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A REFINEMENT OF THE CORE EQUIVALENCE THEOREM

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It is known that if blocking coalitions are restricted to be small relative to the size of the economy then the approximation of core allocations by competitive allocations can still be assured. This paper proves that the same type of approximation result is valid if the *absolute* size of the blocking coalition is bounded. The methods of Anderson (1978) are applied.

1. Introduction

It is an important feature, showed first by Schmeidler (1972), of the Aumann (1964) continuum version of the Core Equivalence Theorem that if an allocation can be blocked at all then it can be blocked by an arbitrarily small coalition.

This is important because the notion of blocking is particularly plausible for small coalitions. Indeed, as with Cournot classical non-cooperative theory, it is precisely when agents (in this case coalitions) are very small when it is justified for them to neglect (as the Core solution concept asks them to) the induced effects a blocking move will generate via the impact on other agents.

But how small is small? The continuum framework is not readily conducive to a very refined answer. After all, as long as a coalition has positive measure it is still arbitrarily large relative to the size of an 'individual'. So, we shall pose our question in the context of finite economies. More precisely, suppose that an allocation in an economy with n traders is bounded away (in some appropriate sense) from being Walrasian. Schmeidler's result suggests that the minimal *relative* size of a blocking coalition goes to zero and indeed a theorem along these lines has been proved by Khan–Rashid (1976 and 1978). But how fast does it go to zero? A little reflection on the Core Equivalence Theorem in replica economies [Debreu–Scarf (1963)]

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indicates it is to be expected that the minimal *absolute* size of a blocking coalition be bounded independently of n. This we shall establish in all generality. As it turns out all that is needed is a small modification of the very elegant proof of a Core Equivalence Theorem provided by Anderson (1978). To minimize changes we pattern our result after his.

2. Model, definition and result

The *commodity space* is R^l. *Trader's characteristics* are pairs $(X, \succ)$ where $X \subset R^l$ is the *trading set* and $\succ \subset X \times X$ is a transitive, locally non-satiated *preference relation*.

For a given $s > 0$ we let our *space of traders characteristics* be $\mathcal{A} = \{(X, \succ): X \geqslant -se\}$, i.e., we will consider only economies with uniformly bounded below trading sets.

An *economy* is a map $\mathcal{E}: I \to \mathcal{A}$, where I is a finite indexing set.

An *allocation* is a map $x: I \to R^l$ such that $\Sigma_{i \in I}\, x(i) = 0$ and $x(i) \in X(i)$ for all $i \in I$.

An allocation x is *blocked* by the coalition $C \subset I$ if $C \neq \phi$ and there is $x': C \to R^l$ such that $\Sigma_{i \in C'}\, x'(i) \leqslant 0$ and $x'(i) \in X_i$, $x'(i) \succ_i x(i)$ for all $i \in C$.

Let $\Delta \subset R^l$ be the unit simplex. Given an economy $\mathcal{E}: I \to \mathcal{A}$, an allocation $x: I \to R^l$ and a price vector $p \in \Delta$ we could measure the (absolute) departure of x from being Walrasian at prices p as in Anderson (1978), i.e., by the expression $c(x, p) = \Sigma_{i \in I} (|p \cdot x(i)| + |\inf\{p \cdot y: y \succ_i x(i)\}|)$. Clearly, the pair (x, p) is in Walrasian equilibrium with the usual definition if and only if $c(x, p) = 0$.

Given any real number r the smallest integer larger than r is denoted $[r]$.

Theorem. For any economy $\mathcal{E}: I \to \mathcal{A}$ and integer $M \geqslant 1$ if $x: I \to R^l$ is an allocation which is not blocked by any coalition $C \subset I$ with $\#C \leqslant M$, then there is $p \in \Delta$ such that

$$c(x, p) \leqslant \left\lceil \frac{\#I}{M} \right\rceil 4ls\,.$$

For $M = \#I$ this is precisely Anderson's (1978) result. Our proof amounts to a slight variation of his.

3. Proof of the theorem

Let $\mathcal{E}: I \to \mathcal{A}, x: I \to R^l$ be the given economy and allocation. Put $\#(I) = N$ and fix $M \geqslant 1$.

Put $H = [N/M]$ and partition I in two different ways $\{I_m: m \in M\}$, $\{I_h: h \in H\}$

in such a manner that $\#I_m \leqslant H$ for all $m \in M$ and $\#(I_m \cap I_h) \leqslant 1$ for all $m \in M$ and $h \in H$.

For each $m \in M$ put $V_m = \cup_{i\in I_m} \{y: y \succ_i x(i)\} \cup \{0\}$. Let V = convex hull $\Sigma_{m\in M} V_m$. Suppose that $y \in V$ and $y < -lse$. By the Shapley–Folkman Theorem [see, for example, Arrow and Hahn (1971)] there are $y_m \in R^l$ such that $\Sigma_{m\in M} y_m = y$, $y_m \in$ convex hull V_m for all $m \in M$, and $\#\{m \in M: y_m \notin V_m\} \leqslant l$. Therefore, denoting $M' = \{m \in M: y_m \in V_m\}$, we have

$$\sum_{m\in M'} y_m = y - \sum_{m\in M\setminus M'} y_m \leqslant y + lse < 0.$$

But then if we let $M'' = \{m \in M': y_m \neq 0\}$ we have $M'' \neq \phi$ and so if for each $m \in M''$ we pick i_m such that $y_m \succ_{i_m} x(i_m)$ we have that $\{i_m : m \in M''\}$ constitutes a blocking coalition with at most m traders. This is a contradiction and we conclude that if $y < -lse$ then $y \notin V$. Therefore, by the separating hyperplane theorem there is $p \in \Delta$ such that for all $v \in V$ we have $p \cdot v \geqslant -lsp \cdot e = -ls$.

Consider any $z: I \to R^l$ such that $\Sigma_{i\in I}\, p \cdot z(i) \leqslant 0$ and $z(i) \in \overline{\{y: y \succ_i x(i)\}}$. Let $I' = \{i \in I: p \cdot z(i) < 0\}$. Then

$$\sum_{i\in I} |p \cdot z(i)| \leqslant -2 (\sum_{i\in I'} p \cdot z(i)) = -2 (\sum_h p \cdot (\sum_{i\in I'\cap I^h} z(i))) \,.$$

But $\Sigma_{i\in I'\cap I^h}\, z(i) \in \overline{V}$ for all $h \in H$. Therefore, $\Sigma_{i\in I} |p \cdot z_i| \leqslant 2Hls$.

By local non-satiation $x(i) \in \overline{\{y: y \succ_i y(i)\}}$ for all $i \in I$. Therefore, $\Sigma_{i\in I} |p \cdot x(i)| \leqslant 2Hls$. Also,

$$|\inf\{p \cdot y: y \succ_i x(i)\}| = |\inf\{p \cdot y: y \in \overline{\{y: y \succ_i x(i)\}}\,,\, p \cdot y \leqslant p \cdot x(i)\}|$$

$$\leqslant \sup\{|p \cdot y|: y \in \overline{\{y: y \succ_i x(i)\}}\,,\, p \cdot y \leqslant p \cdot x(i)\} \qquad \text{for all } i \in I.$$

Hence

$$\sum_{i\in I} |\inf\{p \cdot y: y \succ_i x(i)\}| \leqslant 2Hls \,.$$

Summing up: $c(x, p) \leqslant 4Hls$. Q.E.D.

4. Concluding remark

We have based our interest in small blocking coalitions on strategic game considerations. An alternative and/or complement are coordination costs. However, if coordination costs are the main concern then it seems important to prove not just that blocking can be effected by small coalitions but, besides, that those are easy to form. This suggests a probabilistic approach. We have not explored it. Some of the methods and tools used in Mas-Colell (1978) could conceivably be brought to bear.

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