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Remarks on the Game-Theoretic Analysis of a Simple Distribution of Surplus Problem[1])

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1. Introduction

We will study, from the point of view of the Shapley value for non-side-payments games [*Shapley; Shapley/Shubik; Aumann; Aumann/Shapley*] and, more incidentally, the Core, an economic problem with the following characteristics:

a) There are increasing returns to scale.
b) It is very simple, i.e., there are only one input and one output, and (except in the last two sections) the input does not enter the utility functions.
c) Utility functions exhibit decreasing marginal utility.
d) There is a continuum of traders which, for simplicity, fall into a finite number of types.

An economy with property a) generates a positive surplus in the sense that society as a whole can get more than the sum of what individual components can get by themselves.

We adapt the continuum-of-agents approach and thus rely on the *Aumann/Shapley* theory [1974] because the value becomes then easy to compute and because it is the context where it has a simpler economic interpretation in terms of marginal contributions.

Our purpose is to begin tentatively an analysis of the performance of the value concept in "difficult cases", i.e., cases where competitive equilibrium fails to exist. If there are constant returns to scale, we fall under the domain of the "equivalence theorems" [*Champsaur*, 1975; *Aumann; Mas-Colell*]: The competitive solution is defined and the value is indistinguishable from it. As soon as we have increasing returns, the competitive solution may not exist, but value allocations will still be defined and *they will now be sensitive to the cardinal characteristics of utility functions.* We show, and this is the only result which is new in this note, that this dependence goes in the ex-

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pected direction (Section 7). The question of uniqueness of value allocations is left unsettled.

We end in a somewhat pessimistic note (Sections 8 and 9). If situations are less simple and if there is no transferable utility, it is not clear that, existence-wise, the value performs much better than, say, the Core. Section 10 contains an elementary proof of a result of *Scarf* on the non-emptiness of the Core. Section 11 does the same and slightly generalizes a result of *Champsaur* [1975b] for economies with one public and one private good.

2. The Model

There are only two commodities. One serves as *input* of a *production function* $f: R_+ \to R_+$ while the other is the *output* and shall also be called *income*. Quantities of input (resp. output) are denoted z (resp. x).

We will postulate a continuum of agents, but to stick to essentials and avoid technical problems, we assume that agents characteristics fall into a finite number of types. The characteristics of type i, $1 \leq i \leq m$, are the utility function on income $u_i: R_+ \to R_+$ and a quantity of input $a_i > 0$. So, we are assuming that inputs do not enter the utility function. We will have an opportunity to reconsider this hypothesis in Sections 10 and 11.

The measure of type i is $s_i > 0$, i.e., $s_i / \sum_{j=1}^{m} s_j$ is the fraction of the total mass of agents with characteristics of type i. We normalize by taking $\sum_{i=1}^{m} s_i a_i = 1$.

3. Hypothesis

(H.1) f is C^1 and exhibits non-decreasing average product, i.e., $f(z)/z$ does not decrease with z. See Figure 1.

(H.2) For every i, u_i is increasing, continuous, concave, and C^1 on R_{++}. Also $u_i(0) = 0$.

(H.3) f is C^2 and $f'(0) > 0$.

(H.4) For every i, u_i is C^2 on R_{++} and $u_i''(z) \neq 0$ for all $z > 0$. Also, $u_i'(z_n) \to \infty$ when $z_n \to 0$.

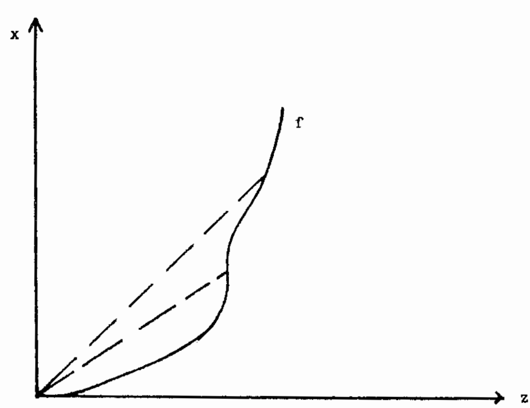


Fig. 1

The essential hypotheses are (H.1) and (H.2); (H.3) and (H.4) are merely convenient ones. With them, every function to be considered will be everywhere differentiable, and we will avoid having to keep track of right and left derivatives. All our subsequent discussion holds true without (H.3) and (H.4) and applies in particular to the important case of linear utility functions.

3. Social Utility Relative to λ

As we are not in a transferable-utility setting, we shall adapt a device proposed by *Shapley* [1969]. Let $\lambda = (\lambda_1, \ldots, \lambda_m) > 0$ be a given vector of utility weights, i.e., of rates of transfer of utility among types. Then, for any vector $\mu = (\mu_1, \ldots, \mu_m) \geqslant 0$ of measures for the m types, the social utility of μ relative to λ, denoted $V_\lambda(\mu)$, is defined

as $V_\lambda(\mu) = \text{Max} \sum_{i=1}^{m} \lambda_i \mu_i u_i(x_i)$

$$\text{s.t.} \sum_{i=1} \mu_i x_i \leqslant f\left(\sum_{i=1}^{m} \mu_i a_i\right).$$

Given (H.1) – (H.4), the solution to this problem, for $\mu \neq 0$ and $\lambda \gg 0$, is characterized by the existence of $p(\lambda, \mu) > 0$ and $x_i(\lambda, \mu) > 0$, $1 \leqslant i \leqslant m$ such that $\lambda_i u_i'(x_i(\lambda, \mu)) = p(\lambda, \mu)$ and $\sum_{i=1}^{m} \mu_i x_i(\lambda, \mu) = f\left(\sum_{i=1}^{m} \mu_i a_i\right)$. Further, $p(\lambda, \mu)$ and $x_i(\lambda, \mu)$ are C^1 functions of their arguments (notice that if $\mu \neq 0$, then $\sum_{i=1}^{m} \mu_i a_i > 0$ and so, by (H.1) and (H.3), $f\left(\sum_{i=1}^{m} \mu_i a_i\right) > 0$). Hence, for $\lambda \gg 0$, $V_\lambda(\mu)$ is a C^1 function whenever $\mu \neq 0$. At $\mu = 0$, V_λ is a linear homogeneous function.

4. The Shapley Value Relative to λ

Let $\lambda \gg 0$ be fixed.

We are in a situation where the total income that society can obtain, i.e., $f\left(\sum_{i=1}^{m} s_i a_i\right)$, is generally larger than the sum of the incomes every agent can obtain by himself, i.e., $\sum_{i=1}^{m} f'(0)\, s_i a_i = f'(0)$. So, in this precise sense, there is a social surplus. We are interested in analysing its distribution according to the game-theoretic concept of the (Shapley) value.

A theory of the value for a continuum of agents has bee presented by *Aumann/Shapley* [1974]. From their work, one gets for the present case a very intuitive and easily computable (much more easily than in the finite-number-of-agents case) formula for the value (Theorem B, p. 23; Prop. 10.17, p. 92). In fact, it is those two virtues which have inclined us to adopt the continuum-of-agents approach. It could be argued that in what concerns economics, Shapley value theory is in its stronger ground in the continuum-of-agents case. There, besides ease of computation, it has an appealing intuitive content as a sort of generalized marginal-productivity theory.

We shall now motivate the value formula. To be self-contained, we avail ourselves of the following "trick": the formula will be derived in intuitive grounds; and, once available, it will be christened "the value". It goes without saying that for a rigorous treatment the *Aumann/Shapley* book must be consulted.

Let us first remember the fomula for the Shapley value of a finite game with side payments. The value of player i is the expectation of his contribution to coalitions not including i. The average is calculated according to the rules: First, a size of coalitions is chosen at random with a uniform probability over sizes. Second, with uniform probability, a coalition of the given size is chosen.

In our case, we have a continuum of sizes $0 \leqslant t \leqslant 1$.

For a fixed $t > 0$, the law of large numbers suggests that all the relevant coalitions will be perfect samples of the whole, i.e., the measure of agents of type i in the coalition will be $t\, s_i$.

The marginal contribution of an agent of type i is, therefore, $(\partial V_\lambda / \partial \mu_i)(ts)$. Taking the average, we *define* the value of an agent of type i as:

$$\nu_i(\lambda) = \int_0^1 \frac{\partial V_\lambda}{\partial \mu_i}(ts)\, dt.$$

As, along any line segment, the function $V_\lambda(\cdot)$ is the integral of its derivative (remember $V_\lambda(\cdot)$ is C^1 everywhere, except possibly at 0) and $V_\lambda(0) = 0$, we have

$$V_\lambda(s) = \sum_{i=1}^{m} s_i \left(\int_0^1 \frac{\partial V_\lambda}{\partial \mu_i}(ts) \right) = \sum_{i=1}^{m} s_i \nu_i(\lambda).$$

This shows both that ν_i is well defined and that the assignment of its value to every agent does exactly exhaust the social utility (everything relative to λ).

Dropping for the moment reference to the fixed λ, put $p(t) = p(\lambda, ts)$, $x_i(t) = x_i(\lambda, ts)$, $0 < t \leqslant 1$. Because $V(\mu)$ is the maximum of $\sum_{i=1}^{m} \lambda_i \mu_i u_i(x_i)$ subject to $\sum_{i=1}^{m} \mu_i x_i \leqslant f(\sum_{i=1}^{m} \mu_i a_i)$, a simple computation gives us:

$$\frac{\partial V_\lambda}{\partial \mu_i}(ts) = \lambda_i u_i(x_i(t)) - p(t) x_i(t) + p(t) f'(t) a_i .$$

The formula has a straightforward interpretation: $\lambda_i u_i(x_i(t))$ is the utility that the agent of type i joining the coalition will enjoy, $-p(t) x_i(t)$ is the decrease in the total utility of the coalition due to the transfer of income to the new agent, $p(t) f'(t) a_i$ is the increase in total utility due to the quantity of input contributed by the new agent.

How do $x_i(t)$ and $p(t)$ depend on t? $x_i(t)$ is inversely related to $p(t)$ and, therefore, all the $x_i(t)$'s move in the same direction. We claim that $x_i(t)$ is non-decreasing with t.

We have

$$\frac{dV_\lambda}{dt}(ts) = \sum_{i=1}^{m} s_i \frac{\partial V_\lambda}{\partial \mu_i}(ts) =$$

$$\sum_{i=1}^{m} \lambda_i s_i u_i(x_i(t)) - p(t)\left(\sum_{i=1}^{m} s_i x_i(t)\right) + p(t) f'(t) =$$

$$\frac{1}{t} V_\lambda(ts) + p(t)(f'(t)\,t - f(t)).$$

By the non-decreasing average product hypothesis $f'(t)\,t \geqslant f(t)$. Hence $(dV(ts))/dt \geqslant (V(ts))/t$. However, if $p'(t) > 0$, then $x_i'(t) < 0$ and so all the marginal increment of utility to a coalition ts is inferior to the utility accruing from the newly joining members, i.e., $(dV_\lambda/dt)(ts) < \sum_{i=1}^{m} s_i \lambda_i u_i(x_i(t)) = (V(ts))/t$. But this is not possible. Hence, $p'(t) \geqslant 0$ for all t.

In fact, the argument of the previous paragraph shows that if $f'(t)\,t > f(t)$ for any $0 < t \leqslant 1$, i.e., if f is not linear on [0, 1], then $p(t)$ cannot be a constant, i.e., $p(1) < \lim_{t \to 0} p(t)$.

5. Value Allocations

An m-tuple $x = (x_1, \ldots, x_m)$ shall be called an *allocation* if $\sum_{i=1}^{m} s_i x_i = f\left(\sum_{i=1}^{m} s_i a_i\right)$.

For a fixed $\lambda \geqslant 0$, there is no reason to expect that the value $\nu(\lambda) = (\nu_1(\lambda), \ldots, \nu_m(\lambda))$ be feasible in the sense that an allocation x exists for which $\lambda_i u_i(x_i) = \nu_i(\lambda)$, all i. Following *Shapley* [1969] and *Aumann* [1975], we *define* an allocation x to be a *value allocation* if for some $\lambda \geqslant 0$, $\lambda_i u_i(x_i) = \nu_i(\lambda)$ for all i; that is to say, with the utility weights λ, the value utility assignments can be reached without actual transfers of utility. See Figure 2.

One has:

(1) Value allocations exists

The proof shall be skipped. It can be carried out with the help of Brouwer's fixed point theorem. See *Shapley* [1969]. There is a difference between the situation here and in *Shapley*'s. Here we want to obtain value allocations with $\lambda \geqslant 0$. What one does is to restrict λ to the unit simplex and then note that if $\lambda_n \to \lambda$ and $\lambda_i = 0$, $\nu_i(\lambda_n)$ remains nevertheless positive and bounded away from zero (remember we assume

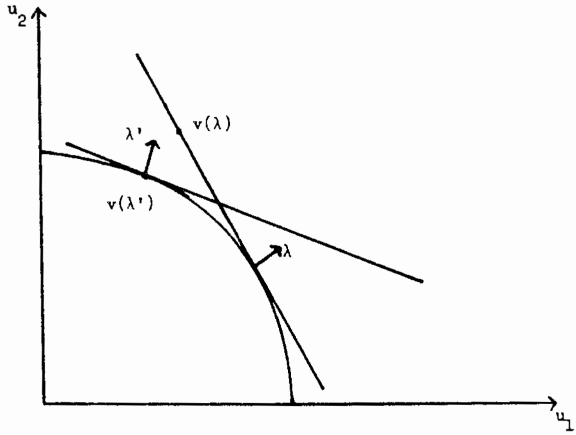


Fig. 2

$a_i > 0$). With the observation in mind *Shapley*'s proof goes through almost *verbatim.*

Open question: Is the value allocation unique?

Value allocations are invariant with respect to positive linear transformations of the utility functions. This is clear with respect to multiplication by positive constant. Since we have normalized to $u_i(0) = 0$ the possibility of adding up a constant is precluded but there is no obstacle in defining value allocations without *a priori* fixing the origin of the utility functions, in which case the invariance with respect to origin also obtains.

6. An Example

Suppose that all utility functions are identical and homogeneous of degree $0 < \sigma < 1$, i.e., $u_i(x_i) = x_i^\sigma$. Normalize λ so that $\sum_{i=1}^m (s_i/(\lambda_i^{1/(\sigma-1)})) = 1$. Then a few computations give $x_i(\lambda, \mu) = (1/(\lambda_i^{1/(\sigma-1)}))f(\sum_{i=1}^m \mu_i a_i)$ and so, $V_\lambda(\mu) = \sum_{i=1}^m \lambda_i s_i (x_i(\lambda, \mu))^\sigma = (f(\sum_{i=1}^m \lambda_i a_i))^\sigma$. Therefore $V_\lambda(ts) = (f(t))^\sigma$ and $(\partial V_\lambda/\partial \mu_i)(ts) = \sigma(f(t))^{\sigma-1}f'(t)\,a_i$. Since the coefficient of a_i is independent of i we must have $v_i(\lambda) = (f(1))^\sigma a_i$.

Suppose that λ gives raise to a value allocation x. Then, of course, $x_i = (1/(\lambda_i^{1/\sigma-1}))f(1)$ and by the definition of value allocation $(f(1))^\sigma a_i = v_i(\lambda) = \lambda_i x_i^\sigma = (1/(\lambda_i^{1/(\sigma-1)}))\,(f(1))^\sigma = (f(1))^{\sigma-1}x_i$. So, $x_i = a_i f(1)$, i.e., the value allocation is unique and it is the allocation of output proportional to the contribution of inputs.

We note that the previous result is also valid for $\sigma = 1$, i.e., for linear utilities.

7. A Result on Value Allocations and the Concavity of Utility Functions

Value allocations depend on cardinal characteristics of preferences. We shall investigate in this section an aspect of this dependence. Specifically, we will prove a not unexpected result: at a value allocation if an agent of type i has no more input than one agent of type j and if the type i utility function is a concave transformation of type j utility then the agent of type i receives no more income than the agent of type j.

More formally, write $i \cdot \geqq j$ if $a_i \geqq a_j$ and $u_j = g \cdot u_i$ where g is C^1 and concave. The interpretation of the relation $u_j = g \cdot u_i$ as u_j being more risk-averse than u_i is well known [see *Pratt; Debreu; Kannai*].

Proposition 1: Let x be a value allocation. If $i \cdot \geqq j$ then $x_i \geqq x_j$.

Proof: We shall argue by contradiction. Suppose that $x_j > x_i$.

Since the property $i \cdot \geqq j$ and the value character of x are invariant under linear rescalings of the utility functions, we can assume that the utility weights associated with λ are all equal to one.

Because $x_j > x_i$ and $u_j'(x_j) = u_i'(x_i)$ we have $u_i'(x_j) < u_j'(x_j)$. But $u_j'(x_j) = = g'(u_i(x_j)) \cdot u_i'(x_j)$ and so, $g'(u_i(x_j)) > 1$. Since g is concave we conclude that $g'(h) > 1$ for $h \leqq u_i(x_j)$.

If $u_j'(y) = u_i'(v)$ for $y \leqq x_j$, $v \leqq x_i$ then $y > z$. Indeed, suppose that $v \leqq y$; by the mean value theorem, $u_j'(w) = u_i'(w)$ for some $w \leqq x_i$. Hence $u_j'(w) = = g'(u_i(w))\, u_i'(w)$ implying $g'(u_i(w)) = 1$ and contradicting therefore the conclusion of the previous paragraph.

Let $p(t), x_i(t), x_j(t)$ be the corresponding size $0 < t \leqq 1$ concepts in the definition of the value. Then $p'(t) \leqslant 0, x_i'(t) \geqq 0, x_j'(t) \geqq 0$.

Denote $\phi_i(t) = u_i(x_i(t)) - p(t)\, x_i(t)$ and, similarly, $\phi_j(t)$. Note that $\phi_i'(t) = -x_i(t)\, p'(t), \phi_j'(t) = -x_j(t)\, p'(t)$.

Since x is a value allocation

$u_i(x_i) = \int_0^1 \phi_i(t)\, dt + \int_0^1 p(t) f'(t)\, a_i dt$, and analogously for j. Therefore

$$u_i(x_i) - u_j(x_j) = \int_0^1 (\phi_i(t) - \phi_j(t))\, dt + (a_i - a_j) \int_0^1 p(t) f'(t)\, dt.$$

Because $p(t) f'(t) \geqq 0$ and $a_i - a_j \geqq 0$, we have $\int_0^1 (\phi_i(t) - \phi_j(t))\, dt \leqq u_i(x_i) - u_j(x_j)$.

Now, $\phi_i'(t) - \phi_j'(t) = (x_j(t) - x_i(t))\, p'(t)$. Since $u_j'(x_j(t)) = u_i'(x_i(t))$ and $x_j(t) \leqq x_j$, $x_i(t) \leqq x_i$ we must have $x_j(t) \geqq x_i(t)$. We conclude that the function $\phi_i(t) - \phi_j(t)$ is non-increasing and therefore $\int_0^1 (\phi_i(t) - \phi_j(t))\, dt \geqq \phi_i(1) - \phi_j(1)$. Hence $u_i(x_i) - u_j(x_j) \geqq \phi_i(1) - \phi_j(1)$. On the other hand, $x_i = x_i(1)$ and $x_j = x_j(1)$ which implies $\phi_i(1) - \phi_j(1) = u_i(x_i) - u_j(x_j) + p(1)(x_j - x_i) > u_i(x_i) - u_j(x_j)$ because $p(1) > 0$ and by assumption $x_j > x_i$. This contradiction proves the proposition. □

The following fact is easily deduced from the last paragraph of section 4 and a careful analysis of the preceding proof.

(2) *Suppose that f is not linear on* [0,1]. *Let $i \cdot \geqq j$ and suppose that either $a_i > a_j$ or $u_j = g \cdot u_i$ where g is C^2 and $g''(v) \neq 0$ for all $v \in R_+$. Then if x is a value allocation $x_i > x_j$.*

See *Kurz* [1977] for a result of the same nature as Proposition 1 in a different model.

8. Some Comments on Incentives, the Value and the Core

An implication of Proposition 1 is that in our model value allocations enjoy the equal treatment property, i.e., if two agents have the same endowments and, up to a strictly monotone linear transformation, the same utility function, then they receive the same amount of income at value allocations.

The Proposition makes also very clear the direction in which incentives work. Suppose that agents were free to choose, i.e., to distort, their characteristics. Then the tendency would be for individual agents (keep in mind that single agents are infinitesimal) to switch to less concave utility functions and to contribute all their endowments of inputs. Going to the limit, we would expect to find all the agents with linea utility functions and not holding back any amount of input. Then the (unique) value allocation is the proportional one, i.e., an agent of type i receives $f(1)a_i$. So, it would appear that from the point of view of the value the proportional allocation plays a distinguished role.

The proportional allocation does also belong to the Core since for any proper subcoalition of agents their average productivity will by necessity be no larger than their average income in the coalition of the whole. So, we see that in the linear utility case the value allocation belongs to the Core. This is not true in general. The Core is an ordinal concept, i.e., invariant with respect to strictly monotonic transformations of utility functions, while value allocations depend on cardinal characteristics. It should be expected that if we begin with linear utility functions and then replace them by concave functions of widely different "degree of concavity" then the value allocation (5) will be pushed away from the Core (which, of course, remains unaltered).

Consideration of an example may help. Let $a_1 = \ldots = a_m$ and f be as in Figure 3. Then the only Core allocation is the proportional one, i.e., every agent gets the same quantity of income. But if two types, say i and j, are such that $u_j = g \cdot u_i$ where g is C^2, concave and $g''(v) \neq 0$ for all $v \in R_+$, then the equal distribution of income cannot be a value allocation; see (2) at the end of section 7. Of course, we can perturb f sligthly so as to avoid the tangency property of Figure 3 without altering the qualitative features of the example.

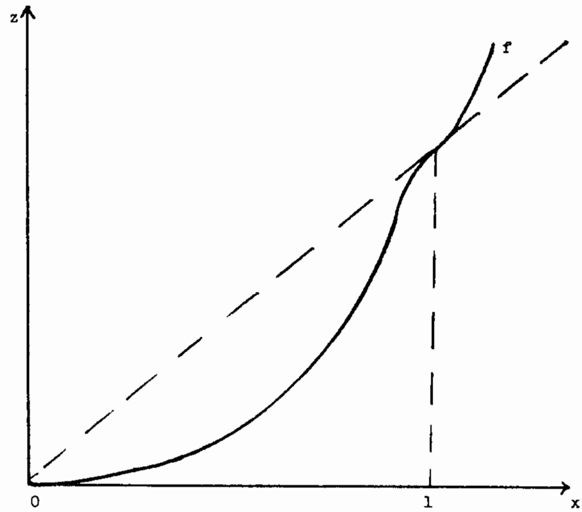


Fig. 3

9. Limited Scope for Generalization

A generalization is readily available. We can postulate that instead of one there are l different inputs. If the non-decreasing average product hypothesis on f is satisfied radially, if $f'(z) \gg 0$ as a vector and if monotonicity and boundary conditions on u_i are adequately defined, then the dicussion of sections 1 – 8 goes through unmodified.

What is crucial is that utility functions do not depend on the input. As soon as utility functions are of the form $u_i(x_i, z_i)$ and z_i enters nontrivially we run into the basic difficulty that value allocations may fail to exist. The reason is clear: the set of utility vectors attainable to the whole society may fail to be convex. Let's consider an example.

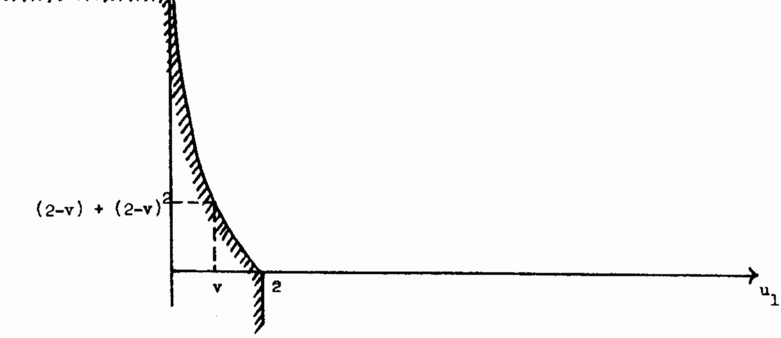


Fig. 4

The production function is $f(z) = z + z^2$. There are two types with utility functions $u_1(x_1, z_1) = z_1$, $u_2(x_2, z_2) = x_2$ and endowments $a_1 = a_2 = 1$. Take $s = (1,1)$. Then for any $\lambda_1 > 0_1$ and $\lambda_2 > 0$ we have $\nu_1(\lambda) > 0$ and $\nu_2(\lambda) > 0$ since marginal contributions are always positive. But the set of utility vectors dominated by some vector feasible to the grand coalition is as in Figure 4. Therefore, for any $\lambda \geqq 0$, $V_\lambda(s)$ is attained at either (2,0) or (0,6), i.e., if a value allocation existed the utility of some of the types would have to be zero.

It would appear then that short of contemplating lotteries on allocations the wide applicability of Shapley value theory to some non-classical economic situations such as increasing returns is limited to the transferable utility case. Unless (as a referee of this Journal has argued) one takes the broad view that putting the finger on an existence failure tells us something and is therefore an application of the theory.

10. A Result of *Scarf* on the Core in Increasing-Returns Economies

As in the previous section, let us assume that utility functions depend on both input and income. It is a remarkable result of *Scarf* [1973, p. 227] that, even without a quasi-concavity assumption on utility, the Core is non-empty [see also *Oddou*]. This is to be contrasted with the negative results for value allocations.

More precisely, assume that $f: R_+ \to R_+$, $f(0) = 0$, is continuous and exhibits non-decreasing average product, i.e., $f(z)/z$ does not decrease. There is a finite set I of agents. Every agent $i \in I$ is characterized by a continuous, monotone preference relation on R_+^2, denoted R_i, and by an amount $a_i \geqslant 0$ of inputs. An allocation $y: C \to R_+^2$ is feasible for coalition C if $y^1(i) \leqslant a_i$ for all i and $\sum_{i \in C} y^2(i) \leqslant f(\sum_{i \in C} (a_i - y^1(i)))$. A feasible allocation y for I belongs to the *Core* is there is no $C \subset I$ and feasible allocation y' for $C \subset I$ such that $y'(i) P_i y(i)$ for every $i \in C$.

Proposition 2: The Core is non-empty

Scarf's proof relies on checking the balancedness conditions and therefore, on non-trivial mathematics. It may be worthwhile to provide an elementary, in fact graphical proof. Although we shall not explicitly do so, we remark that the demonstration to be presented does strengthen *Scarf*'s result in two respects:

(i) it applies as well to the continuum of agents case,

(ii) if a continuous on preferences utility representation is chosen, the utilities associated with the Core allocation whose existence is shown vary continuously with characteristics, i.e., preferences and endowments (see *Champsaur* [1975a] for the analysis of this problem in the context of the next section).

Proof: For every $i \in I$ and $p \geqslant 0$ let $V_i(p) = \{(z, x) \in R_+^2 : z \leqslant a_i$ and $(z, x) R_i (z', x')$ whenever $z' \leqq a_i$ and $pz' + x' \leqq p a_i\} - \{(a_i, 0)\}$. See Figure 5. For every $C \subset I$ put $V_C(p) = \sum_{i \in C} V_i(p)$; see Figure 6. The set $V_C(p)$ depends (close convergence) continuously on $p \geqslant 0$.

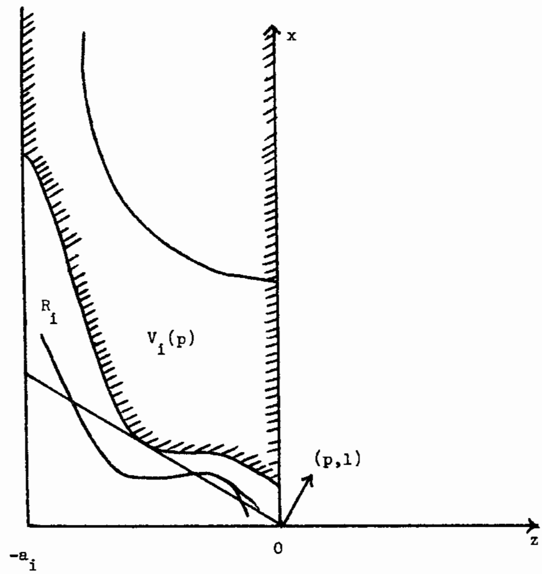


Fig. 5

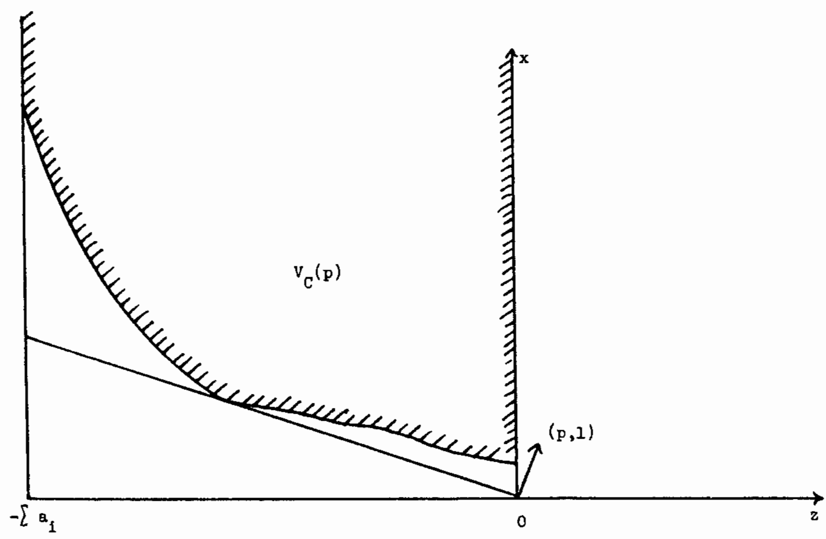


Fig. 6

Let $(v, w)\ R_- \times R_+$ be any non-zero vector with $pv + w \geqq 0$. Then for any $i \in I$ there is $(v'w') \in V_i(p)$ such that $(v'w') = \mu_i(v, w)$ for some $\mu_i \geqslant 0$. See Figure 7. For any $C \subset I$ if $(v, w) \in V_C(p)$ then $pv + w \geqq 0$ and therefore there is $(v'w') \in V_I(p)$ such that $(v, w) = \mu_i(v', w')$ for $\mu_i \leqq 1$. In other words, for any $C \subset I$ $V_C(p)$ is a subset of the closed region $A_p = \bigcup_{\mu \leqslant 1} \mu V_I(p)$. See Figure 8.

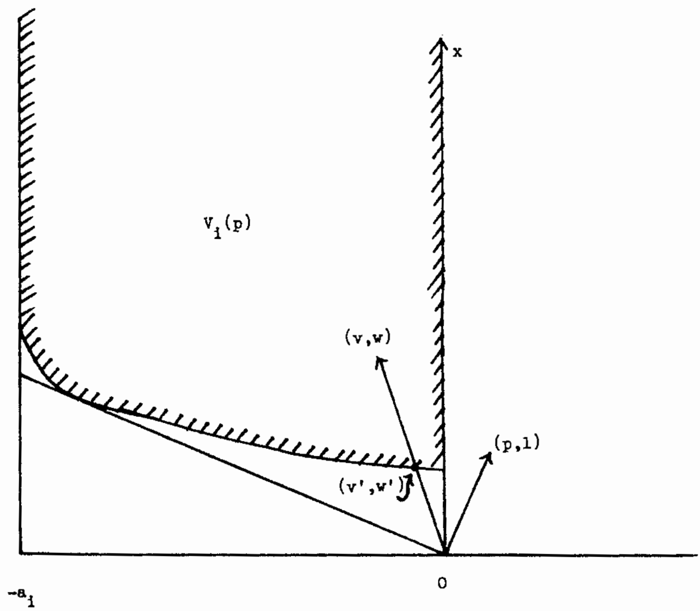


Fig. 7

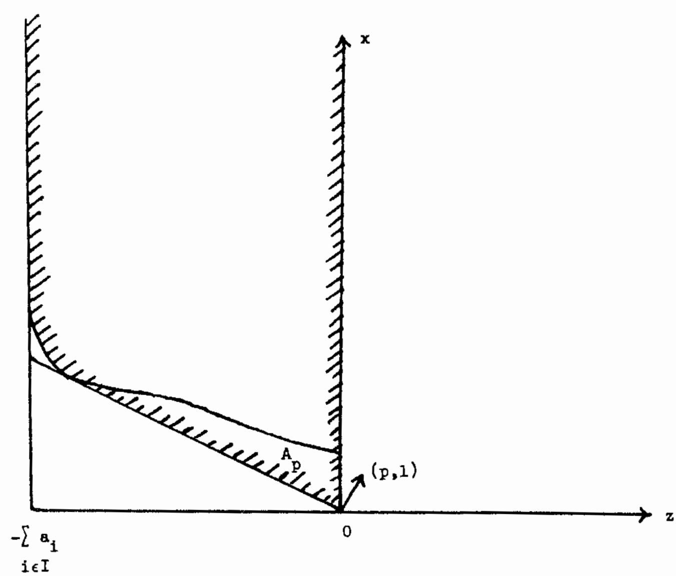


Fig. 8

Let Y be the production set, i.e. $Y = \{(v, w)\colon (v, w) \leqq (-z, f(z))$ for some $z \in R_+\}$. If for some $p \geqq 0$, $Y \cap V_I(p) \neq \emptyset$ then the allocation $(v_i, w_i)_{i\in I}$ such that $(v_i - a_i, w_i) \in V_i(p)$ and $(\sum_{i\in I} (v_i - a_i), \sum_{i\in I} w_i) \in Y$ is feasible and if it is blocked by

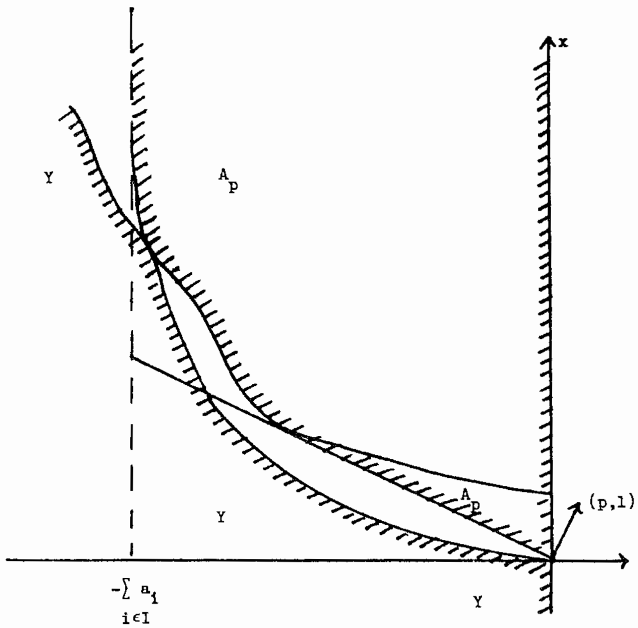


Fig. 9

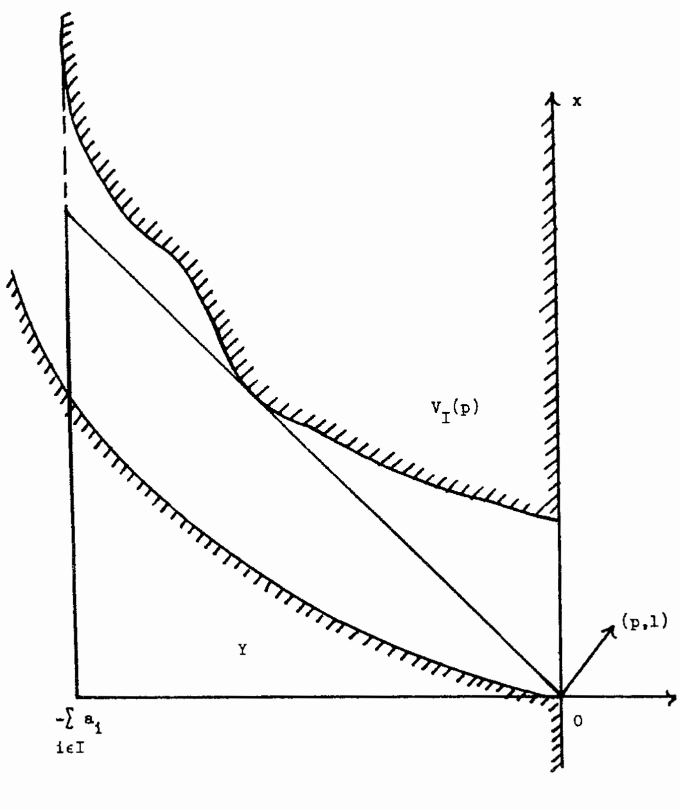


Fig. 10

coalition $C \subset I$ then there must be an allocation $(z_i, w_i')_{i \in C}$ with $\sum_{i \in C} (z_i - a_i, w_i') \in \text{Int } Y \cap V_C(p)$. Since $V_C(p) \subset A_p$ we can conclude: If for some $p \geqq 0$, $Y \cap V_I(p) \neq \emptyset$ and $\text{Int } Y \cap A_p = \emptyset$ then the Core is non-empty. See Figure 9.

By the hypothesis of nondecreasing average product if $\text{Int } Y \cap V_I(p) = \emptyset$ then $\text{Int } Y \cap A_p = \emptyset$; see Figure 9. Hence our problem reduces to show that there is $p \geqq 0$ with $Y \cap V_I(p) \neq \emptyset$ and $\text{Int } Y \cap V_I(p) = \emptyset$. But this follows from the intermediate value theorem. For $p = 0$ $Y \cap V_I(0) \neq \emptyset$ because $(0,0) \in V_I(0)$ and for p sufficiently large $\text{Int } Y \cap V_I(p) = \emptyset$. See Figure 10. Hence, since the dependence of $V_I(p)$ on p is continuous there is some $\bar{p}$ with $Y \cap V_I(\bar{p}) \neq \emptyset$ and $\text{Int } Y \cap V_I(\bar{p}) = \emptyset$. ■

11. A Remark on the Core in Economies with one Private Good and one Public Good.

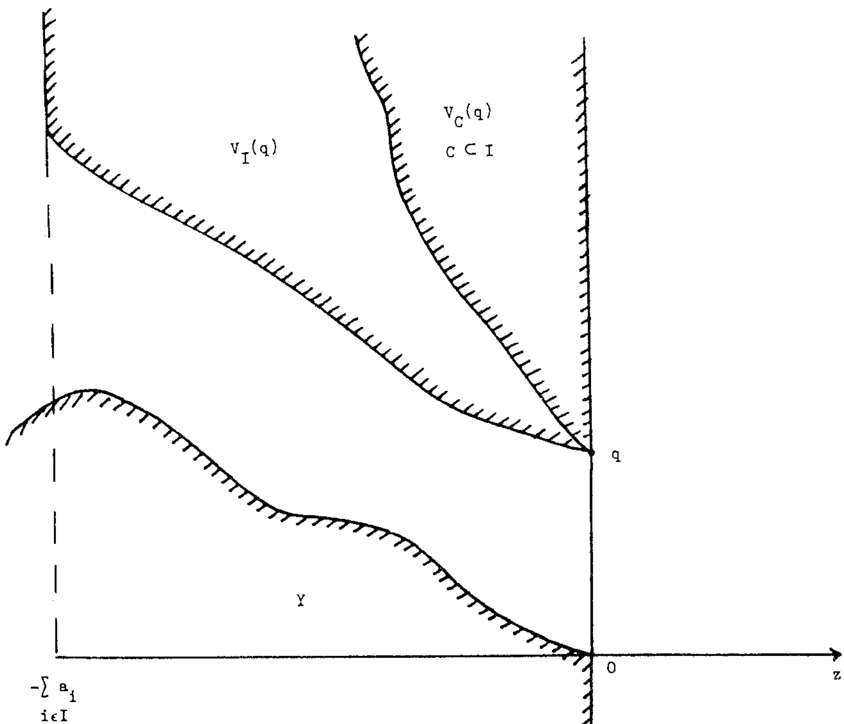
The idea of the proof of Proposition2 can be applied to establish a very general result on the non-emptiness of the Core for economies exhibiting a different variety of increasing returns, i.e., those with a private good and a public good. They have been studied by *Champsaur* [1975b].

Let the public good be denoted by x and the private good by z. There is a finite set I of agents. Every agent, i.e., I, is characterized by a continuous, monotone preference relation R_i on R_+^2 and by an amount a_i of private good. There is a production function $f: R_+ \to R_+$ which transforms private goods into public goods. We assume very little on f. Merely that it is continuous and $f(0) = 0$. A feasible allocation for coalition $C \subset I$ is a function $y: C \to R_+$ and an amount of public good x such that $y(i) \leqslant a_i$ for all i and $x \leqslant f(\sum_{i \in C} (a_i - y(i)))$. A feasible allocation (y, x) for I belongs to the Core if there is no $C \subset I$ and feasible allocation (y', x') such that $(y'(i), x') P_i (y(i), x)$ for all $i \in C$.

Proposition 3: The Core is non-empty

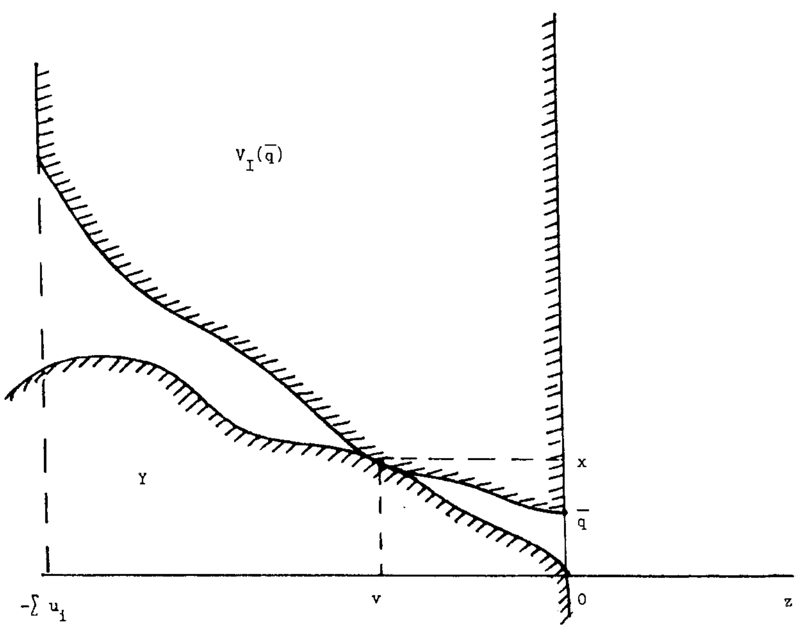
Observation (i) and (ii) on the proof of Proposition 2 can be applied to the present situation as well. Proposition 3 is essentially a result of *Champsaur* [1975b]. Our hypothesis on f are, however, more general.

Proof: As the demonstration builds on the same idea and it is actually even simpler than the proof in Section 10, we will give only the general details. For every $i \in I$ and $q \in R_+$, let $V_i(q) = \{(z, x) \in R_+^2: z \leqslant a_i \text{ and } (z, x) R_i (a_i, q)\} - \{(a_i, 0)\}$. For every $C \subset I$, put $V_C(q) = \{(v, x) \in R \times R_+: v = \sum_{i \in C} v_i, (v_i, x) \in V_i(q) \text{ for all } i \in C\}$. The set $V_C(q)$ is closed, depends continuously on q and it has the property that $V_C(q) \subset V_I(q)$ for all C. See Figure 11. Let $Y = \{(v, w): (v, w) \leqslant (-z, f(z)) \text{ for some } z \in R_+\}$. Then we have that if $(v, x) \in V_I(q) \cap Y$ and $V_I(q) \cap \text{Int } Y = \emptyset$, the allocation (y, x) with $v_i = y(i) - a_i$, $(v_i, x) \in V_I(q)$, $\sum_{i \in I} v_i = v$ belongs to the Core. But, again, by an intermediate value-type argument, there is $\bar{q}$ with $V_I(\bar{q}) \cap Y \neq \emptyset$, $V_I(\bar{q}) \cap \text{Int} = \emptyset$. See Figure 12. ■



$V_C(q)$
$C \subset I$
$V_I(q)$
q
Y
0
z
$-\sum_{i \in I} a_i$

Fig. 11



$V_I(\bar{q})$
x
Y
$\bar{q}$
$-\sum_{i \in I} u_i$
v
0
z

Fig. 12

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