

Two Propositions on the Global Univalence of Systems of Cost Function

ANDREU MAS-COLELL

I. INTRODUCTION AND STATEMENT OF RESULTS

1. Preliminaries and Definitions

Consider a C^1 function $F: Q \rightarrow R^l_+$ where Q is either R^l_{++} or $R^l_+ \setminus \{0\}$. We interpret F as a system of unit cost functions, the arguments being input prices. We assume that F is *linear homogeneous* [i.e., $F(\lambda w) = \lambda F(w)$ for all $\lambda \geq 0$] and call the Jacobian matrix of F at w , denoted $JF(w)$, the *input demand matrix*. If defined, the elasticity matrix $SF(w)$, i.e., the matrix with generic entry

$$s_{ij} = \frac{w}{F^i(w)} \frac{\partial F^i(w)}{\partial w^j},$$

is called the *input share matrix*.

We note that the interpretation of F as a system of cost functions demands that every component function be concave, but formally we will not impose this requirement.

2. The Problem

The factor price equalization problem in the pure theory of international trade (see Samuelson [11, 12], McKenzie [7], and Chipman [2]) prompted in an incidental way the following mathematical question: *Under which hypothesis has the equation $F(w) = v$ at most one solution for all $v \in R^l_+$, i.e., under which hypothesis is F globally univalent?*

With some disrespect for chronology and completeness, the rather complicated history of the analytical treatments of the univalence problem in the economic literature can be summarized thus. Samuelson [11] posed the question and showed that for $l = 2$ the everywhere nonsingularity of the input demand matrix (i.e., the Jacobian matrix) of F was a sufficient condition for univalence. Any hope that nonsingularity could suffice in the general case (Pierce [10]) was quickly dashed by an example of McKenzie [8]). One suggestion of Samuelson [12] involving the positive signedness of principal minors of the Jacobian of F was amended and finally led to the Gale–Nikaido theorem [4] of relevance for general, not just cost, functions. Sticking to cost functions, Samuelson [13, p. 908] suggested that if there existed a nested sequence of principal minor matrices of the input share matrices $SF(w)$ with determinants bounded away from zero uniformly on R^l_{++} , then F would be globally univalent on R^l_{++} . This was later established by Nikaido [9].

In this chapter we reexamine the global univalence problem for cost functions and provide improvements in the following two directions:

(1) In the Samuelson–Nikaido theorem the restrictions on the principal minors of the input share matrix are irrelevant; all that matters is that the determinant of $SF(w)$ be uniformly bounded away from zero (Proposition 1). This provides a global univalence theorem for systems of cost functions with domain R^l_{++} .

(2) In the context of cost functions with domain R^l_{++} , the Gale–Nikaido conditions can be substantially weakened (Proposition 2). The weakened conditions are of the same nature that the ones obtained in Mas-Colell [8] for general C^1 functions defined on compact polyhedra.

3. Conditions on the Input Share Determinant

PROPOSITION 1 *Let $F: R^l_{++} \rightarrow R^l_{++}$ be C^1 . If for some $\varepsilon > 0$ the absolute value of $|SF(w)|$ is larger than ε for all $w \in R^l_{++}$, then F is a homeomorphism onto.*

That is, for all $v \in R^l_{++}$ the equation $F(w) = v$ has a unique solution which in addition depends continuously on v .

Remark 1 Of course, $|SF(w)| \neq 0$ if and only if $|JF(w)| \neq 0$. Therefore McKenzie's example [7] proves that we cannot take $\varepsilon = 0$ in Proposition 1.

Remark 2 Proposition 1 generalizes the Samuelson–Nikaido theorem [9, 13] by dropping all the conditions on minors.

The strong conclusion of Proposition 2 is also its weakness. It yields not only that F is globally univalent but also that $F(R_{++}^l) = R_{++}^l$. This means that the ε restriction on the input share matrix determinant imposes stringent restrictions on the form of the isocost surfaces near the boundary of R_{++}^l .

For example, for no $1 \leq i \leq l$ can an isocost surface be bounded. See Fig. 1, where $(2, 3) \notin F(R_{++}^2)$. Essentially, isocost surfaces must be as in Fig. 2.

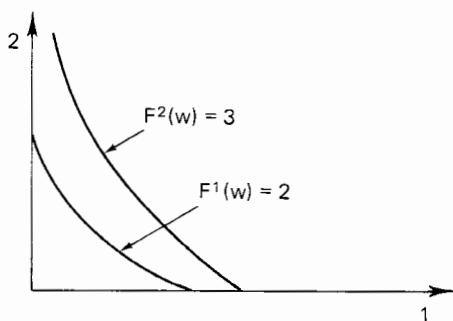


FIGURE 1

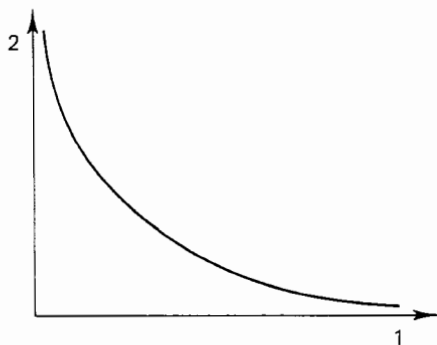


FIGURE 2

Summing up: Proposition 1 can be applied only if isocost surfaces are unbounded. This is an unduly strong requirement.

If isocost surfaces are bounded, then it is natural to assume that F is defined in R_+^l , (see Fig. 3). We can then appeal to the global univalence theorems based on weakened Gale–Nikaido conditions to be discussed in the next section.

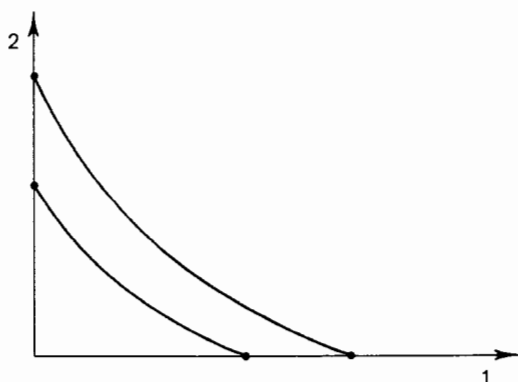


FIGURE 3

4. Generalized Gale–Nikaido Conditions

Let the system of cost functions be defined on R_+^l , i.e., $F: R_+^l \rightarrow R_+^l$.

For every subset $N \subset \{1, \dots, l\}$ of variables, we can define a reduced system of cost functions $F_N: R_+^{l-n} \rightarrow R_+^{l-n}$, where $n = |N|$, by indentifying R_+^{l-n} with the inputs with index not in N , letting the inputs with index in N be free, i.e., $w_i = 0$ for $i \in N$, and disregarding the cost of variables with index in N . For example, if $l = 3$ and $N = \{3\}$, then $F_N: R_+^2 \rightarrow R_+^2$ is defined by $F_N^i(w^1, w^2) = F^i(w^1, w^2, 0)$ for $i = 1, 2$.

Note that for any $w \in R_+^{l-n} \setminus \{0\}$, if we let $\hat{w}^i = 0$ for $i \in N$ and $\hat{w}^i = w^i$ otherwise, then $|JF_N(w)|$ is nothing but the principal minor determinant of $JF(\hat{w})$ obtained by deleting the rows and columns with indices in N .

PROPOSITION 2 *Let $F: R_+^l \rightarrow R_+^l$ be C^1 on $R_+^l \setminus \{0\}$. Suppose that for all $N \subset \{1, \dots, l\}$, $|JF_N(w)|$ is positive for every $w \in R_+^{l-n} \setminus \{0\}$. Then F is a homeomorphism.*

Remark 3 It is clear that Proposition 2 generalizes the Gale–Nikaido (GN) theorem as applied to systems of homogeneous functions on R_+^l . In particular we may note: (i) only infinitesimal conditions are imposed (as in the GN theorem); (ii) for $w \gg 0$ the only condition on $JF(w)$ is that $|JF(w)| > 0$. For an analogous generalization of the GN theorem for the case of arbitrary C^1 functions on a compact convex polyhedron, see Mas-Colell [6].

Remark 4 One could perhaps wonder if Proposition 2 remains valid with the word “positive” replaced by “negative.” It does, but in a vacuous manner. No system of homogenous cost functions $F: R_+^l \rightarrow R_+^l$ can satisfy the negative version of the condition of the theorem.

II. PROOFS

1. Proof of Proposition 1

The euclidean norm in R^l is $\|\cdot\|$. Given a linear map $T: R^l \rightarrow R^l$, its norm is $\|T\| = \max_{\|x\|=1} \|Tx\|$.

Proposition 1 is a consequence of the following theorem, which goes back to Hadamard [5]. It is taken from Berger [1, p. 222], where it is proved for maps on general Banach spaces.

THEOREM 1 *Let $G: R^l \rightarrow R^l$ be C^1 . Suppose there is a number k such that, for all $x \in R^l$, $JG(x)$ is nonsingular and $\|[JG(x)]^{-1}\| \leq k$. Then G is a homeomorphism onto.*

It follows from Cramer's rule that sufficient conditions for the uniform boundedness above of the norm of the inverse of $JG(x)$ are (i) $|JG(x)|$ is bounded away from zero uniformly; (ii) all the entries of the matrix $JG(x)$ are uniformly bounded in absolute value.

Let $F: R^l_{++} \rightarrow R^l_{++}$ satisfy the hypothesis of Proposition 1. Define then $G: R^l \rightarrow R^l$ by letting $G^i(x) = \ln F^i(e^{x^1}, \dots, e^{x^l})$. At any $x \in R^l_{++}$ we have $JG(x) = SF(w(x))$ where $w(x) = (e^{x^1}, \dots, e^{x^l})$. Because of the homogeneity of F , all the entries of $S(w(x))$ are less than or equal to one. Also, by hypothesis, the absolute value of $|S(w(x))| > \varepsilon > 0$. Therefore, by the Theorem, $G: R^l \rightarrow R^l$ is a homeomorphism onto and consequently so is $F: R^l_{++} \rightarrow R^l_{++}$.

Remark 5 Let $G: R^l \rightarrow R^l$ satisfy the hypothesis of the Theorem. Take any $y \in R^l$; then we know that $G(x) = y$ has a unique solution \bar{x} . In fact, $\bar{x} = x(1)$ where for some arbitrary x_0 , $x(t)$ is the solution to the differential equation $\dot{x} = [JG(x)]^{-1} (y - G(x_0))$ with initial condition $x(0) = x_0$.

2. Proof of Proposition 2

Although the content of Proposition 2 is analogous to the one in Mas-Colell [6], we will take a different line of proof. While the demonstration in [6] was based on index-theoretic arguments, we will rely here on the following well-known topological fact (see, for example, Berger [1, p. 221]):

THEOREM 2 *Let U be homeomorphic to R^l and $G: U \rightarrow U$ be given. Then G is a homeomorphism onto if and only if*

- (i) G is proper (i.e., if $K \in U$ is compact, then $G^{-1}(K)$ is compact).
- (ii) G is a local homeomorphism (i.e., for every $x \in U$, G is a homeomorphism on some neighborhood of x).

Let us assume we have given a system of cost functions $F: R_+^l \rightarrow R_+^l$ satisfying the hypothesis of Proposition 2.

Because of the homogeneity hypothesis, F is a homeomorphism of R_+^l if and only if it is a homeomorphism of the unit simplex. Hence F is a homeomorphism if and only if it is one to one.

Note that if F is a homeomorphism of $R_+^l \setminus \{0\}$, then it is a homeomorphism of R_+ because if $F(w) = 0$ and $w \neq 0$, then $F(\frac{1}{2}w) = 0$ by homogeneity; hence F would not be one to one on $R_+^l \setminus \{0\}$.

For any $x \in R^l$ let $\Pi(x)$ be the point of R_+^l closest to x in the euclidean norm. More explicitly, $\Pi^i(x) = \max\{x^i, 0\}$.

Define a map $G: R^l \setminus R_-^l \rightarrow R^l$ by $G(x) = F(\Pi(x)) + x - \Pi(x)$. Of course $G|R_+^l \setminus \{0\} = F$. So it suffices to show that G is a homeomorphism. The region $R^l \setminus R_-^l$ is homeomorphic to R^l . Hence by Theorem 2, our task will be accomplished if we show (i) $G(R^l \setminus R_-^l) \subset R^l \setminus R_-^l$, (ii) $G: R^l \setminus R_-^l \rightarrow R^l \setminus R_-^l$ is proper, and (iii) G is a local homeomorphism.

(i) $G(R^l \setminus R_-^l) \subset R^l \setminus R_-^l$.

LEMMA 1 Let $F: R_+^m \rightarrow R_+^m$ be a system of homogeneous functions. Suppose that for all $x \in R_+^m \setminus \{0\}$, $JF(x)$ is nonsingular. Then $\|x_n\| \rightarrow \infty$ implies $\|F(x_n)\| \rightarrow \infty$.

Proof Suppose there is $\|x_n\| \rightarrow \infty$ such that $\|F(x_n)\| \leq k$ for some k . Then $\|F((1/\|x_n\|)x_n)\| (k/\|x_n\|)$. Without loss of generality we can assume $(1/\|x_n\|)x_n \rightarrow x$. By continuity $\|x\| = 1$ and also $\|F(x)\| = 0$, which implies $F(x) = 0$. By hypothesis $JF(x)$ is nonsingular. By the homogeneity of F we have (using Euler's rule) that $JF(x)x = 0$. So $x = 0$, which contradicts $\|x\| = 1$. Therefore no such sequence x_n can exist, and the lemma is proved. ■

Given $\bar{x} \in R^l \setminus R_-^l$ by relabeling variables if necessary, we can assume that $\bar{x}^i > 0$ for $i \leq k$ and $\bar{x}^i \leq 0$ for $i > k$. Let $N = \{k+1, \dots, l\}$ and Π^* be the linear operator $\Pi^*(x) = (x^1, \dots, x^k, 0, \dots, 0)$. Note that $\Pi(\bar{x}) = \Pi^*(\bar{x})$ and that $\Pi^* \circ F(\Pi^*(x))$, if defined, is naturally identified with $F_N(x^1, \dots, x^k)$. Henceforth, since $\bar{x}^1 > 0$, it follows from Lemma 1 and the homogeneity of F_N that $F_N(\bar{x}^1, \dots, \bar{x}^k) \neq 0$. On the other hand, suppose that $G(\bar{x}) \leq 0$. Then $\Pi^* \circ G(\bar{x}) \leq 0$. However, $\Pi^* \circ G(\bar{x}) = \Pi^* \circ F(\Pi^*(\bar{x}))$. So $\Pi^* \circ F(\Pi^*(\bar{x})) \leq 0$. But $F(\Pi^*(\bar{x})) \geq 0$ by hypothesis (this is the only point in the proof where the assumption $F(R_+^l) \subset R_+^l$ is appealed to). So $\Pi^* \circ F(\Pi^*(\bar{x})) = 0$. Contradiction.

(ii) $G: R^l \setminus R_-^l \rightarrow R^l \setminus R_-^l$ is proper. To verify that $G: R^l \setminus R_-^l \rightarrow R^l \setminus R_-^l$ is proper, it suffices to check that given a sequence $x_n \in R^l \setminus R_-^l$, one has

- (a) if $\|x_n\| \rightarrow \infty$, then $\|G(x_n)\| \rightarrow \infty$;
- (b) if $\max_i x_n^i \rightarrow 0$, then $\max_i G^i(x_n) \rightarrow 0$.

That (b) holds is obvious; if $\max_i x_n^i \rightarrow 0$, then $\Pi(x_n) \rightarrow 0$, and by continuity, $F(\Pi(x_n)) \rightarrow 0$. So $\max_i G^i(x) = \max_i (F^i(\Pi(x_n)) + x_n^i - \Pi^i(x_n)) \rightarrow 0$.

Since there are only a finite number of orthants in R^l , it suffices to prove (a) for a sequence x_n all of which terms belongs to the same orthant. [Indeed, if (a) is not true, then $G(x_n)$ has a bounded subsequence. But any subsequence of x_n has a subsequence all of which terms are in the same orthant. Contradiction.] So, by relabeling variables if necessary, we can assume that for $a, 1 \leq k \leq l$ and all $n, x_n^i \geq 0$ if $i \leq k$, and $x_n^i \leq 0$ for $i > k$.

Let $N = \{k + 1, \dots, l\}$ and Π^* be the linear operator

$$\Pi^*(x) = (x^1, \dots, x^k, 0, \dots, 0).$$

For any $x_n, \Pi(x_n) = \Pi^*(x_n)$. Note, also, that $\Pi^* \circ F(\Pi^*(x_n))$ is naturally identified with $F_N(x_n^1, \dots, x_n^k)$. In particular, by Lemma 1, if $\{\Pi^*(x_n)\}$ is unbounded, $\{\Pi^* \circ F(\Pi^*(x_n))\}$ must be unbounded.

Letting I be the identity map, we have $G(x_n) = \Pi^* \circ F(\Pi^*(x_n)) + (I - \Pi^*)(F(\Pi^*(x_n)) + x_n)$. So

$$\|G(x_n)\| = \|\Pi^* \circ F(\Pi^*(x_n))\| + \|(I - \Pi^*)(F(\Pi^*(x_n)) + x_n)\|.$$

Either $\{\|\Pi^*(x_n)\|\}$ or $\{\|(I - \Pi^*)(x_n)\|\}$ must be unbounded. If $\{\|\Pi^*(x_n)\|\}$ is bounded, then $\{\|(I - \Pi^*)(x_n)\|\}$ is unbounded, and therefore so is the second term of the previous sum. If $\{\|\Pi^*(x_n)\|\}$ is unbounded, then so is the first term of the sum by the observation of the preceding paragraph. In either case $\{\|G(x_n)\|\}$ is unbounded. Since x_n is arbitrary, this yields (b).

(iii) G is a local homeomorphism. To establish that G is a local homeomorphism, we will make use of the following theorem due to Clarke [3]:

Suppose that $G: Q \rightarrow Q, Q \subset R^l$ open, is a Lipschitzian function. Denote by $U \subset Q$ the region of differentiability of G , i.e., $U = \{x \in Q: JG(x) \text{ exists}\}$. For any $x \in U$ define $\hat{J}G(x) = \{A: JG(x_n) \rightarrow A \text{ for some } x_n \rightarrow x, x_n \in U\}$. If every $A \in$ closed convex hull $\hat{J}G(x)$ is nonsingular, then G is a local homeomorphism at x .

Our function $G: R^l \setminus R_-^l \rightarrow R^l \setminus R_-^l$ is piecewise continuously differentiable, hence Lipschitzian. Also, G is C^1 at any $x \in R^l \setminus R_-^l$ such that $x^i \neq 0$ for all i . Henceforth, by Clarke's theorem, we must verify that, given $A \in$ convex hull $\hat{J}G(x) =$ convex hull $\{A: JG(x_n) \rightarrow A, \text{ for some } x_n \rightarrow x, x_n^i \neq 0 \text{ for all } i, n\}$ is nonsingular. Note that $\hat{J}G(x)$ is a finite set.

So let $x \in R^l \setminus R_-^l$. By relabeling variables if necessary, we can assume that $x^i > 0$ for $i \leq k, x^i = 0$ for $k < i \leq h$, and $x^i < 0$ for $h < i \leq l$. Note that $k \geq 1$. In order to compute $\hat{J}G(x)$, it suffices to consider sequences $x_n \rightarrow x$ with all x_n belonging to the same orthant; i.e., we can assume $x_n^i > 0$ for $i \leq f \leq h$ and $x_n^i < 0$ for $i > f \geq k$. Then $JG(x_n)$ exists. In fact, if we denote

by $J_f F(\Pi(x))$ the first f columns of $JF(\Pi(x))$, we have

$$\lim JG(x_n) = \begin{bmatrix} J_f F(\Pi(x)) & 0 \\ & I \\ & & (l-f) \times (l-f) \end{bmatrix}.$$

Therefore, denoting by $\sigma_i F(\Pi(x))$ the i th column of $JF(\Pi(x))$, by e_i the i th column unit vector, and by a_i the i th column of a generic matrix A , we have

$$\begin{aligned} \hat{J}G(x) = \{A : & \text{for } i \leq k, a_i = \sigma_i F(\Pi(x)) \\ & \text{for } i > h, a_i = e_i \\ & \text{for } k < i \leq h, a_i = e_i \text{ or } a_i = \sigma_i F(\Pi(x))\}. \end{aligned}$$

A set of matrices defined as $\hat{J}G(x)$ has an interesting property. The multilinearity of the determinant function implies that if $A \in$ convex hull $\hat{J}G(x)$, then $|A|$ is a positive linear combination of determinants of matrices belonging to $\hat{J}G(x)$. Therefore, if the determinant of the matrices in $\hat{J}G(x)$ have a uniform sign, A must be nonsingular.

As we have just seen, if $A \in \hat{J}F(\Pi(x))$, then

$$A = \begin{bmatrix} J_f F(\Pi(x)) & 0 \\ & I \\ & & (l-f) \times (l-f) \end{bmatrix}, \quad \text{for some } f \geq k.$$

Therefore $|A| = |J_{ff} F(\Pi(x))|$, where $J_{ff} F(\Pi(x))$ is the leading $f \times f$ principal minor of $JF(\Pi(x))$. But $J_{ff} F(\Pi(x)) = JF_{\{f+1, \dots, l\}}(x^1, \dots, x^l)$. So by hypothesis, all the $|A|$, $A \in JG(x)$ have the same sign and this concludes our proof.

REFERENCES

- Berger, M., "Nonlinearity and Functional Analysis." Academic Press, New York, 1977.
- Chipman, J., A survey of the theory of international trade: Part 3, The Modern Theory, *Econometrica* **34**, 18-76 (1966).
- Clarke, F. H., On the inverse function theorem, *Pacific J. Math.* **64**, 97-102 (1976).
- Gale, D., and Nikaido, H., The Jacobian Matrix and the global univalence of mappings, *Math. Ann.* **159**, 81-93. (1965).
- Hadamard, J., Sur les equations fonctionelles, *C. R. Acad. Sci. Paris Ser. A-B* **136**, 351 (1904).
- Mas-Colell, A., Homeomorphism of compact, convex sets and the Jacobian Matrix, *SIAM J. Math. Anal.* (1979).
- McKenzie, L., Equality of factor prices in world trade, *Econometrica* **23**, 239-257 (1955).
- McKenzie, L., The inversion of cost functions: A counterexample, *Internat. Econ. Rev.* **8**, 271-278 (1967).
- Nikaido, H., Relative shares and factor Price equalization, *J. Internat. Econ.* **2**, 257-264 (1972).
- Pierce, I., A further note on factor commodity price relationships, *Econ. J.* **69**, 725-732 (1959).

11. Samuelson, P., International factor-price equalization once again, *Econ. J.* **59**, 181–197 (1949).
12. Samuelson, P., Prices of factors and goods in general equilibrium, *Rev. Econ. Stud.* **21**, 1–20 (1953).
13. Samuelson, P., “Collected Scientific Papers,” Vol. 2, p. 908. MIT Press, Cambridge, Massachusetts, 1966.

DEPARTMENT OF ECONOMICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA