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HOMEOMORPHISMS OF COMPACT, CONVEX SETS AND THE JACOBIAN MATRIX*

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Abstract. Let $K \subset R^n$ be a compact, convex polyhedron and $f: K \to R^n$ a C^1 function. The problem of existence of a global inverse for f is studied. It is shown (Theorem 1) that f has an inverse, if, for every $x \in K$, the Jacobian of f at x, $Jf(x)$, is such that for every linear space spanned by a face of K containing x the determinant of the linear map from L to L formed by projecting $Jf(x)$ on L has positive sign. Theorem 2 is a similar result for K with smooth boundary. The theorems generalize the well-known Gale–Nikaido theorems, which originated in some problems of mathematical economics.

1. Introduction. Let $K \subset R^n$ be a compact, convex set. Without loss of generality we assume that K has a nonempty interior. Let $F: K \to R^n$ be a C^1 function. The derivative map of F at x is denoted $DF(x)$. Given a coordinate system the Jacobian matrix of F at $x \in K$ is denoted $JF(x)$. We want to find sets of local conditions, i.e., conditions on $JF(x)$ only, implying that F is one to one and so, a homeomorphism.

It is well known that the nonsingularity everywhere of $JF(x)$ will not do; see Fig. 1.

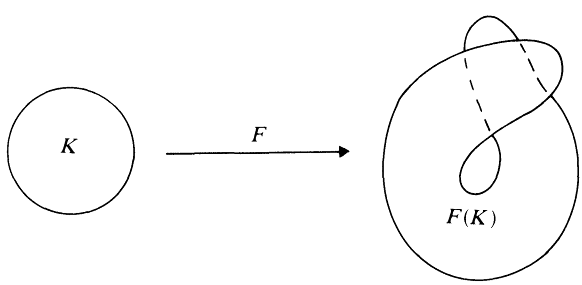

FIG. 1

A set of sufficient conditions is provided by the theorems of Gale and Nikaido ([2], [6, Chap. VII]), which were stimulated by some problems in mathematical economics:

(i) Let K be a rectangle. If for every $x \in K$, $JF(x)$ is a P matrix (i.e., every principal minor of $J(x)$ has positive sign), then F is a homeomorphism.

(ii) If for every $x \in K$, $JF(x)$ is positive quasidefinite (i.e., $v'JF(x)v > 0$ for all $x \in R^n$, $v \neq 0$), then F is a homeomorphism.

It will be shown here that the result can be obtained under substantially weaker hypotheses. In particular, for points $x \in \text{Int}\, K$ our analogue of (i) will impose sign restrictions only on the principal minor of order n.

More specifically, consider (i) above. The set K is a rectangle, i.e., it is of the form $K = \{x \in R^n: s^i \leqq x^i \leqq r^i\}$. For every nonempty subspace $L \subset R^n$ let $\Pi_L: R^n \to L$ denote the perpendicular projection map. The condition that $JF(x)$ be a P matrix is equivalent to the requirement that for every coordinate subspace $L \subset R^n$, the linear map $\Pi_L \cdot DF(x): L \to L$ preserves orientation, i.e., has a positive determinant. We will show that, with K a general polyhedron, F is a homeomorphism if for every $x \in K$ and every subspace $L \subset R^n$ spanned by a face of K which includes x, the linear map $\Pi_L \cdot DF(x): L \to L$ preserves orientation, i.e., has positive determinant (the subspace

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spanned by a convex set is the translation to the origin of the minimal affine space containing the set). So, if K is a rectangle, $JF(x)$ needs to be a P matrix only at the vertices of K and for $x \in \operatorname{Int} K$ the only requirement is that $JF(x)$ have a positive determinant.

Observe also that, in contrast with (i), our conditions are coordinate free, in the sense that their formulation does not rely on a previous choosing of coordinates. This will be emphasized in the statement of the theorem. Consider now (ii) and suppose that the boundary of K, denoted ∂K, is smooth (a C^1 hypersurface, say). For $x \in \partial K$, T_x is the tangent plane of ∂K at x (see Fig. 2). We will show that F is a homeomorphism if: (a) $JF(x)$ has a positive determinant for every $x \in K$, and (b) for every $x \in \partial K$, $JF(x)$ is positive quasidefinite on T_x, i.e., $v'JF(x)v > 0$ whenever $v \neq 0$ and $v \in T_x$.

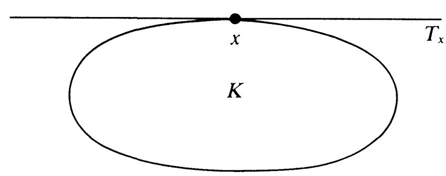

FIG. 2

The mathematical tool for the proofs is fixed-point index theory (see Milnor [5], Guillemin–Pollack [4]), in particular, the powerful Poincaré–Hopf theorem. That index theory could be the key to the sort of generalization of the Gale–Nikaido theorem given here was surmised by H. Scarf [8] in view of the Eaves and Scarf analysis in [1] of the index theory associated with the linear (and nonlinear) complementarity problem (see also Saigal and Simon [7]).

It is worth emphasizing that our results are not of a purely differential topological nature; they hold for domains K which are *convex* sets. It should be clear from the inspection of Fig. 3 how counterexamples can be constructed for nonconvex K and maps F satisfying (a) and (b) of the paragraph previous to the last.

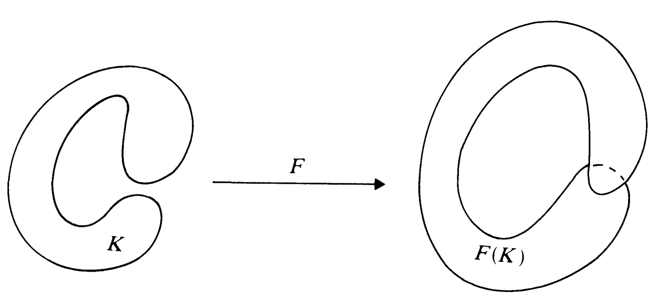

FIG. 3

2. Statement of theorems. Terminology and notation are as in the Introduction.

THEOREM 1. *Let $K \subset R^n$ be a compact, convex polyhedron of full dimension and $F: K \to R^n$ a C^1 function. If for every $x \in K$ and subspace $L \subset R^n$ spanned by a face of K which includes x, the map $\Pi_L \cdot DF(x)$: $L \to L$ has a positive determinant, then F is one-to-one and so, a homeomorphism.*

THEOREM 2. *Let $K \subset R^n$ be a compact, convex set of full dimension with a C^1 boundary ∂K and $F: K \to R^n$ a C^1 function. If for every $x \in K$, $DF(x)$ has a positive determinant and if for all $x \in \partial K$, $DF(x)$ is positive quasidefinite on T_x (i.e., $v'DF(x)v > 0$ for $v \in T_x$, $v \neq 0$), then F is one-to-one and so, a homeomorphism.*

3. Demonstration.

1. It may be useful if we first sketch the main idea of the proof, which is very simple. We first extend F to the whole of R^n in a certain simple manner which preserves differentiability except in a set of measure zero and has the property that whenever differentiable the extended function has a positive Jacobian determinant. Now take any point of R^n, say, $0 \in R^n$. It turns out that for our purposes we can assume that $F^{-1}(0)$ lies entirely in the region of differentiability. This means that the sum of the indexes of F at points $x \in F^{-1}(0)$ equals the sum of the signs of the Jacobian determinant, i.e., the sum is $\geqq 1$. But, after verifying that the extended F satisfies appropriate boundary conditions, we appeal to a topological index theorem to conclude that the sum must be $\leqq 1$. Hence $F^{-1}(0)$ is a singleton set.

2. We let $K \subset R^n$ be a general compact, convex polyhedron of full dimension and prove Theorem 1. We shall see at the end that Theorem 2 is essentially a corollary of Theorem 1.

We note first that it suffices to prove that $F|\text{Int } K$ is one-to-one. Indeed, we can always extend F to a K' containing K in its interior and sufficiently similar to K for all hypotheses on $DF(x)$ to be still satisfied.

3. For every $x \in R^n$ let $z(x) \in K$ be the foot of x, i.e., $z(x)$ is the (unique) element of K minimizing $\|x-z\|$ for $z \in K$. Of course, $z(x) = x$ for $x \in K$; see Fig. 4.

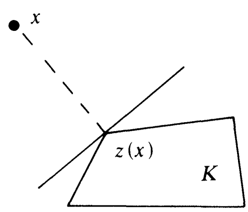

FIG. 4

We now extend $F: K \to R^n$ to the whole of R^n by letting a function $\hat{F}: R^n \to R^n$ be defined by $\hat{F}(x) = F(z(x)) + x - z(x)$; see Fig. 5. For any $y \in F(K)$ define $\hat{F}_y: R^n \to R^n$ by $\hat{F}_y(x) = \hat{F}(x) - y$.

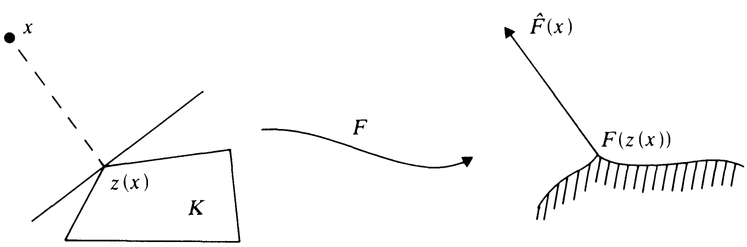

FIG. 5

4. Let S_r, B_r be, respectively, the sphere and ball of radius r. We claim that for any $y \in F(K)$ and any r sufficiently large, $\hat{F}_y | S_r$ has degree one, i.e., it is homotopic, with respect to $R^n \backslash \{0\}$, to the identity in S_r. Indeed, it suffices to verify that for r sufficiently large and any $y \in F(K)$, if $x \in S_r$, then $x'\hat{F}_y(x) > 0$. Take $r > \max_{z \in K, y \in F(K)} \|F(z) - z - y\| = s$. Then

$$x'\hat{F}_y(x) = \|x\|^2 - x'(z(x) + y - F(z(x)))$$
$$\geqq \|x\|^2 - \|x\| \, \|z(x) + y - F(z(x))\| \geqq r^2 - rs > 0.$$

5. The region $A = \{x \in R^n : \hat{F} \text{ is not } C^1 \text{ at } x\}$ contains no open set. This is clear since $z(x)$ is C^1 everywhere except at $x \in K$ with $x - z(x)$ perpendicular at $z(x)$ to more than one face of K and those x are contained in a finite number of hyperplanes. Since $\hat{F}$ is Lipschitzian, $\hat{F}(A)$ contains no open set.

6. We now state the basic lemma. The proof is postponed to 8.

LEMMA 1. *Let K be a polyhedron and F satisfy the hypothesis of Theorem* 1. *Then if $x \notin A$, $|D\hat{F}(x)|$ is positive.*

Of course, $|D\hat{F}(x)|$ denotes the determinant of the linear map $D\hat{F}(x)$.

7. Let $r > 0$ be a fixed number with $K \subset B_r$ and $\hat{F}_y | S_r$ of degree one for any $y \in F(K)$. By the Poincaré–Hopf theorem (see Milnor [5]), if $\hat{F}_y^{-1}(0) \cap A = \emptyset$, then $\sum_{x \in \hat{F}_y^{-1}(0) \cap B_r} \operatorname{sign} |D\hat{F}_y(x)| = 1$, which, by the lemma, means that $\hat{F}_y^{-1}(0) \cap B_r$ is a singleton set.

Suppose now that $F | \operatorname{Int} K$ were not one-to-one, i.e., there are $x_1, x_2 \in \operatorname{Int} K$ with $x_1 \neq x_2$ and $F(x_1) = F(x_2)$. By the implicit function theorem there are disjoint open sets $V_1, V_2 \subset \operatorname{Int} K$ with $x_1 \in V_1$, $x_2 \in V_2$ and $F(V_1) \cap F(V_2) \neq \emptyset$ open. Since $\hat{F}(A)$ contains no open set, there is $y \in F(V_1) \cap F(V_2)$ such that $y \notin \hat{F}(A)$. But then $\hat{F}_y^{-1}(0) \cap A = \emptyset$ and $F^{-1}(y) \subset \hat{F}_y^{-1}(0) \cap B_r$ is not a singleton set. This contradiction establishes that $F | \operatorname{Int} K$ must be one-to-one and concludes the proof of Theorem 1.

8. We now prove Lemma 1.

Let $x \notin A$. Then $x - z(x)$ is perpendicular to a single face of K, which, of course, includes $z(x)$. Let L be the subspace spanned by this face and $L^\perp$ the subspace orthogonal to L. We then have that for small $v \in L$, $z(x+v) = z(x) + v$ and so, $\hat{F}(x+v) = F(z(x)+v) + x - z(x)$; hence, $D\hat{F}(x)v = DF(z(x))v$. For $v \in L^\perp$, $z(x+v) = z(x)$ and so, $\hat{F}(x+v) = F(z(x)) + x + v - z(x)$; hence $D\hat{F}(x)v = v$. Therefore, if we choose an orthogonal coordinate system whose k first coordinates generate L, $J\hat{F}(x)$, the matrix of $D\hat{F}(x)$ with respect to this coordinate system, takes the form

$$J\hat{F}(x) = \begin{bmatrix} J_k F(z(x)) & \begin{matrix} 0 \\ I \end{matrix} \end{bmatrix}, \qquad \text{where } J_k F(z(x)) \text{ are}$$

the first k columns of $JF(x)$. So $|D\hat{F}(x)| = |J\hat{F}(x)| = |J_{kk}F(z(x))|$, where $J_{kk}F(z(x))$ are the first k rows of $J_k F(z(x))$. But $J_{kk}F(z(x))$ is the matrix of $\Pi_L \cdot DF(z(x)): L \to L$ which by hypothesis is positive.

9. We now prove Theorem 2.

LEMMA 2. *Under the hypothesis of Theorem* 2, *if $x \in \partial K$ and $L \subset T_x$ is a subspace, then $\Pi_L \cdot DF(x): L \to L$ has a positive determinant.*

If the lemma holds, the proof is concluded since we can approximate K by a polyhedron K' and if K' is sufficiently close to K, Lemma 2 implies that the hypotheses of Theorem 1 are satisfied.

Lemma 2 is a well-known fact. Choose an orthogonal coordinate system such that the first k coordinates generate L and the nth is perpendicular to T_x and let $JF(x)$ be the matrix of $DF(x)$ in this coordinate system. Then $J_{n-1,n-1}F(x)$, the matrix formed by deleting the nth row and column is positive quasidefinite. This is the assumption of the theorem. But any principle minor of a positive quasidefinite matrix is positive (see, for example, Nikaido [6, p. 374]); this applies to $J_{kk}F(x)$, the matrix of $\Pi_L \cdot DF(x): L \to L$, and yields the lemma. Q.E.D.

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