NOTES ON THE SMOOTHING OF AGGREGATE DEMAND*

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1. Introduction

The problem of smoothing (of excess demand) by aggregation was posed by Debreu (1972) and Hildenbrand (1974). It can be informally described as: 'find natural frameworks in which even if preferences are not convex, one can reasonably expect that if there are a continuum of consumers and their characteristics are disperse, in some appropriated sense, then aggregate demand will be a continuous, or C^1, function.' Contributions have been made by Sondermann (1975a,b), Chichilnisky (1974), Ichiishi (1976), Araujo (1974), Mas-Colell (1973), Mas-Colell and Neuefeind (1977). Sondermann's work is especially important for these notes.

An economy with a continuum of traders is formalized as follows: there is given a metric space $\mathscr{A}$ of consumer characteristics; $\mathscr{A}$ will consist, for example, of preferences–endowments pairs or utility functions–endowments pairs; by using, for example, utility functions we are free to assume when necessary that $\mathscr{A}$ possess a C^r type metric or even a C^r linear structure. There is given a space $E \subset R^m$ of parameters (or 'names') and a (Borel) probability measure v on R^m with compact support contained in E and, for definiteness, absolutely continuous with respect to Lebesgue measure on R^m. A (continuum) *economy* will then be specified by E, v, and suitably restricted function $f : E \to \mathscr{A}$, i.e., an assignment of characteristics to parameters.

Two approaches have been proposed for the definition and topologization of a space of economies. The first we will call the *distribution approach* and is due to Aumann (1964) and Hildenbrand (1974); since for the purposes of equilibrium theory only the distribution induced by f on $\mathscr{A}$ is relevant, two triples (E, v, f), (E', v', f') are viewed as close if mean endowments are close and if the measures induced in $\mathscr{A}$ are close in the weak convergence for

measures. Note that the space of parameters becomes an inessential construct. The second approach we will call *parametric* and is due to Araujo (1974) and Sondermann (1975a). The space of parameters E and the measure v are kept fixed and two economies $f, f : E \rightarrow \mathscr{A}$ are viewed as C^r close if they are C^r close as functions. Of course, if $r \geq 1$, the space of characteristics under consideration has to possess some sort of C^r linear structure. Clearly, the parametric approach leads to a stronger topology than the distribution approach.

For the purposes to which the concept of a continuum economy has been so far put to use, the distribution approach has been the appropriate one [see Hildenbrand's book (1974)]. It is the case, however, that the distribution topology on economies is quite weak and that the aggregation properties concerning us are quite delicate; if we strive for results asserting that the continuity, or smoothness, of aggregate demand is a stable property, i.e., it is preserved (except in borderline cases) under small perturbations of the function f, then consideration of the parametric approach is inescapable. Of course, in the distribution approach, density theorems are easier to come by. While to have them is better than nothing, the density without the stability of an aggregation property can hardly be considered as a very successful solution to the smoothing by aggregation problem.

Let us consider first the continuity of aggregate (or mean) excess demand. It is well known [see Hildenbrand (1974)] that under standard conditions mean excess demand will be an upper hemicontinuous correspondence of prices. Hence, finding sufficient conditions for the continuity of mean demand is tantamount to finding sufficient conditions for the uniqueness of mean demand at every price.

In the parametric context, Dieter Sondermann has given a solution to the problem of the following nature: let X be the consumption set of the consumers in the economy (for example, $X = R^l_+$) and $U(x, a)$ a C^1 function of commodity bundles $x \in X$ and parameters $a \in E$. For fixed $a \in E, U(\cdot, a)$ is interpreted as the utility function of a. To focus matters on essentials, we assume that all consumers have the same initial endowments. If U satisfies the (transversality) condition, which we will call the Sondermann condition, 'if $U(x, a) = U(y, a)$, $x \neq y$, then $\partial_a(U(x, a) - U(y, a)) \neq 0$', then aggregate mean demand is a singleton for every price. In section 2 of these notes we point out:

(a) Without direct reference to the demand problem, the Sondermann condition is sufficient for the uniqueness of mean maximizers under quite general conditions. Further, our sufficiency proof is completely elementary.

(b) The Sondermann condition is a strong one and in the parametric approach is not even C^0 dense. The problem of finding (if they exist)

weaker transversality-like conditions leading, for some r (most likely, r = 2), to a C^r open-dense set of functions with unique mean maximizers remains unanswered.

The Sondermann condition is dense (but not open) in the distribution approach. We could use this to establish the distribution-density of economies having continuous excess demand, a fact which was proved in Mas-Colell and Neuefeind (1977) by different methods.

Let us now consider the smoothness properties of mean demand. In section 3, we point out:

(c) Even requiring, as it should be done, that v possess a smooth density, the Sondermann condition is not sufficient for the smoothness (C^1) of mean demand. The counterexample is very simple. It would appear that it is not reasonable to expect a C^1 mean excess demand. It is very doubtful that in the parametric approach the C^1 property be, for some r, C^r open-dense or even dense. In fact, no significant sufficient condition (parametric-open or not) is known to yield smooth demand, not even for very simple sorts (indivisible bundles of commodities) of consumption sets.

(d) Economies with C^∞ mean excess demand are dense in the distribution approach. In view of (c), this is the stronger result available in the area. The strategy of our proof proceeds by approximating the given commodity space by one having all commodities but one available only in discrete amounts and then proving the theorem in this new context.

2. Uniqueness of mean maximizers

Let X be a topological space. The space of parameters will be E, an open set of $\mathbf{R}^n$, $1 \leq n \leq \infty$. $\mathbf{R}^\infty$ is the countable product of the reals, $\mathbf{R}$, with the product topology. Finally, v will denote a Borel probability measure on $\mathbf{R}^n$.

We will make use of the following assumptions:

Assumption 1. $(X \times X) \backslash \varDelta$ is a Lindelöf space, where $\varDelta = \{(x, y) \in X \times X : x = y\}$.

Assumption 2. $F : X \times E \to \mathbf{R}$ is a continuous function.

Assumption 3. For every i, $x \in X$ and $a \in E$, $\partial_{a_i} F(x, a)$ exists and depends continuously on x and a.

Assumption 4. v is a product probability measure, each factor being absolutely continuous with respect to Lebesgue measure.

The basic hypothesis is:

Sondermann Condition (SC). If $F(x,a)=F(y,a)$, $x \neq y$, then $\partial_{a_i}(F(x,a)-F(y,a))$ $\neq 0$ for some i.

Theorem 1.   If Assumptions 1–4 and the Sondermann Condition hold, then, for v-a.e., $a \in E$, the function $F(\cdot,a):X \to R$ has at most one maximizer.

Proof.   The proof will be organized in four steps.

(1) For every $a \in E$ and non-empty, open set $U \subset X$, let $v_U(a)=\sup\{F(x,a): x \in U\}$. Because of the continuity of F, the possibly extended real valued function $a \mapsto v_U(a)$ is lower semicontinuous, hence, measurable.

(2) Let $x,y \in X$, $\bar{a} \in E$ be such that $F(x,\bar{a})=F(y,\bar{a})$, $x \neq y$. By the SC there is i such that $\partial_{a_i}(F(x,\bar{a})-F(y,\bar{a})) \neq 0$. Without loss of generality we take $\partial_{a_i}(F(x,\bar{a})-F(y,\bar{a}))>0$. For any $a \in E$ define $I_a^i = \{a + \lambda e_i : \lambda \in R\}$ where $e_i^i=1$ and $e_i^j=0$ for $j \neq i$. We claim:

> There are open sets $x \in U_x$, $y \in U_y$, $\bar{a} \in A$ such that for
> every $a \in A$ and open sets $x \in U_x' \subset U_x$, $y \in U_y' \subset U_y$, we
> have $\#\{a' \in I_a^i \cap A : v_{U_x'}(a')=v_{U_y'}(a')\} \leq 1$.   (*)

Indeed, suppose that (*) is false. Take the nets, directed by inclusion, of open neighbourhoods of $x,y,\bar{a}$, denoted $U_{x,\beta}$, $U_{y,\beta}$, A_β. Then, for every β, there are $a_\beta, a_\beta' \in A_\beta$ and $U_{x,\beta}' \subset U_{x,\beta}$, $U_{y,\beta}' \subset U_{y,\beta}$ such that

$$a_\beta - a_\beta' = \lambda_\beta e^i, \qquad \lambda_\beta > 0,$$

and

$$v_{U_{x,\beta}'}(a_\beta) = v_{U_{y,\beta}'}(a_\beta), \qquad v_{U_{x,\beta}'}(a_\beta') = v_{U_{y,\beta}'}(a_\beta').$$

Hence, there exists $y_\beta \in U_{y,\beta}'$, $x_\beta \in U_{x,\beta}'$ such that

$$F(y_\beta, a_\beta) + \lambda_\beta^2 \geq F(z, a_\beta) \quad \text{for all} \quad z \in U_{x,\beta}',$$

and

$$F(x_\beta, a_\beta') + \lambda_\beta^2 \geq F(z, a_\beta') \quad \text{for all} \quad z \in U_{y,\beta}'.$$

In particular,

$$F(x_\beta, a_\beta) - F(y_\beta, a_\beta) \leq \lambda_\beta^2,$$

and

$$-(F(x_\beta, a'_\beta) - F(y_\beta, a'_\beta)) \leqq \lambda_\beta^2.$$

So,

$$(1/\lambda_\beta)((F(x_\beta, a_\beta) - F(y_\beta, a_\beta)) - (F(x_\beta, a'_\beta) - F(y_\beta, a'_\beta)) \leqq 2\lambda_\beta.$$

By the Mean Value Theorem $\partial_{a_i}(F(x_\beta, a''_\beta) - F(y_\beta, a''_\beta)) \leqq 2\lambda_\beta$ for some a''_β in the segment $[a'_\beta, a_\beta]$. But $x_\beta \to x$, $y_\beta \to y$, $a_\beta, a'_\beta \to \bar{a}$. Hence, by continuity, $\partial_{a_i}(F(x, \bar{a}) - F(y, \bar{a})) \leqq 0$ which contradicts our hypothesis and establishes the claim.

(3) We claim now:

> For any $\bar{a} \in E$ and $x, y \in X$, $x \neq y$, there are open sets $x \in U_x, y \in U_y$, $\bar{a} \in A \subset E$ such that for any open sets $x \in U'_x \subset U_x$, $y \in U'_y \subset U_y$, we have $v(\{a \in A : v_{U'_x}(a) = v_{U'_y}(a)\}) = 0$.   (**)

By the continuity of F this is true if $F(x, \bar{a}) \neq F(y, \bar{a})$. If $F(x, \bar{a}) = F(y, \bar{a})$, (**) follows from (*) and Fubini's theorem. (Note that by step 1 the set $\{a \in A : v_{U'_x}(a) = v_{U'_y}(a)\}$ is measurable for any open U'_x, U'_y.)

(4) Let $E' \subset E$ be compact and take some fixed $x, y \in X \times X \backslash \Delta$. For every $a \in E'$ let $U_{x,a}$, $U_{y,a}$, A_a be as in (**). Since E' is compact there are $\{a_1, \ldots, a_m\} \subset E'$ such that $E \subset \bigcup_{j=1}^m A_{a_j}$. Put $U_x = \bigcap_{j=1}^m U_{x,a_j}$, $U_y = \bigcap_{j=1}^m U_{y,a_j}$. Then by (**), $v(\{a \in E' : a)\}) = 0$. Since, being a Lindelöf space, $X \times X \backslash \Delta$ can be covered by a countable number of open sets of the form $U_x \times U_y$ it follows that for v-a.e. $a \in E'$, $F(\cdot, a) : X \to R$ has at most one maximizer. Taking into account that E is σ-compact this yields the theorem. ∎

Example 1. Let $X \subset R^m, m < \infty$, and E be an open subset of R^m. Suppose that $f : X \to R$ is a continuous function. Take $F(x, a) = f(x) + ax$. Then SC is satisfied. This was the case studied in Mas-Colell (1973).

Example 2. Let $X = (0, 1)$, $E = l_\infty$, and $F(x, a) = \sum_{i=1}^\infty a_i x^i$. Then, for any product measure v on $\mathbf{R}^\infty$, satisfying Assumption 4, and such that $v(l_\infty) = 1$, the power series $F(\cdot, a)$ has a unique maximizer (which is a well-known fact) for v a.e. $a \in l_\infty$.

Now let X be compact and $C^0(X)$ be the set of continuous functions on X with the supremum norm. Then a $F : X \times E \to R$ satisfying Assumption 2 induces a (Borel) distribution μ_F on $C^0(X)$, i.e., $\mu_F(U) = v\{a \in E : F(\cdot, a) \in U\}$ for every Borel set. Suppose that we regard two $F : X \times E \to R$, $F' : X \times E' \to R$ (E, v possibly different from E', v') as close if $\mu_{\bar{F}}, \mu_{F'}$ are close in the weak convergence for measures. Then we can approximate any F fulfilling Assumption 1 by one fulfilling Assumption 3, and the Sondermann

Condition; indeed, it suffices to approximate μ_F by a μ having finite support, smooth the functions in the support of μ and apply to each one the perturbations described in Example 1.

Let us now consider what in the introduction has been called the parametric approach. Assume that X is a non-empty, compact subset of R^m, $m < \infty$, and v has compact support and satisfies Assumption 4. Let $\mathcal{U}^r, r \geq 0$, be the space of C^r functions $F: X \times E \to R$ with the topology of C^r uniform convergence on $X \times \text{supp}(v)$. We note first, in Example 3 below, that the Sondermann condition is not a dense one for any $r \geq 0$.
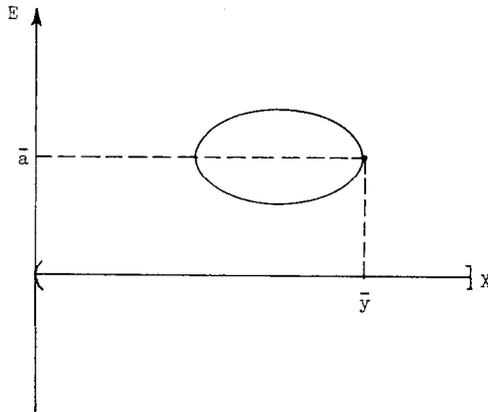

Fig. 1

Example 3. Let $X = [0,1]$, $E = (-\infty, \infty)$, and v be the uniform distribution on $(-2,2)$. Let $F: X \times E \to R$ be a C^∞ function such that the set $H = \{(y,a) \in (0,1] \times (-\infty, \infty): F(0,a) - F(y,a) = 0\}$ has the form pictured in fig. 1 and $DF(0,a) \neq DF(y,a)$ whenever $(y,a) \in H$. It is clear that such a function exists. At $(0, \bar{y}, \bar{a})$ the Sondermann Condition fails, since, if $\partial_a(F(0,\bar{a}) - F(\bar{y},\bar{a})) \neq 0$, we could solve a as a function of y (by the implicit function theorem). Further, a point such as $(\bar{y}, \bar{a})$ will exist for any small C^0 perturbation of F.

Observe that the characteristics of the example do not depend on the dimensionality of X and E.

It would appear that the conclusion of Theorem 1 obtains under weaker transversality conditions than the SC, conditions perhaps involving higher derivatives of F. In particular, we are willing to make the following conjecture:

Conjecture. Let $n \times m$. Then there is an *open and dense* subset of $\mathcal{U}^2$ such that for every $F: X \times E \to R$ in this set $F(\cdot, a)$ has only one maximizer for v-a.e. $a \in E$.

Of course, the above conjecture is only a representative, the simplest one, of a whole class of questions waiting for answers. In the economic context, a positive answer to the conjecture would deal with a single budget. It would still have to be dilucidated in order to prove or disprove a continuity property, if, for an open-dense set of economies (assume all the consumers have the same initial endowment), mean maximizers are unique for *all* budgets. This is a rather heavy further requirement.

The need of *some* transversality condition (i.e., some condition involving the derivatives of F) can be gauged from the next example.

Example 4. Let $X = [0, 1]$, $E = (0, 1)$, and $K \subset E$ be a Cantor middle-third set. For $a \in E$, let $g(a)$ be the distance from a to K. Put $F(x, a) = xg(a)$. Then F is continuous and if $F(x, a) = F(y, a)$, $x \neq y$, then $F(x, a') \neq F(y, a')$ for a' arbitrarily close to a. However, the set of a', for which $F(\cdot, a')$ does not have a unique maximizer is K which may have Lebesgue measure as close to 1 as we wish.

The last example spells out the relevance of Theorem 1 for the problem of continuous mean demand. The result is due to Sondermann.

Example 5. Let $F: \mathbf{R}^l_+ \times \mathbf{R}^n \to \mathbf{R}$, $F(x, a) = u_a(x)$, where u_a is a continuous utility function depending on n parameters a. Let v be a Borel probability measure on $\mathbf{R}^n$. Then, under Assumptions 2, 3, 4, and SC, and if the economy is represented by a measure $v \times \delta_{\{\omega\}}$ where ω are arbitrary initial endowments, the excess demand correspondence, as usually defined, is in virtue of Theorem 1 (let $X = \{y : py \leq p\omega\}$ for fixed p) a continuous function.

3. Smooth excess demand

In this part we prove that the set of economies (defined as measures on a space of characteristics) yielding C^∞ excess demand is dense with respect to the topology of weak convergence for measures. This is done in subsections 3.2 and 3.5 (the central ones) for a model where all commodities but one are indivisible. In subsections 3.1, 3.3 and 3.4, which are rather technical and contain nothing new, the discrete commodities result is applied to yield the same theorems for the divisible commodities model. This generalizes results of Mas-Colell and Neuefeind (1977). In the parametric approach, subsection 3.6 establishes, by example, and in the context of the discrete model, that the Sondermann Condition does not guarantee C^1 mean demand. Some comments on sufficient conditions for smoothness of mean demand are also included.

3.1

For the set-up of this subsection, see Hildenbrand (1974).

Let the consumption set be R^l_+: $\mathscr{P}$ denotes the space of monotone, continuous preference relations on R^l_+. We endow $\mathscr{P}$ with the closed convergence topology. The space of agents' characteristics is $\mathscr{A} = \mathscr{P} \times R^l_{++}$, with the product topology. Generic elements of $\mathscr{A}$ are denoted by $a = (\succsim_a, \omega_a)$.

Let $S = \{p \in R^l_{++}: \|p\| = 1\}$. The excess demand correspondence $\phi: \mathscr{A} \times S \to R^l_+$ is defined as usual: $(a, p) \to \{x \in R^l_+: px \leqq p\omega_a$ and '$py \leqq p\omega_a$ implies $x \succsim_a y$'$\} - \{\omega_a\}$.

A (continuum) economy is a Borel measure v on $\mathscr{A}$ with $\int \omega_a \, dv < \infty$. Let $\mathscr{E}$ be the space of economies; $\mathscr{E}$ is endowed with the topology which induces the following notion of convergence: v_n tends to v if $v_n \to v$ in the weak convergence for measures and if $\int \omega_a \, dv_n(a) \to \int \omega_a \, dv(a)$.

Given $v \in \mathscr{E}$ the mean excess demand correspondence $\Phi_v: S \to R^l$ is defined by $p \to \int \phi(\succsim_a, \omega_a, p) \, dv(a)$. It is in general a set-valued map. If it happens to be singleton-valued, we regard it as a function.

Theorem 2. *There is a dense set $\mathscr{E}^* \subset \mathscr{E}$ such that for every $v \in \mathscr{E}^*$, $\Phi_v: S \to R^l$ is a C^∞ function.*

Remark 1. We could strengthen the theorem by asserting that for every $v \in \mathscr{E}^*$ the boundary condition '$p_n \to p \in \partial S$ implies $\inf\{\|x\|: x \in \Phi_v(p_n)\} \to \infty$' is satisfied. Indeed, given any $v \in \mathscr{E}$ we can transfer an arbitrarily small amount of mass to a Cobb–Douglas consumer, the resulting economy will satisfy the above condition [see Mas-Colell and Neuefeind (1977)].

3.2

We will describe a model where all commodities but one are discrete and a theorem similar to Theorem 2 will be stated.

Let $K \subset R^{l-1}_+$ be a fixed, compact rectangle containing $0 \in R^{l-1}_+$. Denote $\mathscr{K}^* = \{H \subset K: H$ is a finite rectangular lattice such that $coH = K\}$. A generic element of $\mathscr{K}^* \cup \{K\}$ will be denoted J; J^+ will be the strictly positive elements of J.

For every J let $\mathscr{P}(J)$ be the space of continuous preference relations on $J \times [0, \infty)$ satisfying:

(a) If $(x, b) > (x', b')$ then $(x, b) \succ (x', b')$ (strict monotonicity).

(b) For every (x, b) there is $b' > b$ such that $(0, b') \succ (x, b)$.

Denote $\mathscr{A}(J) = \mathscr{P}(J) \times (J^+ \times (0, \infty))$ and define a set of (continuum) economies $\mathscr{E}(J)$ as in subsection 3.1. Let $\hat{S} = \{p \in \bar{S}: p^l > 0\}$. The excess demand

correspondence $\phi: \mathcal{A}(J) \times \hat{S} \to R^l$ and the mean excess demand correspondence $\Phi_\nu: \hat{S} \to R^l$ for $\nu \in \mathcal{E}(J)$ are defined again as in the previous subsection. Let $\mathcal{A}_b(J)$, and accordingly $\mathcal{E}_b(J)$ be the set of $a \in \mathcal{A}(J)$ satisfying:

(c) There is $\varepsilon > 0$ such that for all $p \in \hat{S}$ and $(x, b) \in \phi(\ _a, \omega_a, p) + \{\omega_a\}$ we have $b \geq \varepsilon$.

Theorem 2'. For every $H \in \mathcal{K}^*$ there is a dense set $\mathcal{E}_b^*(H) \subset \mathcal{E}_b(H)$ such that for every $\nu \in \mathcal{E}_b^*(H)$, $\Phi_\nu: S \to R^l$ is a C^∞ function.

Remark 2. It is false that the set of ν having C^1 excess demand is dense in $\mathcal{E}(H)$. Examples are not difficult to come by. This is to be contrasted with the C^0 case; it is, implicitly, proved in Mas-Colell (1975, th. 1) that the set of ν having continuous excess demand is dense in $\mathcal{E}(H)$.

Remark 3. Theorem 2' provides an alternative approach to the proof of Theorem 3 in Mas-Colell (1975); this theorem dealt with establishing 'generic' properties of the equilibrium correspondence in a model with discrete commodities.

3.3

We will show how Theorem 2' implies Theorem 2.

Let $\mathcal{K}^*$ be as above and denote $\mathcal{A}' = \bigcup_{H \in \mathcal{K}^*} \mathcal{A}(H) \cup \mathcal{A}(K)$. $\mathcal{A}'$ becomes a topological space in the obvious manner (i.e., by means of the closed convergence).

Lemma 1. Let $\succsim \in \mathcal{P}(K)$: then there is $H_n \in \mathcal{K}^*$ and $\succsim_n \in \mathcal{P}(H_n)$ such that $\succsim_n \to \succsim$.

Lemma 2. If $a \in \mathcal{A}_b(J)$, $a_n \in \mathcal{A}(H_n)$, $a_n \to a$ then eventually $a_n \in \mathcal{A}_b(H_n)$.

Lemma 3. There is a continuous function $\Psi: \mathcal{A}' \to \mathcal{A}(K)$ such that (i) $\Psi(a) = a$ if $a \in \mathcal{A}(K)$, (ii) for every $a \in \mathcal{A}'$ and $p \in S$, $\phi(\Psi(a), p) = \phi(a, p)$.

It is well known that to prove Theorem 2 it suffices to consider a Dirac measure δ_a, $a \in \mathcal{A}$. Let u be a utility function for $\succsim_a$; we can assume that u is C^2. If $\varepsilon > 0$ is small enough the utility function $u'(y) = u(y) + \varepsilon \prod_{i=1}^l y^i + \varepsilon y^l$ represents a preference relation $\succsim'$ which is arbitrarily close to $\succsim$, strictly monotone and satisfies (a), (b) and (c). Let K be as in subsection 3.2 but otherwise arbitrary. Assuming $\omega_a \in K \times [0, \infty)$ and putting $\succsim'' = \succsim' \cap (K \times [0, \infty)) \times (K \times [0, \infty))$ we have $\bar{a} = (\succsim'', \omega_a) \in \mathcal{A}_b(K)$ and $\succsim''$ is arbitrarily close to $\succsim_a$ in closed convergence. Since $\succsim''$ is strictly monotone the preference relation $\succsim''' \in \mathcal{P}$ defined by $z' \succsim''' z$ if for every $y \in K \times [0, \infty)$

$z \in (\{y' \succsim'' y\} + R^l_+)$ implies $z' \in (\{y' \succsim'' y\} + R^l_+)$ extends $\succsim''$ to R^l_+ and is such that $\phi(\succsim''', \omega_a, p) = \phi(\succsim'', \omega_a, p)$ for any $p \in S$. By taking K large enough $\succsim'''$ is arbitrarily close to $\succsim_a$. Therefore our problem reduces to prove: let $a \in \mathscr{A}_b(K)$, then there are $v_n \in \mathscr{E}(K)$ such that $v_n \to \delta_a$ and Φ_{v_n} is C^∞. By Lemmas 1 and 2 there are $H_n \in \mathscr{K}^*$ and $\succsim_n \in \mathscr{P}(H_n)$ such that $\omega_a \in H_n$, $\succsim_n \to \succsim$ and $a_n = (\succsim_n, \omega_a) \in \mathscr{A}_b(H_n)$. So, $\delta_{a_n} \to \delta_a$. By Theorem 2' we can replace δ_{a_n} by $v'_n \in \mathscr{E}(H_n)$ where $\Phi_{v'_n}$ is C^∞. By Lemma 3 the measures $v_n = v'_n \circ \Psi^{-1}$ on $\mathscr{A}(K)$ are well defined and $\Phi_{v_n} = \Phi_{v'_n}$. Since Ψ is continuous and $v'_n \to \delta_a$, we have $v_n \to \delta_{\Psi(a)}$, since $a \in \mathscr{A}_b(K)$, $\Psi(a) = a$ and so $v_n - \delta_a$ as we wanted to prove.

3.4

In this subsection we prove the lemmas used in subsection 3.3.

Proof of Lemma 1. Let $H_n \to K$ in closed convergence and put $\succsim_n = \succsim \cap (H_n \times [0, \infty)) \times (H_n \times [0, \infty))$. ∎

Proof of Lemma 2. Suppose it is false, i.e., $a_n \to a \in \mathscr{A}_b(J)$ and there is $p_n \in \hat{S}$, $p_n \to p \in \bar{S}$, such that for $(x_n, b_n) \in \phi(a_n, p_n) + \{\omega_{a_n}\}$, $b_n \to 0$. With no loss of generality we can assume $x_n \to \bar{x}$, we distinguish two cases:

(i) $p \in \hat{S}$; let $(\bar{x}, \bar{b}) \in \phi(a, p) + \{\omega_a\}$, since $a \in \mathscr{A}_b(J)$ we have $\bar{b} > \varepsilon$ for some $\varepsilon > 0$. Pick $0 < \bar{b} < \bar{b}$. Since $p^l > 0$ we can assume that $p_n(\bar{x}, \bar{b}) < p_n(x_n, b_n)$ which implies $(x_n, b_n) \succsim_{a_n} (\bar{x}, \bar{b})$. Therefore $(x, 0) \succsim_a (\bar{x}, \bar{b})$ which yields $(\bar{x}, 0) \succsim_a (\bar{x}, b)$. But then $(\bar{x}, 0) \in \phi(a, p) + \{\omega_a\}$ which is a contradiction.

(ii) $p^l = 0$. Let b' be such that $(0, b') \succ_a (\bar{x}, 0)$. By continuity we can assume $(0, b') \succ_{a_n} (x_n, b_n)$. Of course $p_n(x_n, b_n) \to p(\bar{x}, 0) > 0$ and $p(0, b') = 0$, so, for n large enough $(x_n, b_n) \succsim_{a_n} (0, b')$, a contradiction. ∎
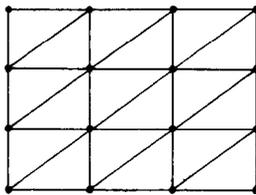

Fig. 2.

Proof of Lemma 3. For every $H \in \mathscr{K}^*$ let $\delta(H)$ be the minimum distance between points of H. We shall convene that every H is accompanied by a simplicial subdivision of K with set of vertices H and mesh $< l\delta(H)$; further the simplicial subdivision is chosen so that it varies continuously with H (see fig. 2 for an example).

Obviously, to prove the lemma we should only worry about preferences.

For any $H \in \mathscr{K}^*$, $\succsim \in \mathscr{P}(H)$ (or $\succsim \in \mathscr{P}(K)$) and $(x, b) \in H \times [0, \infty)$ let $U(\succsim, x, b)$ be the amount of commodity l such that $(x, b) \sim (0, U(\succsim, x, b))$; $U(\succsim, \cdot)$ is a utility function for $\succsim$ and depends continuously on $\succsim$ and (x, b).

Let $\succsim \in \mathscr{P}(H)$; the definition of Ψ will proceed in four steps by $\succsim \to \succsim^1 \to \succsim^2 \to \succsim^3 \to \succsim^4 \equiv \Psi(\succsim)$. The reader will easily verify that every step modifies $\succsim$ in a continuous manner.

(i) $\succsim \to \succsim^1$; $\succsim^1$ is a preference relation defined on $H \times [-1, \infty)$ as follows: if $x \in H$ and $b \geq 0$ then $U(\succsim^1, x, b) = U(\succsim, x, b)$; if $-1 \leq b \leq 0$ then $U(\succsim^1, x, b) = (1 + b)U(\succsim, x, 0) + b$; $U(\succsim^1, \cdot)$ is the utility function for $\succsim^1$; $\succsim^1$ is strictly monotone for $b \neq -1$.

(ii) $\succsim^1 \to \succsim^2$; for every $x \in H$ the range of $U(\succsim^1, x, \cdot)$ is the interval $[-1, \infty)$ hence for every $c \in [-1, \infty)$ there is a unique $b(x, c)$ such that $U(\succsim^1, x, b(x, c)) = c$. We define $\succsim^2$ on $K \times [-1, \infty)$ simply by letting the indifference surface corresponding to a utility level $c \in [-1, \infty)$ to be the graph of the function from K to R obtained by linear expansion of the function $x \to b(x, c)$ from H to R.

(iii) $\succsim^2 \to \succsim^3$; let $\succsim^3 = \succsim^2 \cap (K \times [0, \infty)) \times (K \times [0, \infty))$. Note that at the end of step (iii) or (ii) we have for any $\omega \in H$ and $p \in S$, $\phi(\succsim^3, \omega, p) \subset \phi(\succsim, \omega, p) \subset \mathrm{co}\phi(\succsim, \omega, p)$.

(iv) $\succsim^3 \to \succsim^4$; for any $x \in K$ let $g(x)$ be the distance from x to the set H; $\succsim^4$ is defined on $K \times [0, \infty)$ by the utility function $u(x, b) = U(\succsim^3, x, b) - \delta(H)g(x)$. We have then $\phi(\succsim^4, \omega, p) = \phi(\succsim^3, \omega, p) \cap (H \times [0, \infty)) = \phi(\succsim, \omega, p)$ for any $\omega \in H$ and $p \in S$.

3.5

In this section we prove Theorem 2'. It suffices to consider a Dirac measure in $\mathscr{E}_b(H)$ supported at $\bar{a}$. Let $H = \{x_1, \ldots, x_m\}$, and δ_j be the indicator of the set $\{x_j\} \times [0, \infty)$. For a $\gamma > 0$ and every v in $[-\gamma, \gamma]^m$ let u_v be the utility function $u_v = u \times \sum_{j=1}^m v^j \delta_j$, where u represents $\succsim_{\bar{a}}$. By repeated application of Lemma 2 we can assume that u is C^∞ and that for every $v \in [-\gamma, \gamma]^m$, u_v represents an element $\succsim_v$ in $\mathscr{A}_b(H)$; further the ε of condition (c) can be taken to hold uniformly in v. Let $f: R \to [0, \infty)$ be a C^∞ density function with support contained on $[-\gamma, \gamma]$. Finally, let μ be the probability measure on $[-\gamma, \gamma]^m$ defined by the density $\prod_{j=1}^m f(v^j)$. It is enough to show that for such μ, Φ_ν is $C^\times$, where ν is the measure in $\mathscr{A}_b(H)$ induced by $\mu \times \delta_{\{\omega_a\}}$ via the continuous function $(v, \omega) \mapsto (\succsim_v, \omega)$.

For each j and $p \in S$, let

$$z_j(p) = \left(x_j, \frac{p\omega_{\bar{a}} - p(x_j, 0)}{p^l} \right);$$

$z_j\colon S \to R^l$ is obviously C^∞.

For $p \in S$ let $I(p) = \{j\colon z_j^l(p) > \varepsilon/4\}$ and

$$A_j(p) = \{v \in [-\gamma, \gamma]^m, u_v(z_j(p)) \geqq u_v(z_k(p)) \text{ all } k \text{ in } I(p)\}.$$

We want to show that $\Phi_v(p)$ is C^∞ in a neighborhood of $\bar{p}$. But in a neighborhood of $\bar{p}$ we have

$$\Phi_v(p) = \int_{[-\gamma,\gamma]^m} \phi(\geqq_v, \omega_{\bar{a}}, p) = \sum_{j \in I(\bar{p})} z_j(p)\mu(A_j(p)) - \{\omega_{\bar{a}}\},$$

since

$$\mu(A_j(p)) = \int_{-\infty}^{\infty} \prod_{\substack{k \in I(\bar{p}) \\ k \neq j}} \int_{-\infty}^{v^j - u(z_k(p)) + u(z_j(p))} f(v^k)\, \mathrm{d}v^k f(v^j)\, \mathrm{d}v^j$$

is a C^∞ function of p we have proved the result.

3.6

We first present an example, given in parametric form, where the Sondermann Condition is satisfied but mean demand fails to be C^1. The example is for a model with discrete commodities and it suggests the difficulty in obtaining reasonable sufficient conditions for the aggregate demand to be differentiable.

Example 6.   Let the commodity space be $\{1, 2, 3\} \times (0, \infty)$ and let the agents utility function, parametrized by α in $(0, 2)$, be

$$u_\alpha(1, t) = \alpha t,$$

$$u_\alpha(2, t) = \alpha^2 t,$$

$$u_\alpha(3, t) = t^2.$$

Finally let μ be the uniform distribution in $(0, 2)$. We claim that $u_\alpha(i, t)$ satisfy the SC: if $\psi_\alpha(i, j, t, s) = u_\alpha(i, t) - u_\alpha(j, s) = 0$ and $(i, t) \neq (j, s)$ then $\partial_\alpha \psi_\alpha(i, j, t, s) \neq 0$. Since if $i = j$ and $\psi_\alpha(i, j, t, s) = 0$ then $t = s$, we can assume $i \neq j$. Moreover, $\partial_\alpha \psi_\alpha(1, 3, t, s) = t \neq 0$; $\partial_\alpha \psi_\alpha(2, 3, t, s) = 2\alpha t \neq 0$, and we just have to check the case in which $\psi_\alpha(1, 2, t, s) = \alpha t - \alpha^2 s = 0$, or $t = \alpha s$. As $\partial_\alpha \psi_\alpha(1, 2, t, s) = t - 2\alpha s = -\alpha s \neq 0$, the claim is proved. Let the indivisible commodity be taken as numéraire and suppose that the consumer has wealth $= 4$ (i.e., he has 4 units of numéraire). We are going to show that the aggregate demand function is C^0 but not C^1.

Given a price $(1, p)$ each consumer faces three alternatives:

> *Bundle A:* $(1, 3/p)$,
>
> *Bundle B:* $(2, 2/p)$,
>
> *Bundle C:* $(3, 1/p)$.

Consumer α chooses A if

$$\alpha \frac{3}{p} > \alpha^2 \frac{2}{p},$$

and

$$\alpha \frac{3}{p} > \left(\frac{1}{p}\right)^2,$$

that is if $1/3p < \alpha < 3/2$. Consumer α chooses B if

$$\alpha^2 \frac{2}{p} > \alpha \frac{3}{p},$$

and

$$\alpha^2 \frac{2}{p} > \left(\frac{1}{p}\right)^2,$$

i.e., if $\alpha > 3/2$ and $\alpha > 1/\sqrt{2p}$, that is, if $\alpha > 3/2$ for $p > 2/9$ and if $\alpha > 1/\sqrt{2p}$ for $p < 2/9$. Finally consumer α chooses C if

$$\left(\frac{1}{p}\right)^2 > \alpha \frac{3}{p},$$

and

$$\left(\frac{1}{p}\right)^2 > \alpha^2 \frac{2}{p},$$

i.e., if $\alpha < 1/3p$ and $\alpha < 1/\sqrt{2p}$, that is, if $\alpha < 1/3p$ for $p > 2/9$ and if $\alpha < 1/\sqrt{2p}$ for $p < 2/9$. It follows that for $1/8 < p < 2/9$,

$$\Phi^1(p) = \int_0^{1/\sqrt{2p}} 3\frac{1}{2}\,d\alpha + \int_{1/\sqrt{2p}}^2 2\frac{1}{2}\,d\alpha = 2 + \frac{1}{2}(1/\sqrt{2p}),$$

and for $p > 2/9$,

$$\Phi^1(p) = \int_0^{1/3p} 3\frac{1}{2}\,d\alpha + \int_{1/3p}^{3/2} \frac{1}{2}\,d\alpha + \int_{3/2}^2 2\frac{1}{2}\,d\alpha = 5/4 + 1/3p.$$

Φ^1 is continuous at $p=2/9$ since $\Phi^1(2/9)^- = \Phi^1(2/9)^+ = 11/4$. But Φ^1 is not C^1 at $p=2/9$ since $\partial_-\Phi^1(2/9) = -27/16 \neq -27/4 = \partial_+\Phi^1(2/9)$. Fig. 3 makes it very obvious why Φ is not C^1 at $2/9$.
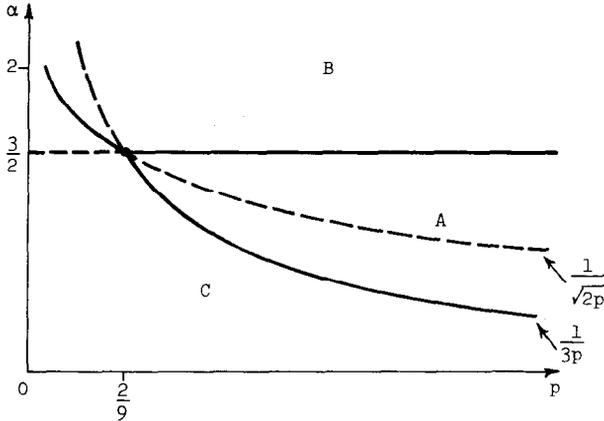

Fig. 3

We finish by making some remarks about obtaining a C^1 mean excess demand. As in the introduction, $E \subset R^m$ is an open set of parameters and $X \subset R^l$ the consumption set. There is given a probability measure μ on R^m having compact support contained in E and a C^∞ density with respect to Lebesgue measure in R^m. Assuming, to reduce things to essentials, that every consumer has the same initial endowment ω an economy is described by a function $U: X \times E \to R$; $U(\cdot, a)$ is the utility function corresponding to $a \in E$. We assume that U is C^1. It has been seen in section 2 that if U satisfies the SC, mean demand is a continuous function of prices. It is not hard to verify that to get a C^1 dependence on prices the proper analog of the SC is:

Generalized SC (GSC): If $\quad U(x,a)=U(y,a)=\ldots=U(z,a), \quad x \neq y, \ldots, x \neq z$, then $\{\partial_a(U(x,a)-U(y,a)), \ldots, \partial_a(U(x,a)-U(z,a))\}$ are linearly independent vectors.

[Of course, a number of monotonicity and boundary conditions on $U(\cdot, a)$ are also necessary.] The GSC is exceedingly strong and of not much interest; in particular, it *cannot* be satisfied in the standard divisible commodity model where $X = R^l_+$. In the discrete commodity model of section 3 where $X = H \times [0, \infty)$, the GSC can be satisfied in general only if m, the number of parameters, is larger than $\# H$. Observe that the proof of Theorem 2' proceeds precisely by creating a very large space of perturbation parameters (certainly one with $m > \# H$) in which the GSC is satisfied.

Summing up: the suggestion is strong that, at least in the standard model, a finite number of parameters is not a rich enough space to sustain an expectation that mean excess demand will be C^1.

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