Journal of Mathematical Economics 4 (1977) 117–126. © North-Holland Publishing Company

ON THE EQUILIBRIUM PRICE SET OF AN EXCHANGE ECONOMY

Andreu MAS-COLELL*

University of California, Berkeley, CA, U.S.A.

Received August 1975, final version received October 1976

1. Introduction

Consider a pure exchange economy composed of a finite number of traders with strictly positive initial endowments and continuous, monotone, strictly convex preferences. It is well known that the intersection of the unit sphere with the set of price equilibria for such an economy is non-empty and compact. Under smoothness hypotheses on preferences and generic (i.e., non-degeneracy) conditions further (fixed-point index type) strong restrictions on the equilibrium price set can be derived; this was done by Dierker (1972). In this paper we provide converses to the above statements, i.e., we prove that, imprecisely speaking, if the number of commodities is greater than two, then every pattern of equilibria compatible with the above referred to properties can arise from an economy in the class we consider.

It is obvious that this characterization of the equilibrium price set problem [already studied by Sonnenschein (1972)] is closely related to the problem of characterizing excess demand functions defined on compact sets of prices. In fact, the solution to the first problem has had to wait for a solution to the second to be found; this has been accomplished only recently; see Sonnenschein (1973), Mantel (1974, 1976), Debreu (1974). Our proof here amounts to a refinement of the one by Debreu (1974) leading to a sharper version of his result, sharp enough for the purposes of this paper.

2. Statement of results

The commodity space is R^l; $R^l_+ = \{p \in R^l : p \geqq 0\}$, $P = \{p \in R^l : p \gg 0\}$, $\hat{P} = R^l_+ \setminus \{0\}$, $S = \{p \in P : \|p\| = 1\}$, $S_\varepsilon = \{p \in S : p^i \geqq \varepsilon \text{ for all } i\}$; for $p \in R^l$, $T_p = \{x \in R^l : px = 0\}$; $e = (1, \ldots, 1) \in R^l$.[1]

*I am very indebted to G. Debreu, L. Shapley, and R. Mantel for their comments. Financial support from NSF grants SOC72–05551A02 and SOC73–05650A01 is gratefully acknowledged.

[1] $\| \ \|$ denotes the euclidean norm. Subscripts indicate vectors and superscripts components of vectors. The boundary, interior, and closure of a set A are denoted, respectively, by ∂A, Int A, and $\bar{A}$. Neighborhood always mean open neighborhood.

In our context an *excess demand function* $f: S \to R^l$ is a continuous function satisfying:

(W) $pf(p) = 0$ for every $p \in S$,
(BB) there is $k \in R$ such that for every $p \in S$, $f(p) > ke$,
(BC) if $p_n \to p \in \partial S$, $p_n \in S$, then $\lim n \|f(p_n)\| = \infty$.

Note that the sum of two excess demand functions is an excess demand function.

For every excess demand function f let the *set of equilibria* be $E_f = \{p \in S: f(p) = 0\}$; under the assumptions made E_f is non-empty and compact [see Debreu (1970)].

All the economies to be considered will be of pure exchange and of the form $\mathscr{E} = \{(\succsim_i, \omega_i, R^l_+)\}^l_{i=1}$, where R^l_+ is the consumption set, $\succsim_i$ is a continuous, monotone, strictly convex preference relation on R^l_+, and $\omega_i \in P$. To avoid repetition, from now on an *economy* is defined to have those properties (hence, economies always have l participants).

Given an economy $\mathscr{E}$, an (aggregate) excess demand function f corresponding to $\mathscr{E}$ (we also say that $\mathscr{E}$ generates f) is defined in the usual manner, i.e.,

$$f(p) = \sum_{i=1}^{l} \{x \in R^l_+ \mid px \leqq p\omega_i \text{ and } py \leqq p\omega_i,\ y \geqq 0,$$
$$\text{implies } x \succsim_i y''\} - \sum_{i=l}^{l} \{\omega_i\}.$$

It is known that f will in fact be an excess demand function, i.e., satisfy (W), (BB), and (BC) – see Arrow and Hahn (1971, ch. 4, theorem 8).

It is not known if every excess demand function can be generated by an economy. Following the work of Sonnenschein (1972, 1973), and Mantel (1974), Debreu (1974) showed that for any excess demand f and $\varepsilon > 0$ there is an excess demand f^* such that $f^* |S_\varepsilon = f|S_\varepsilon$ and f^* is generated by an economy [of course, Debreu does not impose conditions (BB) and (BC)]. There is no presumption, however, to the effect that $E_{f^*} \subset E_f$, which makes the result fall just short of what we need for the characterization of the equilibrium price set problem. The main result of this paper is an extension of Debreu's theorem which renders it readily usuable for our purposes.

Theorem. If $f: S \to R^l$ is an excess demand function and $\varepsilon > 0$, then there is $\mu < \varepsilon$ and an excess demand function f^* such that $f^*|S_\mu = f|S_\mu$, $E_{f^*} = E_f \subset S_\mu$ and f^* is generated by an economy.

The following corollary is immediate:

Corollary 1. Let $A \in S$ be a non-empty, compact set; then there is an excess demand f generated by an economy $\mathscr{E}$ such that $A = F_f$.

Proof. Pick some $z \in A$; for every $p \in S$ let $h(p)$ be the projection of $z-p$ on T_p. Define $\alpha(p) = \min\{\|p-q\|: q \in A\}$; $p+\gamma(p)h(p) \in \partial P$. The function $\gamma(p)$ is continuous on S since $S \subset P$, P is open, convex, and h is continuous. Define $f(p) = \alpha(p)\gamma(p)h(p)$. It is easily checked that f is an excess demand function and $f(p) = 0$ if and only if $p \in A$. Apply, then, the theorem. ∎

As a characterization of the equilibrium price set result, Corollary 1 is not yet very satisfactory; on the one hand, many of the equilibria in $A = E_f$ will typically be 'degenerate'; on the other hand, plenty of relevant information about the nature of an equilibrium price vector is contained not only in its position in S but also in the form of the linear derivative map of the excess demand function at this price vector.[2]

Let f be a C^1 excess demand function. At every $p \in E_f$ the map $Df(p): T_p \to R^l$ carried T_p into itself. So, from now on we shall regard $Df(p)$ as a map from T_p into T_p (if $p \in E_f$!). If f satisfies

(R) $\quad |Df(p)| \neq 0 \quad \text{for every} \quad p \in E_f,$[3]

then E_f is finite, and Dierker (1972) has shown that

$$\sum_{p \in E_f} \operatorname{sign} |Df(p)| = (-1)^{l+1}.$$

It follows from results of Debreu (1970) that condition (R) is generic (i.e., it holds for 'almost every' economy, where 'almost every' is appropriately defined) in the class of economies having C^1 demand functions.

The next corollary shows, in a particularly strong form, that (at least, if the number of commodities is greater than two) the fixed-point-index condition of Dierker exhausts the restrictions on equilibrium prices derivable from the underlying economic model of this paper.

Corollary 2. Suppose we are given a compact set $K \subset S$ and a function $f: K \to R^l$ such that, letting $E = \{p \in K: f(p) = 0\}$: (i) K is an $l-1$ smooth, compact manifold (for example, K may be a union of a finite collection of pairwise disjoint closed discs); (ii) $S \setminus K$ is connected; (iii) f is C^1; (iv) $pf(p) = 0$ for every $p \in K$; (v) $|Df(p)| \neq 0$ for $p \in E$; and (vi) $E \neq \phi$, $E \subset \operatorname{Int} K$, and $\sum_{p \in E} \operatorname{sign} |Df(p)| = (-1)^{l+1}$. Then there is an excess demand function f^* such that f^* is generated by an economy $f^* \mid K = f$ and $f^*(p) = 0$ only if $p \in K$.

[2]In contrast, the properties of the derivative map of excess demand at non-equilibrium prices are very sensitive to such irrelevant matters as normalization.

[3]$|Df(p)|$ is the determinant of $Df(p)$; sign $|Df(p)| = +1, 0, -1$, according to if $|Df(p)| > 0, = 0, < 0$; f is C^1 on a closed set K if it can be extended to a C^1 function in a neighborhood of K.

Proof. Let $B \subset S$ be a $(l-1)$ ball (up to diffeomorphism) which contains in its interior $K \cup \{(1/l^{\frac{1}{2}})e\}$. Define $g\colon \partial B \to R^l$ by $g(p) = e-(pe)p$. Parametrizing B by the first $(l-1)$ coordinates, we can regard B as an open subset of R^{l-1}; analogously, we replace $f(p)$, $g(p)$ by their projections on the first $l-1$ coordinates, which, abusing notation, we denote by the same symbol. Let $W = \overline{B \backslash K}$; then W is a compact, connected, oriented $(l-1)$ manifold and $\partial W = \partial B \cup \partial K$. Define $F\colon \partial W \to S^{l-2}$ by letting $F(p)$ equal $f(p)/\|f(p)\|$ if $p \in \partial K$ and $g(p)/\|g(p)\|$ if $p \in \partial B$. Clearly, the proof is finished if the map F can be extended from ∂W to W or equivalently [this is Hopf's theorem; see Guillemin and Pollack (1974, p. 145)] if the degree of F is zero. But

$$\begin{aligned}\deg F &= -\deg f\,|\,\partial K + \deg g\,|\,\partial B \\ &= -\sum_{p \in E} \operatorname{sign}|Df(p)| + \deg g \mid \partial B \\ &= -(-1)^{l+1} + (-1)^{l+1} = 0. \quad \blacksquare\end{aligned}$$

No attempt has been made to state the most general possible version of the corollary. The requirement that $S \backslash K$ be connected has strong implications only for $l = 2$. It is clear in this case that if we represent prices in a line, then the equilibrium prices associated with positively sloped excess demand should alternate with the negatively sloped ones.

3. Proof of the theorem

(1) For every $p \in S$ let $g_e(p) = e-(pe)p$. The fact proved in the following lemma is well known:

Lemma 1. If f is an excess demand function there is $\delta > 0$ such that $f(p)g_e(p) > 0$ whenever $p \in S \backslash S_\delta$.

Proof. By (BB) and (BC) there is $\delta > 0$ such that if $p \in S \backslash S_\delta$, then

$$\sum_{i=1}^{l} f^i(p) > l.$$

But then

$$f(p)g_e(p) = \sum_{i=1}^{l} f^i(p) > 0. \quad \blacksquare$$

In view of Lemma 1 the theorem is an obvious corollary of the next proposition; its statement is somewhat technical but it yields results which are important in applications [see Mas-Colell (1977)]; among other things it makes clear that the norm of the initial endowments of the economy to be constructed only depends on the norm of the values of excess demand (and not, for example, on the norm of derivatives were they to exist).

Proposition. For every $\varepsilon > 0$ *there is* $0 < \mu < \varepsilon$ *and a function* $k: (0, \infty)$ *such that if* $f: S \to R^l$ *is a continuous function satisfying* $pf(p) = 0$ *for every* $p \in S$ *and* $f(p)g_e(p) > 0$ *for* $p \in S \backslash S_\varepsilon$ *then there exists an excess demand function* f^* *satisfying: (i)* $f^* | S_\varepsilon = f | S_\varepsilon$, *(ii)* $f^*(p)g_e(p) > 0$ *for* $p \in S \backslash S_\varepsilon$, *(iii)* f^* *is generated by an economy* $\{(R_+, \succsim_i, \omega_i)\}_{i=1}^l$ *with, for all i,* $\|\omega_i\| \leqq l(2+k(r))$ *for any* $r \geqq \operatorname{Max}_{p \in S_\mu} \|f(p)\|$, *and* $\succsim_i$ *continuous, monotone, strictly convex.*

Notice that in the proposition μ and k depend only on ε. No attempt is made to obtain a sharp bound for $\|\omega_i\|$.

(2) We proceed now to prove the proposition:

We state first some properties of functions from P, or S, to R.

(I) $v: P \to R$ is proper, strictly convex, and C^2. Moreover, $p_n \to p \in \partial P$ implies $\|\nabla v(p_n)\| \to \infty$ and there is an $r > 0$ such that if $\|p\| \leqq 1$, then $p\nabla v(p) \leqq r$ and $\nabla v(p) \leqq 0$.

(II) $v: S \to R$ is the restriction to S of a function on P satisfying (I); see fig. 1.

Note that the sum of functions satisfying (I) [resp. (II)] satisfies (I) [resp. (II)]. A function is proper if inverse images of compact sets are compact.

Let $v: S \to R$ satisfy (II); then v is minimized in at most one point and for every $r \in v(S)$ the cone spanned by $v^{-1}((-\infty, r])$ is strictly convex [indeed, let v be defined on P and satisfy (I); if $v(p), v(q) \leqq r, p, q \in S, p \neq q$, then, for $0 < t < 1$, $\|tp+(1-t)q\| < 1$ and $v(tp+(1-t)q) < r$; since v is decreasing on $\{q': \|q'\| < 1\}$, $v((1/\|tp+(1-t)q\|)\ (tp+(1-t)q)) < r$]. Therefore, if v satisfies (II), then for all $p \in S$, $\nabla v(p) \in T_p$ is normal to a supporting hyperplane at p of the cone spanned by $v^{-1}((-\infty, v(p)])$; hence, $v(p) > v(q)$ implies $\nabla v(p)(q-(pq)p) < 0$. See fig. 1.

Observe that if $v: S \to R$ satisfies (II), then $p_n \to p \in \partial S$ implies $\|\nabla v(p_n)\| \to \infty$ because, letting $v = v' \mid S$ for a $v': P \to R$ satisfying (I), $\nabla v(p_n) = \nabla v'(p_n) - (p_n \nabla v'(p_n))p_n$.

Lemma 2. Let $v: S \to R$ *satisfy* (II). *Suppose that* $f: S \to R^l$ *is an excess demand function such that, for all* $p \in S$, $f(p) > \alpha e$ (where $\alpha > -\infty$) and $f(p) = -\beta(p) \nabla v(p)$ *where* $\beta(p)$ *is positive and uniformly bounded away from* 0. *Then f is generated by a* $(\succsim, \omega, R_+^l)$ *where* $\succsim$ *is a continuous, monotone, strictly convex preference relation on* R_+^l *and* $\omega \in P$, $\|\omega\| \leqq l\,|\alpha|$.

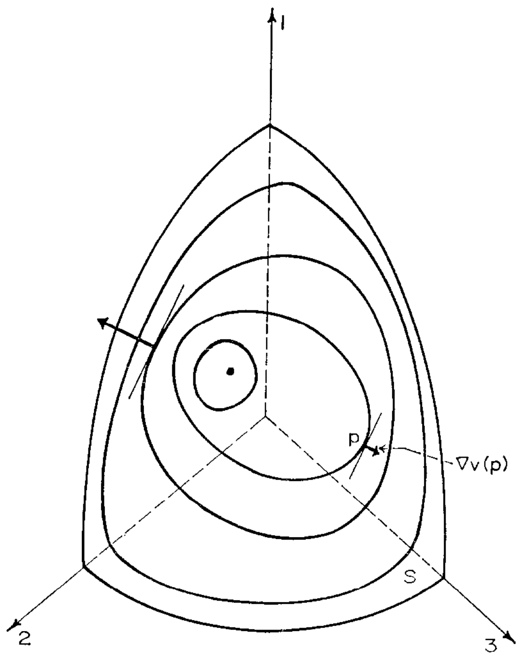

Fig. 1

Proof. A careful reading of Debreu's proof (1974) would validate Lemma 2. For the sake of completeness, and skipping details, we outline here a somewhat different demonstration.

Obviously, we can take $\omega = 0$ and look for a $\succsim$ defined on R^l. We will show only the existence of a continuous and monotone $\succsim$ generating f; $\succsim$ could then be 'convexified' by using standard methods [see Hildenbrand (1975, 1.1, problem 7)]; in order to have $\succsim$ strictly convex one would have to apply, in a second step, Debreu's (1974) technique.

There is a unique $\bar{p} \in S$ for which $f(\bar{p}) = 0$. Divide R^l in three regions: $\hat{P}$, $A = \{x \mid x \notin \hat{P},\ \bar{p}x > 0\}$, $B = \{x \mid \bar{p}x \leqq 0\}$. Let u: $S \to [0, 1]$ be a continuous function such that $u(\bar{p}) = 0$ and $u(p) \geqq u(q)$ if and only if $v(p) \geqq v(q)$.

By the properties of v there is for every $x \in A$ a unique $p(x) \in S$ such that x belongs to the positive ray spanned by $f(p(x))$. The function $x \mapsto p(x)$ is continuous. Let $x_n \in A$, $x_n \to x$, $x \notin A$, then $p(x_n) \to \partial S$ if $x \in \hat{P}$ or $p(x_n) \to \bar{p}$ if $x \in B$. One also sees that $p \in S$, $x \in A$, $px \leqq 0$, and $x \notin \{tf(p)\colon t \geqq 0\}$ implies $u(p(x)) < u(p)$ (see fig. 2).

Define a function $\xi\colon R^l \to R$ as follows: for $x \in A$ write $x = tf(p(x))$ and take $\xi(x) = tu(p(x))$ if $t \leqq 1$ or $\xi(x) = u(p(x)) - \|x - f(p(x))\|$ if $t > 1$; for $x \in B$ take $\xi(x) = -\|x\|$; for $x \in \hat{P}$ take $\xi(x) = \min x^i$.

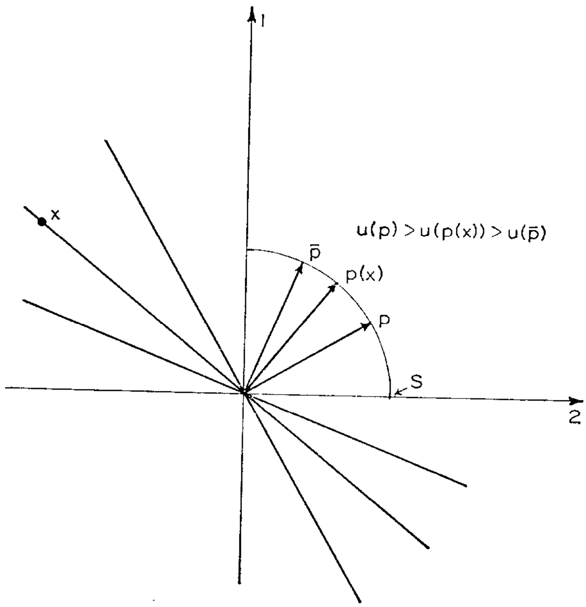

Fig. 2

The function ξ satisfies: (i) ξ is continuous [this is so because u is bounded, $u(\bar{p}) = 0$, f is continuous, and $\lim \|f(p_n)\| \to \infty$ as $p_n \to \partial S$]; (ii) ξ has no local maxima; (iii) for any $p \in S$, if $px \leqq 0$ and $x \neq f(p)$, then $\xi(x) < \xi(f(p))$. Let $\hat{\xi}(x) = \max \{\xi(y): y \leqq x\}$; the function $x \mapsto \hat{\xi}(x)$ is well defined and, by (i) above, continuous. By (ii), $x \gg y$ implies $\hat{\xi}(x) > \hat{\xi}(y)$. It is, moreover, not difficult to check that, for all $p \in S$, $px \leqq 0$ and $x \neq f(p)$ implies $\hat{\xi}(x) > \hat{\xi}(f(p))$. Hence the preference relation given by $x \succsim y$ if and only if $\hat{\xi}(x) > \hat{\xi}(y)$ is continuous, monotone, and generates f. ▮

For any $a \in P$ define $u_a: S \to R$ by $u_a(p) = (1/2\|a\|)\ \|p-a\|^2$; if $a \geqq e$, u_a satisfies (II). Observe that $-\nabla u_a(p) = (1/\|a\|)\ (a-(pa)p)$; see fig. 3.

Lemma 3. For any $a \in P$ with $a \geqq e$ and compact K with $(1/\|a\|)\ a \in K$, there is a function v: $S \to R$ satisfying (II) *and such that*: (*i*) $v \mid K = u_a$ *and* $v(p) \geqq u_a(p)$ *for all* p; (*ii*) $\nabla v(p) \leqq 2e$ *for all* p; *see fig*. 4.

Proof. Define ξ: $P \to R$ by $\xi(p) = -(1/l)\,(\sum_{i=1}^{l} \log p^i)$ and $\hat{u}_a: P \to R$ by $\hat{u}_a(p) = (1/2\|a\|)\ \|p-a\|^2$ (so, $u_a = \hat{u}_a \mid S$). Both ξ and $\hat{u}_a$ satisfy (I). Clearly, for $p \in S$, $\nabla(\xi \mid S)(p) = p-(1/l)(1/p^1, \ldots, 1/p^l) < 2e$.

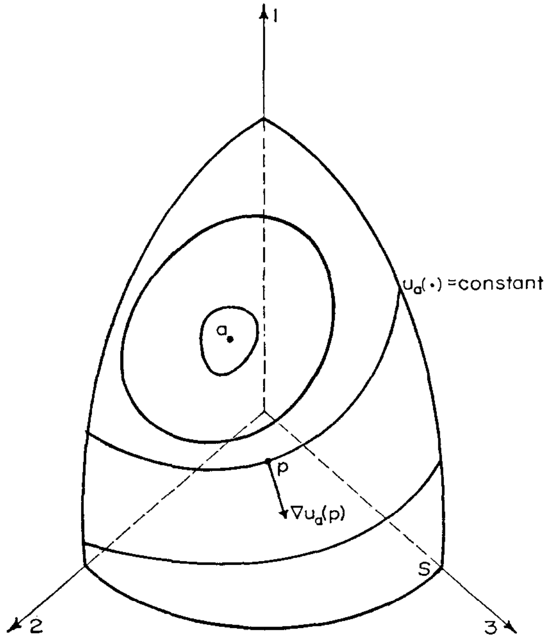
1
a
$u_a(\cdot)$ =constant
p
$\nabla u_a(p)$
S
2
3

Fig. 3

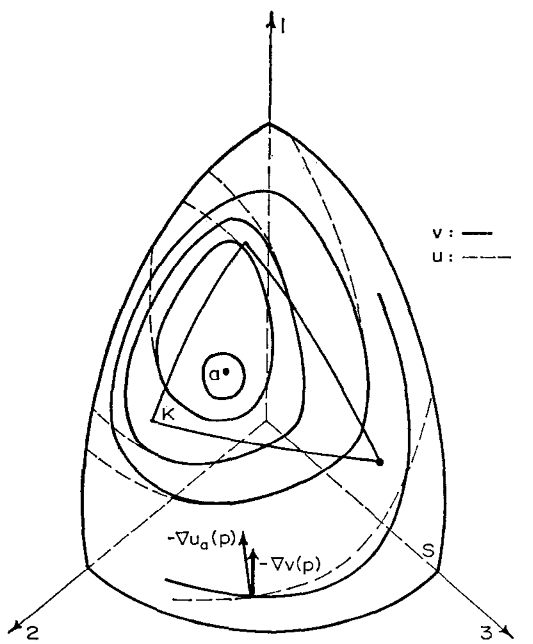
1
v :
u :
a
K
$-\nabla u_a(p)$
$-\nabla v(p)$
S
2
3

Fig. 4

Let $B \subset S$ be a neighborhood of K with $\bar{B} \subset S$ and define γ: $P \to R$ by $\gamma(p) = \max\{0, \xi(p) - s\}$ where s is chosen large enough to have $\gamma(p) = 0$ on a neighborhood of $\bar{B}$; γ is convex and we can smooth it out around $\gamma^{-1}(s)$ so as to obtain a convex function $\hat{\gamma}$; $P \to S$ which satisfies (I) – except for strict convexity – and $\hat{\gamma}(p) \geqq 0$, $\nabla(\hat{\gamma} \mid S)(p) \leqq e$ for $p \in S$, $\hat{\gamma}(p) = 0$ for $p \in B$. Take then $v = \hat{u}_a + \hat{\gamma} \mid S = u_a + \hat{\gamma} \mid S$; it is clear that v satisfies (II) and $\nabla v(p) = (1/\|a\|)\,((pa)p - a) + \nabla(\hat{\gamma} \mid S)(p) \leqq 2e$. ∎

Let $\varepsilon > 0$. This ε will remain fixed for the rest of the proof. For a $\delta > 0$ choose $a_i \in S$ such that $a_i^j \geqq e$ and $(1/\|a^i\|)\, a_i^j = \delta$ for every $j \neq i$; δ is chosen small enough to guarantee that $S_{\varepsilon/2}$ is contained in the interior of the convex cone $\Gamma \subset R^l$ spanned by $\{a_1, \ldots, a_l\}$. Choose $\mu > 0$ such that $\Gamma < S \subset S\mu$.

For any $r > 0$ let $B_r \subset R^l$ be the closed r ball. Given $r > 0$ pick $\theta > 0$ such that, for every $x \in B_r$ and $p \in S_{\varepsilon/2}$, $x + \theta p \in \Gamma$. Such θ does obviously exist (remember $S_{\varepsilon/2} \subset \operatorname{Int} \Gamma$). We put $k(r) = \theta + r$.

For every i pick a function v_i: $S \to R$ satisfying the conclusions of Lemma 3 with respect to a_i and $\Gamma \cap S$. The function $\sum_{i=1}^{l} u_{a_i}$ attains its minimum value at the unique point $p = (1/l^{\frac{1}{2}})\, e \in S$. Since $v_i \geqq u_{a_i}$ and $v_i\,|S_\varepsilon = u_{a_i}|\, S_\varepsilon$ we get $\sum_{i=1}^{l} v_i(p) > \sum_{i=1}^{l} v_i((1/l^{\frac{1}{2}})e)$ for all $p \neq (1/l^{\frac{1}{2}})\, e$; since $\sum_{i=1}^{l} v_i$ satisfies (II), we conclude that $\sum_{i=1}^{l} \nabla v_i(p)(e - (pe)p) = \sum_{i=1}^{l} \nabla v_i(p) g_e(p) < 0$ for all $p \neq (1/l^{\frac{1}{2}})e$.

Let f be an excess demand function satisfying $f(p)g_e(p) > 0$ for $p \in S \backslash S_\varepsilon$. To obtain a f^* fulfilling the conclusions of the proposition we proceed as in Debreu (1974).

Pick $\bar{r} > \operatorname{Max}_{p \in S_\mu} \|f(p)\|$. Then, for every $p \in S_{\varepsilon/2}$, $f(p) + (k(\bar{r}) - \bar{r})p \in \Gamma$. Since the vectors $\{a_1, \ldots, a_l\}$ are linearly independent we can write, for $p \in S_{\varepsilon/2}, f(p) + (k(\bar{r}) - \bar{r})p = \sum_{i=1}^{l} \beta_i(p) a_i$, $\beta_i(p) > 0$ in a unique and continuous fashion; moreover, for all i, $\|\beta_i(p)a_i\| \leqq \|f(p) + (k(\bar{r}) - \bar{r})p\| \leqq k(\bar{r})$. Observe that $-\nabla u_{a_i}(p) = a_i - (pa_i)p$ is nothing but the perpendicular projection of a_i on T_p. So, projecting perpendicularly into T_p we get, for $p \in S_{\varepsilon/2}$, $f(p) = \sum_{i=1}^{l} \beta_i(p)(-\nabla v_i(p))$ and, of course, $\|\beta_i(p)\nabla v_i(p)\| \leqq \|\beta_i(p)a_i\| \leqq k(\bar{r})$ for all i.

Let η: $S \to [0, 1]$ be C^2 and such that $\eta(p) = 1$ for $p \in S \backslash S_{\varepsilon/2}$ and $\eta(p) = 0$ for $p \in S_\varepsilon$. Define f_i: $S \to R^l$ by $f_i(p) = -(\eta(p) + (1 - \eta(p))\beta_i(p))\nabla v_i(p)$. Let $f^* = \sum_{i=1}^{l} f_i$ and note that every f_i, and so f^*, are excess demand functions.

By Lemmas 2 and 3 every f_i can be generated by a $(\succsim, \omega, R_+^l)$ with $\succsim$ continuous, monotone, strictly convex and with $\|\omega\| \leqq l \max\{k(\bar{r}), 2\} \leqq l(k(s(\bar{r})) + 2)$. By construction $f^*(p) = f(p)$ for $p \in S_\varepsilon$. If $p \in S \backslash S_\varepsilon$, then either $p \in S \backslash S_{\varepsilon/2}$ in which case $-f^*(p)g_e(p) = \sum_{i=1}^{l} \nabla v_i(p) g_e(p) < 0$, or $p \in S_{\varepsilon/} \backslash S_{\varepsilon 2}$ in which case we also have $f^*(p)g_e(p) = -\eta(p)(\sum_{i=1}^{l} \nabla v_i(p) g_e(p)) + (1 - \eta(p))\, f(p)g_e(p) > 0$ because both terms of the sum are positive. Q.E.D.

References

Arrow, Kenneth and Frank Hahn, 1971, General competitive analysis (Holden-Day, San Francisco, CA).

Debreu, Gerard, 1970, Economies with a finite set of equilibria, Econometrica 38, 387–392.
Debreu, Gerard, 1974, Excess demand functions, Journal of Mathematical Economics 1, 15–23.
Dierker, Egbert, 1972, Two remarks on the number of equilibria of an economy, Econometrica 40, 951–955.
Guillemin, Victor and Alan Pollack, 1974, Differential topology (Prentice-Hall, Englewood Cliffs, NJ).
Hildenbrand, Werner, 1975, Core and equilibria of a large economy (Princeton University Press, Princeton, NJ).
Mantel, Rolf, 1974, On the characterization of aggregate excess demand, Journal of Economic Theory 7.
Mantel, Rolf, 1976, Homothetic preferences and community excess demand functions, Journal of Economic Theory 12, 197–202.
Mas-Colell, Andreu, 1977, Regular, nonconvex economies, Econometrica, 45.
Sonnenschein, Hugo, 1972, Market excess demand functions, Econometrica 40, 549–563.
Sonnenschein, Hugo, 1973, Do Walras identity and continuity characterize the class of community excess demand functions?, Journal of Economic Theory 6, 345–354.