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ON THE CONTINUOUS REPRESENTATION OF PREORDERS*

BY ANDREU MAS-COLELL[1]

1. STATEMENT OF THE PROBLEM AND RESULT

Let X be a topological space. A preorder $\succsim$ on X is a reflexive, transitive, complete relation on $X \times X$ (we look at $\succsim$ as a subset $\succsim \subset X \times X$). If for a $\succsim$, x, $y \in X$ stand in the relation, we write $x \succsim y$; $x \succ y$ means $\neg(y \succsim x)$.

Denote by $\mathscr{P}$ the set of continuous (i.e., closed) preorders on X. A real valued function $u: X \to R$ is a *utility* for $\succsim$ if it is order-preserving, i.e., "$u(x) \geq u(y) \Leftrightarrow x \succsim y$." A basic result due to Debreu [**1**] is: *if X is 2^{nd} countable, then every $\succsim$ has a continuous utility function.* From now on we assume that X is 2^{nd} countable.

Suppose that $\mathscr{P}$ itself is made into a topological space. Then one may wish to find a function $U: \mathscr{P} \times X \to R$ jointly continuous and such that, for all $\succsim \in \mathscr{P}$, $U(\succsim, \cdot)$ is a utility for $\succsim$. The following theorem gives a set of sufficient conditions for the existence of such a function:

THEOREM. *If X is a 2^{nd} countable, locally compact topological space and $\mathscr{P}$ has the topology induced by the closed convergence on the space of closed, nonempty subsets of X, then there is a jointly continuous function $U: \mathscr{P} \times X \to R$ such that for every $\succsim \in \mathscr{P}$, $U(\succsim, \cdot)$ is $\succsim$-order-preserving.*

In the next section we discuss the topological conditions of the Theorem in more detail. There is probably not a simple constructive ("canonical") method to find a U function; we will proceed, nonconstructively, by combining Debreu's representation theorem with a selection argument.

In economics the problem of this paper arose in the context of models of large economies. It was first treated, implicitly, by Kannai [**5**] and, explicitly by Hildenbrand [**3**]; they gave a solution for, essentially, $X = R^l_+$, $\mathscr{P}$ = set of "monotone" preorders and $\mathscr{P}$ endowed with the closed convergence topology. Their result was generalized by Neuefeind [**10**] and Mount and Reiter [**8**] by allowing more general $X \subset R^l$ and by replacing "monotone" by, respectively, "null Lebesgue measure of indifference classes" and "nonsaturated." All those authors have been able to write down an explicit formula for U, the strategy of their proofs is, precisely, to find sufficient conditions making a candidate U into a utility function; this is also the approach pursued in the recent work of Mount and Reiter [**9**]. In the general formulation of this paper the problem has been posed by

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Kannai [**6**]; the relevance of selection arguments was already pointed out by Mount and Reiter [**8**].

There is a difference between the setup of this paper and the one of Neuefeind [**10**]; while we consider only a fixed consumption set X, he allows for it to vary and, endowing the space of consumption sets with the closed convergence topology, he looks for a utility continuous jointly on consumption sets and preferences. The possibility of extending the theorem here to a more general situation of this kind does certainly deserve further study.

2. SOME REMARKS ON THE TOPOLOGY FOR $\mathscr{P}$

For the definition of the closed convergence see Hildenbrand [**4**, (18)]; note that, because X is locally compact, the closed convergence does, indeed, induce a topology.

The use of the closed convergence to topologize $\mathscr{P}$ can be argued on intuitive grounds (it very straightforwardly captures what one would understand for two sets to be close to each other "away from infinity") or simply on the basis that it is reasonably weak. It is also the topology which, with an economic motivation, has been used in the previous treatments of the problem. Another justification will be mentioned later on.

The closed convergence is not, however, *in general* the minimal topology on $\mathscr{P}$ for which the problem of finding a continuous U has a solution. In fact, it does not appear that, in general (i.e., for arbitrary X), such a minimal topology exists. We devote the next paragraphs to elaborate, somewhat loosely, this point.

If $U: \mathscr{P} \times X \to R$ is continuous and order-preserving for every $\succsim \in \mathscr{P}$, then the set $A = \{(\succsim, x, y): x \succsim y\} \subset \mathscr{P} \times X \times X$ is closed. Hence the closedness (in the product topology) of A is a necessary condition a topology on $\mathscr{P}$ should satisfy for the representation problem to have a solution. Let $\mathscr{T}$ be the space of (non-empty) closed subsets of $X \times X$. A minimal topology on $\mathscr{P}$ making A closed exists, it is the relativization to $\mathscr{P}$ of the topology on $\mathscr{T}$ having as subbasis sets of the form $\{T \in \mathscr{T}: T \cap K = \emptyset\}$ where K is compact (see Kannai [**5**, (787)]). Let us denote this topology on $\mathscr{T}$ and $\mathscr{P}$ by $\mathscr{K}$. For the purposes of the representation problem the $\mathscr{K}$ topology is, however, too weak; it is not separated and one easily verifies that if $U: \mathscr{P} \times X \to R$ is continuous then since $\# U(\{X \times X\} \times X) = 1$, $\# U(\mathscr{P} \times X) = 1$.

To proceed with the discussion it will be convenient to work with a particular, important example. Take $X = R^l$; then we are in the usual economic situation and we can let $\mathscr{P}_m \subset \mathscr{P}$ be the set of monotone preorders ("preference" relations). As shown by Kannai and Hildenbrand the $\mathscr{K}$ topology coincides on $\mathscr{P}_m$ with the closed convergence topology and, so, the latter is the minimal topology for which there is a continuous order-preserving $U: \mathscr{P}_m \times X \to R$; this is, incidentally, a very good reason to restrict oneself to the closed convergence topology on the whole of $\mathscr{P}$.

We argue now that, for $X = R^l$, there is no minimal topology. Denote the closed convergence one by $\mathscr{C}$ and let $\bar{\mathscr{P}}_m$ be the $\mathscr{C}$-closure of $\mathscr{P}_m$ on $\mathscr{T}$. Denote $\mathscr{S}=\bar{\mathscr{P}}_m\backslash\mathscr{P}_m^*$; due to the nonclosedness of the transitivity property $\bar{\mathscr{S}}=\bar{\mathscr{P}}_m$, (where $\bar{\mathscr{S}}$ is the $\mathscr{C}$-closure of $\mathscr{S}$). It is easily verified that if $S\in\mathscr{S}$ and $S'\subset S$ is closed, then $S'\notin\mathscr{P}$. Hence, for every $S\in\mathscr{S}$, there is a $\mathscr{K}$-continuous function α_S: $\mathscr{T}\to[0, 1]$ such that $\alpha_S(S)=0$ and $\alpha_S(\mathscr{P})>0$.

Let $\mathscr{K}\subset\mathscr{M}\subset\mathscr{C}$ and U: $\mathscr{P}\times X\to X$ be order-preserving and $\mathscr{M}$-continuous. Suppose that for every $\succsim_n\underset{\mathscr{S}}{\to}S$, $\succsim_n\in\mathscr{P}_m$, $S\in\mathscr{S}$, one has $U(\succsim_n,\cdot)\to$constant, uniformly on compacta; then it is easily checked that $U(\succsim,\cdot)=$constant, for every $\succsim\in\mathscr{P}_m$. Hence there should be $\succsim_n\in\mathscr{P}_m$, $S\in\mathscr{S}$, such that $\succsim_n\underset{\mathscr{C}}{\to}S$ and $\succsim_n$ does not $\mathscr{M}$-converge to $X\in\mathscr{P}$. We modify the $\mathscr{M}$ topology by redefining the neighborhoods of $X\times X$ on $\mathscr{P}$ to be $\{\mathscr{B}_1\cup(\mathscr{B}_2\cap\mathscr{B})$: $X\times X\in\mathscr{B}_1$, $\mathscr{B}_1$ is $\mathscr{M}$-open, $\mathscr{B}_2\in\mathscr{T}$ is $\mathscr{K}$ open and $S\in\mathscr{B}_2\}$, i.e., we are identifying X and S. The new topology, say $\mathscr{M}'$, is weaker than $\mathscr{M}$ since now $\succsim_n$ $\mathscr{M}'$-converges to X. Let $U'(\succsim, x)=\alpha_S(\succsim)U(\succsim, x)$; it is easily verified that U' is $\mathscr{M}'$-continuous and order-preserving for every $\succsim$. Henceforth, no minimal topology on $\mathscr{P}$ exists.

3. PROOF OF THE THEOREM

Since X is locally compact it is regular and, therefore, being 2^{nd} countable, metrizable (Dugundji [**2**, (XI 6.4, IX 9.2)]); $\mathscr{P}$ is also a metrizable space (Hildenbrand [**4**, (19)]).

Let $\{B_i\}$ be a countable basis for the topology of X composed of relatively compact open sets; such a basis exists (Dugundji [**2**, (Chapter XI, 6.3)]).

Let $T=\{ij: \bar{B}_i\cap\bar{B}_j=\emptyset\}$. For every $ij\in T$ define $\mathscr{P}_{ij}=\{\succsim\in\mathscr{P}: \bar{B}_i\succ\bar{B}_j\}$. Since $\bar{B}_i\cap\bar{B}_j=\emptyset$ and X is normal, $\mathscr{P}_{ij}\neq\emptyset$, and because $\bar{B}_i$, $\bar{B}_j$ are compact, $\mathscr{P}_{ij}$ is open.

Let $\mathscr{F}(X)$ be the Fréchet space of continuous functions f: $X\to R$ with the maximum on compacta seminorms.

A function g: $X\to R$ is a p-utility (pseudoutility) for $\succsim$ if "$x\succsim y\Rightarrow g(x)\geq g(y)$."

For every $ij\in T$, define a correspondence Ψ_{ij}: $\mathscr{P}_{ij}\to\mathscr{F}(X)$ by $\Psi_{ij}(\succsim)=\{u$: $u(x)\subset[0, 1]$, $u(\bar{B}_i)=1$, $u(\bar{B}_j)=0$ and u is a continuous p-utility for $\succsim\}$.

PROPOSITION. *There is a continuous function* F_{ij}: $\mathscr{P}_{ij}\to\mathscr{F}(X)$ *such that* $F_{ij}(\succsim)\in\Psi_{ij}(\succsim)$ *for every* $\succsim\in\mathscr{P}_{ij}$.

We show first how the Proposition yields the Theorem. Let α_{ij}: $\mathscr{P}\to[0, 1]$ be a continuous function such that $\alpha_{ij}(\mathscr{P}_{ij})>0$, $\alpha_{ij}(\mathscr{P}\backslash\mathscr{P}_{ij})=0$. Since $\mathscr{P}_{ij}$ is open, such a function exists. Define $\hat{F}_{ij}$: $\mathscr{P}\to\mathscr{F}(X)$ by $\hat{F}_{ij}(\succsim)=\alpha_{ij}(\succsim)F_{ij}(\succsim)$ if $\succsim\in\mathscr{P}_{ij}$ and $\hat{F}_{ij}(\succsim)=0$ otherwise (0 is here the zero function of $\mathscr{F}(X)$). $\hat{F}_{ij}$ is continuous, and, for all $\succsim\in\mathscr{P}$, $\hat{F}_{ij}(\succsim)$ is a p-utility for $\succsim$. If $\succsim\in\mathscr{P}_{ij}$, then $\hat{F}_{ij}(\succsim)(\bar{B}_i)>\hat{F}_{ij}(\succsim)(\bar{B}_j)$.

Define U_{ij}: $\mathscr{P}\times X\to[0, 1]$ by $U_{ij}(\succsim, x)=\hat{F}_{ij}(\succsim)(x)$; U_{ij} is continuous (Dugundji [**2**, (XII, 3.1)]).

Define U: $\mathscr{P}\times X\to R$ by $U(\succsim,x)=\sum_{ij\in T}(1/2^{i+j})U_{ij}(\succsim, x)$. Then U is continuous and, for all $\succsim$, $U(\succsim,\cdot)$ is a utility of $\succsim$. Indeed, every $U_{ij}(\succsim,\cdot)$ is a p-

utility for $\succsim$ and, so, it suffices to see that if $x \succ y$, then $U_{ij}(\succsim, x) > U_{ij}(\succsim, y)$ for some ij. However, this is readily apparent because, by the continuity of $\succsim$, we can find ij such that $x \in B_i$, $y \in B_j$ and $\bar{B}_i \succ \bar{B}_j$ (Dugundji [2, (XI, 6.2)]).

We prove now the proposition; for notational economy the subindices ij are dropped.

For every $\succsim$, $\Psi(\succsim)$ is a nonempty, closed, convex set (nonemptiness follows from Debreu's theorem). We shall show that Ψ is a lower semicontinuous correspondence. Since every nonempty, closed, convex-valued lower semicontinuous correspondence from a metric space to a Fréchet space has a continuous selection (see Michael [7, (Theorem 3.2″)] where the statement is given for Banach rather than Fréchet spaces but, as Michael himself points out, the proof, which is not difficult, applies as well to Fréchet spaces), the proof will be concluded.

Let $\succsim_n \to \succsim$ and $u \in \Psi(\succsim)$. We need to find $u_n \in \Psi(\succsim_n)$ converging to u uniformly on compacta (both $\mathscr{P}$ and $\mathscr{F}(X)$ are separable).

Let $J \subset X$ be a compact set with $\bar{B}_i, \bar{B}_j \subset \operatorname{Int} J$. We need to show that for every $\varepsilon > 0$ there is N such that if $n > N$, then there exists a continuous p-utility for $\succsim_n$, u_n: $X \to R$ with $\max_{x \in J} |u(x) - u_n(x)| \leq \varepsilon$.

Choose a metric d for X.

Let $\varepsilon > 0$. Since J is compact, $u|J$ is uniformly continuous and, so, we can pick $\delta > 0$ such that $x, y \in J$ and $d(x, y) \leq \delta$ implies $|u(x) - u(y)| < \varepsilon/16$. Since $\succsim_n \to \succsim$ we can choose N such that if $n > N$ and $x \succsim_n y$, $x, y \in J$, then $d(x, x') \leq \delta$, $d(y, y') \leq \delta$ for some $x', y' \in J$ with $x' \succsim y'$.

For every n and $x \in X$ let $L_n(x) = \{z \in X | x \succsim_n z\}$, and, for every n, denote $X'_n = \{z \in X: L_n(z) \cap J \neq \emptyset\}$. For every n, define g_n: $X_n \to [0, 1]$ by $g_n(x) = \max\{u(z): z \in L_n(x) \cap J\}$ and then extend g_n to X by letting $g_n(x) = \inf\{u(z): z \in X_n\}$.

It is immediately verified that g_n satisfies: (i) g_n is a p-utility for $\succsim_n$, (ii) g_n is upper-semicontinuous, and (iii) $g_n(x) \geq u(x)$ for all $x \in J$. On the other hand, if $n > N$, we should have $g_n(x) \leq u(x) + \varepsilon/4$ for all $x \in J$; otherwise $u(y) \geq u(x) + \varepsilon/4$ for some $x, y \in J$ with $x \succsim_n y$ and we get $x' \succsim y'$ for some $x', y' \in X$ with $d(x, x') \leq \delta$, $d(y, y') \leq \delta$. This yields $u(x') \geq u(y')$ and, so, $u(x) \geq u(y) - \varepsilon/8$, a contradiction. We conclude that, if $n > N$, then $\sup_{x \in J} |g_n(x) - u(x)| \leq \varepsilon/4$.

From now on we consider a fix $n > N$; when there is no ambiguity we drop the subindex n. By Debreu's theorem there is a continuous utility v: $X \to [0, \varepsilon/4]$ for $\succsim$. Let $\bar{g} = g + v$; then $\bar{g}$ is an upper-semicontinuous utility for $\succsim_n$ such that $\sup_{x \in J} |\bar{g}(x) - u(x)| \leq \varepsilon/2$. Let $\{I_i\}$, $I_i \subset R$ be the collection of distinct, nondegenerate intervals of the form $[\lim_m \bar{g}(x_m), \bar{g}(x)]$ where $x_m \to x$. By definition, every I_i is closed.

LEMMA 1. *The intervals I_i are pairwise disjoint.*

PROOF. It suffices to show that if $[a, b] = [\lim \bar{g}(x_m), \bar{g}(x)]$, $x_m \to x$, $a \neq b$ and $y \in X$, then $\bar{g}(y) \notin [a, b)$. Let $y \in X$, $\bar{g}(y) \in [a, b)$. If $\bar{g}(y) \in (a, b)$ then, by the closedness of $\succsim_n$, $y \succsim x$ which contradicts $\bar{g}(y) < b$. If $\bar{g}(y) = a$, then we can assume by the preceding argument that $\bar{g}(x_m) \leq a$ for all m and we get again the contradiction $y \gtrsim x$.

It is an easy exercise to verify that every I_i can be realized by considering sequences x_m with $x_m \in J$. Therefore, length $I_i \leq \varepsilon$ for all i.

LEMMA 2. *Let* $\{I_i\}$ *be a collection of pairwise disjoint, closed, nondegenerate intervals* $I_i \subset R$ *such that length* $I_i \leq \varepsilon$ *for all* i. *Then there is a continuous nondecreasing function* $\xi: R \to R$ *such that* $\sup_t |\xi(t) - t| < \varepsilon/2$ *and* ξ *is constant on every* I_i.

PROOF. Let t_i be the midpoint of I_i and put $\xi(t) = t_i$ for $t \in I_i$. Observe that the ξ so defined on $\bigcup_i I_i$ extends continuously to $\overline{\bigcup_i I_i}$. For every maximal interval $G \subset R \setminus \overline{\bigcup_i I_i}$ extend, then, ξ linearly from the endpoints of G.

Let $\xi: R \to R$ be as in Lemma 2 and put $u_n = \xi \circ \bar{g}_n$. Since ξ is nondecreasing, u_n is a p-utility for $\gtrsim_n$. Moreover, $\sup_{x \in J} |u_n(x) - u(x)| \leq \varepsilon$. To verify that u_n is continuous let $x_m \to x$, $\lim_m \bar{g}_n(x_m) = t < \bar{g}_n(x)$. Since ξ is constant on $[t, \bar{g}_n(x)]$ we get $\lim_m u_n(x_m) = \xi(\lim_m \bar{g}_n(x_m)) = \xi(t) = \xi(\bar{g}_n(x)) = u_n(x)$. Therefore, the Proposition is proved.

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REFERENCES

[1] DEBREU, G., "Continuity Properties of Paretian Utility," *International Economic Review*, V (September 1964), 285–293.

[2] DUGUNDJI, J., *Topology* (Boston: Allyn and Bacon, 1966).

[3] HILDENBRAND, W., "On Economies with Many Agents," *Journal of Economic Theory*, II (June 1970), 161–188.

[4] ———, *Core and Equilibria of a Large Economy* (Princeton: Princeton University Press, 1974).

[5] KANNAI, Y., "Continuity Properties of the Core of a Market," *Econometrica*, XXXVIII (November, 1970), 791–815.

[6] ———, "Representability of Preferences by Utility Functions," Address given at the Berkeley Colloquium on Mathematical Economics, August, 1974.

[7] MICHAEL, E., "Continuous Selections, I," *Annals of Mathematics*, LXIII (1956), 361–382.

[8] MOUNT, K., AND S. REITER, "Continuous Representation of Preferences," (mimeographed), *Center for Mathematical Studies in Economics and Management Science*, Discussion Paper No. LXXXI (revised), Northwestern University, Evanston, Illinois (1974).

[9] ———, AND ———, "Construction of a Continuous Utility Function for a Class of Preferences," *Journal of Mathematical Economics*, III (December, 1976), 227–246.

[10] NEUEFEIND, W., "On Continuous Utility," *Journal of Economic Theory*, V (December, 1972), 174–179.

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References

[1] **Continuity Properties of Paretian Utility**
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http://links.jstor.org/sici?sici=0020-6598%28196409%295%3A3%3C285%3ACPOPU%3E2.0.CO%3B2-X

[5] **Continuity Properties of the Core of a Market**
Yakar Kannai
Econometrica, Vol. 38, No. 6. (Nov., 1970), pp. 791-815.
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http://links.jstor.org/sici?sici=0012-9682%28197011%2938%3A6%3C791%3ACPOTCO%3E2.0.CO%3B2-S

[7] **Continuous Selections. I**
Ernest Michael
The Annals of Mathematics, 2nd Ser., Vol. 63, No. 2. (Mar., 1956), pp. 361-382.
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