr{E}$ be given. We are only interested in finding a dense set of economies satisfying certain properties, so we can alter slightly the economy ν to suit our purposes. In this proof we shall repeatedly do so without always explicitly replacing symbols; that the given ν has possibly (and legitimately) been substituted by an approximating one will be indicated by the locution "we can assume..."

We can assume that ν has a finite support (this is well known; see Hildenbrand [11]), i.e., $\text{supp}(\nu) = \{c_1, ..., c_m\}$. We can also assume that $m \geqslant l$ and that the $c_j = (\succsim_j, \omega_j)$, $1 \leqslant j \leqslant l-1$, satisfy the following property: for a compact $K \subset R^l_{++}$ such that $\Pi(V) \subset K$ for an open neighborhood $\nu \in V \subset \mathscr{E}'$, one has that for all $p \in K$ and all $1 \leqslant j \leqslant l-1$, $\varphi^j(\succ_j, \omega_j, p) > 0$ and $\varphi^i(\succsim_j, \omega_j, p) = 0$ if $1 \leqslant i \leqslant l-1$, $i \neq j$. In fact, by Theorem 2, there is K such that $\Pi(V) \subset K$ for some open $\nu \in V$; denoting $e = (1, ..., 1) \in R^{l-1}$, let ξ be a constant with $pe + 1 < \xi$ for all $p \in K$; put $\omega_j = (0, \xi)$ and let $\succsim_j$ be represented by the utility function $u_j(x,s) = s(\xi x^j + \delta ex + 1)$, if δ is sufficiently small $c_j = (\succsim_j, \omega_j)$ is as desired.

Denote $\nu(\{c_j\}) = \nu_j$.

A subset $J \subset Z^{l-1}$ will be called a *simplex* if $\dim \text{co}\, J = \#J - 1$;[3] we write $\dim J$ for $\dim \text{co}\, J$. For an m-tuple of finite sets $L = (L_1, ..., L_m)$, $L_j \subset Z^{l-1}$, we say that L is in *general position* if every L_j is a simplex and $\sum_{j=1}^m \dim L_j = \dim \text{co}\,(\sum_{j=1}^m \nu_j L_j) = l-1$; $L' \subset L$ means $L_j' \subset L_j$ for all j.

For $1 \leqslant j \leqslant l-1$ let $e_j \in Z^{l-1}$ be the vector: $e_j^j = 1$, $e_j^i = 0$ if $i \neq j$. From now on the symbol L will denote m-tuples of finite subsets of Z^{l-1}. We will consider L satisfying:

(3) for $1 \leqslant j \leqslant l-1$, $L_j = \{v_j e_j\}$, where v_j is a positive integer. Note that if $p \in K$, then $F_\nu(p) = \sum_{j=1}^m \nu_j(\text{co}\, L_j - \omega_j')$ where $\omega_j' = (\omega_j^1, ..., \omega_j^{l-1})$ and $L = (L_1, ..., L_m)$ satisfies (3).

Lemma 1. *For every L satisfying* (3) *the set* $\theta(L) = \{\alpha \in R^m : 0 \in \text{Bdry} \sum_{j=1}^m \alpha_j(\text{co}\, L_j - \omega_j')\}$ *has (Lebesgue) measure zero.*

Proof. $\theta(L)$ has measure zero if for every $(\alpha_l, ..., \alpha_m)$, the set $\{(\alpha_1, ..., \alpha_{l-1}) \in R^{l-1} : -\sum_{j=1}^{l-1} \alpha_j v_j e_j \in \text{Bdry} \sum_{j=l}^m \alpha_j(\text{co}\, L_j - \omega_j)\} \subset R^{l-1}$ has measure zero, but this is obvious since $\text{Bdry} \sum_{j=l}^m \alpha_j(\text{co}\, L_j - \omega_j) \subset R^{l-1}$ has measure zero and $(\alpha_1, ..., \alpha_{l-1}) \to -\sum_{j=1}^{l-1} \alpha_j v_j e_j$ is a nonsingular linear map. ∎

There is only a countable number of distinct L. Hence, by Lemma 1, we can approximate the vector $(\nu_1, ..., \nu_m)$ by a strictly positive one $(\alpha_1, ..., \alpha_m)$ not belonging to any $\theta(L)$ (L satisfying (3)). Replace ν by a new probability measure ν' defined by $\nu'(\{c_j\}) = (1/\sum_{j=1}^m \alpha_j)\, \alpha_j$; for this ν' we can assume $\Pi(\nu') \in K$ and so, $0 \in F_{\nu'}(p)$ does then imply $0 \in \text{Int}\, F_{\nu'}(p)$. Hence, from now on we will assume that ν satisfies:

(4) *for all $p \in R^{l-1}_{++}$, if $0 \in F_\nu(p)$, then $0 \in \text{Int}\, F_\nu(p)$.*

The previous conclusion finishes the first step of the proof. Note that so far

[3] "co" denotes convex hull; of course, when we write co J, we are regarding J as a subset of R^{l-1}.

all the adjustments have taken place on the weights ν_j; preferences, or endowments, have not been altered. From now on the weights remain fixed and only preferences will be adjusted.

Let u_j, $1 \leqslant j \leqslant m$, be utility functions for $\gtrsim_j$, $1 \leqslant j \leqslant m$. It is easily seen that if a sequence of utility functions approximate u_j uniformly on compacta, then the induced preferences approximate $\gtrsim_j$ in closed converge. Hence, we can assume that u_j is of class C^∞ and from now on by a perturbation of the u_j's we always mean a C^∞ approximation uniform on compacta.

For any $1 \leqslant j \leqslant m$, $p \in R_{++}^{l-1}$ and $x \in Z^{l-1}$ let $x(p, j) \in Z^{l-1} \times R$ be given by $x(p, j) = (x, \sum_{i=1}^{l-1} p^i \omega_j^i + \omega_j^l - px)$. Let $\Omega^\circ = Z_+^{l-1} \times (0, \infty)$ and for every L define $\Lambda(L) \subset R^{l-1}$ by $\Lambda(L) = \{p \in R^{l-1}$ for all j, if $x, x' \in L_j$, then $x(p, j), x'(p, j) \in \Omega^\circ$ and $u_j(x(p, j)) = u_j(x'(p, j))\}$. Note that if $L' \subset L$, then $\Lambda(L) \subset \Lambda(L')$.

Lemma 2. *We can assume that for every L, $\Lambda(L)$ is a C^∞ manifold of dimension $(l - 1) - r$*, where $r = \sum_{j=1}^{m} r_j$ and $r_j = \# L_j - 1$.

Proof. Let $L_j = \{x_{j0}, \ldots, x_{jr_j}\}$.

The set $Q = \{p \in R_{++}^{l-1}: x(p, j)\ e\ \Omega^\circ$ for all j and $x \in L_j\}$ is open. For every $p \in Q$, j, and $0 \leqslant h \leqslant r_j$ denote $u_{jh}(p) = u_j(x_{jh}(p, j))$ and, for every j and $1 \leqslant h \leqslant r_j$, define $\eta_{jh}: \Omega \to R$ by $\eta_{jh}(x, s) = s$ if $x = x_{jh}$ and $\eta_{jh}(x, s) = 0$, otherwise.

If $r = 0$, then $\Lambda(L) = Q$ and the lemma is proved. Let $I = \{j : r_j \neq 0\} \neq \varnothing$ and define $\Psi: Q \times \prod_{j \in I} R^{r_j} \to \prod_{j \in I} R^{r_j}$ by $\Psi^{jh}(p, t) = u_{j0}(p) - u_{jh}(p) - t^{jh} \eta_{jh}(x_{jh}(p, j))$. Observe that, for every $p, t \in Q \times \prod_{j \in I} R^{r_j}$, $D_t \Psi(p, t)$ has full rank (since $\eta_{jh}(x_{jh}(p, j)) > 0$). Hence for a $\bar{t} \gg 0$ arbitrarily close to 0 we have rank $D\Psi_{\bar{t}}(p) = r$ whenever $\Psi_{\bar{t}}(p) \equiv \Psi(\bar{t}, p) = 0$ (we are here applying the Transversality Theorem; see, for example, Guillemin and Pollack [8, p. 68]). By redefining the utility functions for $j \in I$ to be $u_j \times \sum_{h=1}^{r_j} \bar{t}^{jh} \eta_{jh}$ we can assume rank $D\Psi_0(p) = r$ whenever $\Psi_0(p) = 0$. Then $\Lambda(L) = \Psi_0^{-1}(0)$, and $\Lambda(L)$ is a C^∞ manifold of dimension $= \dim Q - r = l - 1 - r$. ∎

Lemma 2 implies that ν satisfies (i) and (iii) in the definition of regular economy. Since $\Pi(\omega_j') \subset K$ the collection $\mathscr{L} = \{L : 0 \in F_\nu(p) = \sum_{j=1}^{m} \nu_j(\text{co } L_j - \omega_j')$, $p \in \Pi(\nu)\}$ is finite. If $L \in \mathscr{L}$, i.e., $0 \in F_\nu(p) = \sum_{j=1}^{m} \nu_j(\text{co } L_j - \omega_j)$ for some p, then $\Lambda(L) = \varnothing$, because $p \in \Lambda(L)$, and, by (4), $r = \sum_{j=1}^{m} (\# L_j - 1) \geqslant l - 1$; by Lemma 2, $r > l - 1$ implies $\Lambda(L) = \Phi$, hence $L \in \mathscr{L}$ implies $r = l - 1$ and so, $\Lambda(L)$ is a discrete set. Since one has $\Pi(\nu) \subset \bigcup_{L \in \mathscr{L}} \Lambda(L) \cap K$, $\#\Pi(\nu)$ is finite. Also, $0 \in \text{Int} \sum_{j=1}^{m} \nu_j(\text{co } L_j - \omega_j')$ and $r = l - 1$ imply that L_j are all simplices and that 0 can be written in a unique way in the form $0 = \sum_{j=1}^{m} \nu_j(x_j - \omega_j')$, $x_j \in \text{co } L_j$. In turn, since L_j is a simplex, x_j can be written in a unique way as a convex combination of the points in L_j; so, if $p \in \Pi(\nu)$ there is a unique τ such that $(p, \tau) \in W(\nu)$.

For every $p \in R_{++}^{l-1}$ let $L(p)$ be the maximal m-tuple of finite subsets of

Z^{l-1} such that $F_\nu(p) = \sum_{j=1}^m \nu_j(\text{co } L_j(p) - \omega_j)$. For every L define $\Gamma(L) = \{p \in R_{++}^{l-1}: L \subset L(p)\}$, $\Gamma_e(L) = \{p \in R_{++}^{l-1}: L = L(p)\}$. In the last paragraph it has been shown that if $p \in \Pi(\nu)$, then we can assume $L(p)$ is in general position.

By a simplex in a sphere $S \subset R^{l-1}$ we mean a set homeomorphic to a simplex; if $T \subset S$ is a simplex, then Bd T and In T will denote, respectively, the boundary and interior of T when regarded as a simplex.

LEMMA 3. *We can assume that the conclusion of Lemma 2 holds and that for every* $\epsilon > 0$ *there is* $0 < \delta < \epsilon$ *such that if* $p \in \Pi(\nu)$, *then* $S_{p\delta} \subset R_{++}^{l-1}$ *and, for every* $L' \subset L(p)$, $\Gamma(L') \cap S_{p\delta}$ *is a simplex of dimension* $(l-2) - \sum_{j=1}^m \dim L_j'$ *and interior* $\Gamma_e(L') \cap S_{p\delta}$.

Proof. It suffices to prove the lemma with the further conditional $L(p) = L$, where L is in general position.

Let $L' \subset L$; if $L' = L$, then the conclusion follows from Lemma 1 because $\Gamma(L) \subset \Lambda(L)$ and $\Lambda(L)$ is discrete. Suppose $L' \subsetneq L$ and let $I = \{j: \dim L_j' < \dim L_j\}$, $r_j = \dim L_j - \dim L_j'$, $r = \sum_{j=1}^m r_j$. For every $1 \leqslant j \leqslant m$ pick $x_{j0} \in L_j'$ and for every $j \in I$ let $L_j \backslash L_j' = \{x_{j1}, ..., x_{jr_j}\}$.

Observe that $\Lambda(L')$ remains unaltered by any changes in the u_j's which let $u_j \mid L_j'$ unaltered. For every $p \in \Lambda(L')$, $1 \leqslant j \leqslant m$ and $0 \leqslant h \leqslant r_j$, denote $u_{jh}(p) = u_j(x_{jh}(p, j))$ and for every $j \in I$ and $1 \leqslant h \leqslant r_j$ define $\eta_{jh}: \Omega \to R$ by $\eta_{jh}(x, s) = s$ if $x = x_{jh}$, $\eta_{jh}(x, s) = 0$, otherwise. Define now $\Psi: \Lambda(L') \times \prod_{j \in I} R^{r_i} \to \prod_{j \in I} R^{r_j} \equiv R^r$ by $\Psi^{jh}(p, t) = u_{j0}(p) - u_{jh}(p) - t^{jh}\eta_{jh}(x_{jh}(p, j))$. Observe that for every (p, t) the map $D_t\Psi(p, t)$ has full rank. Hence, as in the proof of Lemma 2, for a $\bar{t} > 0$ arbitrarily close to 0, $D\Psi_{\bar{t}}(p)$ has rank $= r$ whenever $\Psi_{\bar{t}}(p) = 0$. By redefining, for $j \in I$, the utility functions to be $u_j + \sum_{h=1}^{r_j} \bar{t}^{jh}\eta_{jh}$ we can assume rank $D\Psi_0(p) = r$ whenever $\Psi_0(p) = 0$ and so, we have: If $p \in \Pi(\nu)$ and $L = L(p)$, then $\Psi_0(p) = 0$ and, for an open set $V \subset R_{l-1}^{++}$, $p \in \Gamma(L') \cap V = \Psi_0^{-1}(R_+^r) \cap V$ and $\Psi_0 \mid V$ is a diffeomorphism. Since $\Pi(V) \subset K$, it is clear that the previous conclusion remains true if the utility functions are slightly perturbed and so we can assume that the conclusion of Lemma 2 is satisfied.

Let $\bar{p} \in \Pi(\nu)$, $L = L(p)$, and take V as above. Let $S \subset \Psi_0(V)$ be a sphere centered at the origin and radius $\xi > 0$. For any sufficiently small $\delta > 0$, $S_{\bar{p}\delta} \subset V$ and the function $\hat{\Psi}: S_{\bar{p}\delta} \to S$ defined by $\hat{\Psi}(p) = (\xi/\| \Psi(p)\|)\, \Psi(p)$ is a diffeomorphism.[4] We have then that $\Psi_0(S_{\bar{p}\delta}) \cap R_+^r$ is diffeomorphic to

[4] Indeed, it has degree ± 1; so, it suffices to show it is a local diffeomorphism. Suppose it is not so for any δ. Then we have $p_n \to \bar{p}$, $p_n \neq \bar{p}$, $v_n \to v \in R^{l-1}$, $\| v_n \| = 1$ such that $(1/\| \Psi_0(p_n)\|)(p_n - \bar{p}) \to z \neq 0$, $v_n(p_n - \bar{p}) = 0$, and

$$\frac{D\Psi_0(p_n)v_n}{\| D\Psi_0(p_n)v_n \|} = \frac{\Psi(p_n)}{\| \Psi(p_n) \|} = \frac{D\Psi(\bar{p})(p_n - \bar{p}) + o(\| p_n - \bar{p} \|)}{\| \Psi(p_n) \|};$$

but then, $(1/\| D\Psi(\bar{p})v \|)v = z$ and $vz = 0$, a contradiction.

$S \cap R_+^r (q \to \xi(q/\|q\|)$ is a diffeomorphism) and since $S_{\bar{p}\delta} \cap \Gamma(L')$ is diffeomorphic to $\Psi_0(S_{\bar{p}\delta}) \cap R_+^r$ and $S \cap R_+^r$ is a $(r-1)$ simplex, we conclude that $S_{\bar{p}\delta} \cap \Gamma(L')$ is a simplex of dimension $r - 1 = \sum_{j=1}^m (\dim L_j - \dim L_j') - 1 = l - 2 - \sum_{j=1}^m \dim L_j'$. One proves in exactly the same way that $\text{In}(S_{\bar{p}\delta} \cap \Gamma(L')) = S_{\bar{p}\delta} \cap \Gamma_e(L')$. Noting that $\Pi(v)$ is a finite set the proof is finished. ∎

We are now ready to prove that every $p \in \Pi(v)$ is an essential solution of $0 \in F_v(p)$. Let $\bar{p} \in \Pi(v)$, $\epsilon > 0$, and take $0 < \delta < \epsilon$ as in Lemma 3.

For $L \subset L(\bar{p})$ denote $\hat{\Gamma}(L) = \Gamma(L) \cap S_{\bar{p}\delta}$. By the definitions and Lemma 3, the collection $\{\hat{\Gamma}(L): L \subset L(p)\}$ satisfies the following properties: (i) $\hat{\Gamma}(L)$ is a simplex of dimension $(l-2) - \sum_{j=1}^m \dim L_j$; (ii) $S_{\bar{p}\delta} \subset \bigcup_{L \subset L(\bar{p})} \hat{\Gamma}(L)$; (iii) if $L \neq L'$, then $\text{In}\, \hat{\Gamma}(L) \cap \text{In}\, \hat{\Gamma}(L') = \varnothing$; (iv) $\text{Bd}\, \hat{\Gamma}(L) = \bigcup_{L' \subsetneq L} \hat{\Gamma}(L)$.

For every $L \subset L(p)$ define $J(L) = \{q \in S_{\bar{p}\delta} : qx \geqslant qy$ for all $x \in \sum_{j=1}^m (\text{co}\, L_j - \omega_j')$ and $y \in F_v(p)\}$, i.e., $J(L)$ is the set of unit normals to hyperplanes which support the polyhedron $F_v(p)$ at every point of the face determined by L. Since $L(\bar{p})$ is in a general position, for every $L \subset L(\bar{p})$ $\dim \sum_{j=1}^m (\text{co}\, L_j - \omega_j') = \sum_{j=1}^m \dim L_j$ and so, $\dim J(L) = l - 2 - \sum_{j=1}^m L_j$. In fact, it is trivially verified that the collection $\{J(L): L \subset L(\bar{p})\}$ satisfies conditions (i)–(iv) of the last paragraph (with, of course, the symbol $\hat{\Gamma}$ replaced by J) and so, a standard recursive argument (star with 1-simplices) yields the existence of a homeomorphism $g: S_{\bar{p}\delta} \to S_{\bar{p}\delta}$ carrying every $\hat{\Gamma}(L)$ onto $J(L)$.

Noting that $0 \notin F_v(S_{\bar{p}\delta})$, we can define $\hat{F}_v: S_{\bar{p}\delta} \to S_{\bar{p}\delta}$ by $\hat{F}_v(p) = \{p/\|p\|: p \in F_v(p)\}$. For any $p \in S_{\bar{p}\delta}$ we have $p \in \hat{\Gamma}(L(p))$ and so, $g(p) \in J(L(p))$ which implies $g(p)\, x \geqslant g(p)\, y$ for every $x \in F_v(p)$ and $y \in F_v(\bar{p})$; since $0 \in F_v(\bar{p})$, we get $g(p)\, x \geqslant 0$ for all $x \in F_v(p)$. In particular, $-g(p) \notin \hat{F}_v(p)$ for all $p \in S_{\bar{p}\delta}$ and we can conclude that $\hat{F}_v$ and g are homotopic; hence $\deg \hat{F}_v = \deg g \neq 0$, because g is a homeomorphism. Therefore, $\bar{p}$ is an essential solution.

Q.E.D.

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