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EFFICIENCY AND DECENTRALIZATION IN THE PURE THEORY OF PUBLIC GOODS*

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Some basic facts of public goods theory are presented in the primitive set-up of a collection of projects devoid of any linear structure. There is a single private good. Characterizations of Pareto optimal and core states in terms of valuation functions (i.e. supporting "prices") are obtained. Voluntary financing schemes are discussed.

I. INTRODUCTION, MODEL, AND ASSUMPTIONS

I.1. Introduction

It is the intention of this paper to gather and present the basic facts of public goods theory in the primitive setup of a collection of projects devoid of any linear structure (and where, therefore, the notion of "price per unit" is meaningless). The main point to be made is that, in contrast to what could be called the minimal dimension of informational variables, the efficiency and decentralizability results of the theory are quite fundamental and hold with great generality.

There are at least two reasons why we believe that on this occasion the disadvantages of generality do not outweigh the advantages. First, as compared with standard public goods theory, there is no loss of substance. The concept of Lindahl prices, for example, is primarily of theoretical interest, as it is not devised to model any existing market or even, if one takes the position that markets are in essence mass phenomena, potentially existing ones. Second, it is not uncommon that a public decision problem be given in terms of a choice among a few (say six or seven) projects.

Not much will be found here that is very new or deep. Perhaps

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the only result that will not sound familiar is part (C) of Proposition 1, which provides a decentralized characterization of the core. Our treatment is connected directly to the work of Dubins [1977] and to the theory of the allocation of public goods, particularly the static theory of Lindahl equilibrium (see Milleron [1972]) and the dynamic mechanisms leading to efficient allocations proposed by Malinvaud [1971] and Dréze-de la Vallée Poussin [1971] (see also Champsaur, Drèze, and Henry [1977]).

Formally our presentation is of the general equilibrium type, but the best interpretation is in partial-equilibrium terms. Indeed, we postulate, as an essential component of the theory, the existence of a single private good that can, and probably should, be thought of as a Hicksian composite commodity (for explicit partial-equilibrium approaches see Groves-Loeb [1975] and Green-Laffont [1977]).

The next subsection describes the model and basic assumptions. The exposition is organized in three parts. Section II presents the analog of "personalized prices" equilibrium theory. Section III considers decision devices obtained by putting "prices" under the control of agents. Section IV iterates the functioning of the decision device.

I.2. The Model and Assumptions

There is given a nonempty, compact, metric space K of *projects* and a finite collectivity of agent $I = \{1, \ldots, n\}$. It is appropriate to think of K as a finite, not very large, set. Similarly, I should also be thought of as not too large. Every agent i has preferences $\succsim_i$ defined on tuples (x,m) of projects and amounts of a unique private good (to be called "money"). The existence of a "money" commodity makes possible, to some extent, the transfer of welfare among agents and it is basic to the approach to social decision theory taken here. It constitutes an interesting specialization of the general theory of social choice (see Mueller [1976] for a recent survey).

A.1. It will *always* be assumed that agents' preferences satisfy: $\succsim_i$ is a continuous, reflexive, complete, transitive preorder on $K \times [0,\infty)$.

A.2. (a) $\succsim_i$ is continuous and strictly monotone in money, i.e., for all $x \in K$ if $m' > m$, then $(x,m') \succ_i (x,m)$.
(b) (Indispensability of money) If $m > 0$, then for any $x, x' \in K$ and $i \in I$, $(x',m) \succ_i (x,0)$.

A.2. (b) is obviously very strong, but it will make our analysis simpler. We let $\{u_i:K \times [0,\infty) \rightarrow R: i \in I\}$ be a family of continuous utility functions for the $\succsim_i$. They will be useful later on. Every $i \in I$ is endowed with a nonnegative w_i amount of money.

To complete the description of the model, we introduce a cost function $c:K \to (-\infty,\infty]$ that shall be assumed continuous. Note that $c(x) = +\infty$ is allowed; so if $c(x) = +\infty$ and $x_n \to x$, then $c(x_n) \to +\infty$. Occasionally we shall also assume that $c(x) \geq 0$.

Very often we shall postulate the existence of a distinguished project, denoted $0 \in K$, such that $c(0) = 0$. It is to be interpreted as the "status quo," i.e., as the situation from which a change is being contemplated. When there is a $0 \in K$, we assume that $u_i(0,w_i) = 0$ for all $i \in I$.

DEFINITION 1. A *state* is a project $x \in K$ and an assignment of money to agents $m:I \to R$. It is denoted (x,m).

DEFINITION 2. A state (x,m) is feasible if

$$c(x) \leq \sum_{i \in I} w_i - \sum_{i \in I} m_i.$$

DEFINITION 3. A state (x,m) is Pareto optimal (P.o.) if it is feasible and if there is no feasible state (x',m') such that $(x',m_i') \succ_i (x,m_i)$ for all $i \in I$.

Under hypothesis A.2 this definition is equivalent to the more usual strong version that allows some agents to remain as well off. The same is true of the next definition.

DEFINITION 4. A state (x,m) belongs to the *core* if it is feasible and there is no $S \subset I$ such that $S \neq \emptyset$ and for some state (x',m'),

$$c(x') \leq \sum_{i \in S} (w_i - m_i')$$

and

$$(x',m_i') \succ_i (x,m_i)$$

for all $i \in S$.

DEFINITION 5. Let there be a status quo $0 \in K$. A state is maximally Pareto improving (m.P.i.) if it is P.o. and $(x,m_i) \succsim_i (0,w_i)$ for all $i \in I$.

Remark 1. The term "maximally Pareto improving" is preferred to the more usual of "individually rational Pareto optimal" because the latter, which is an import from game theory, would be misleading in our context.

II. VALUATION EQUILIBRIUM

DEFINITION 6. A valuation system is a vector $v = (v_1, \ldots, v_n)$ of upper semicontinuous functions $v_i: K \rightarrow [-\infty, \infty)$, $i = 1, \ldots, n$. We allow $v_i(x) = -\infty$.

From the standpoint of an individual $i \in I$ a valuation function v_i is given as a datum, and it is to be interpreted analogously to prices, i.e., $v_i(x)$ is the amount of money to be relinquished for the right to enjoy the project x. As the source of valuation systems we should think of a coordinating center in charge of announcing and enforcing them. To appeal to impersonal markets, as one does with the usual prices, would not be a sensible thing to do here.

DEFINITION 7. A state $(\bar{x}, \bar{m})$ is *supported by a valuation system* v if for some $\Pi = (\Pi_1, \ldots, \Pi_n) \in R^n$ with

$$\sum_{i \in I} \Pi_i = \sum_{i \in I} v_i(\bar{x}) - c(\bar{x}):$$

(a) for every $i, (\bar{x}, \bar{m}_i)$ maximizes $\succsim_i$ on

$$\{(x, m_i): v_i(x) + m_i = w_i + \Pi_i\},$$

(b) $\bar{x}$ maximizes $\Sigma_{i \in I} v_i(x) - c(x)$ on K.

Parts (a) and (b) are, respectively, utility and profit maximization conditions; Π is a vector of distribution of profits or losses. Note that if $(\bar{x}, \bar{m})$ is supported by a valuation system v, then it is supported by a valuation system v' with total profit zero; setting $v_i'(x) = v_i(x) - \Pi_i$ and $\Pi_i' = 0$ will suffice.

DEFINITION 8. A state $(\bar{x}, \bar{m})$ ia a *valuation equilibrium* if it can be supported by a valuation system.

Note that a valuation equilibrium is automatically a feasible state.

Remark 2. If K is some cube in R^m and a valuation system is restricted to be linear homogeneous, i.e., $v_i(x) = p_i x$, then the concept of a valuation equilibrium coincides with the concept of Lindahl equilibrium (see Milleron [1972] for a survey of the use of this concept in public goods theory). So, valuation systems are a sort of "nonlinear" personalized prices.

Of the following characterization results only (C) can make any claim to novelty.

PROPOSITION 1. (1) The state $(\bar{x}, \bar{m})$ is a valuation equilibrium if and only if $(\bar{x}, \bar{m})$ is Pareto optimal.

(2) Let there be a status quo project $0 \in K$. The state $(\bar{x},\bar{m})$ is a valuation equilibrium with respect to a valuation system v with $v_i(0) \leq 0$ for all i, if and only if $(\bar{x},\bar{m})$ is maximally Pareto improving.
(3) Let $c(x) \geq 0$ for all $x \in K$. The state $(\bar{x},\bar{m})$ is a valuation equilibrium with respect to a nonnegative valuation system and zero profits, i.e., $\Pi = 0$, if and only if $(\bar{x},\bar{m})$ is in the core.

Proof of Proposition 1. We first verify the only if part of (1)–(3). Take a state (x,m) supported by a valuation system v. Suppose that for some $C \subset I$, $C \neq \emptyset$, there is (x',m') such that $(x',m_i') \succ_i (\bar{x},m_i)$ for every i and

$$\sum_{i \in C} (w_i - m_i') \geq c(x').$$

If Π is the vector of distribution of profits, then by (a) and (b) of Definition 7,

$$\sum_{i \in I} v_i(x) - c(x) \geq \sum_{i \in I} v_i(x') - c(x'),$$

$$\sum_{i \in C} (v_i(x') + m_i') > \sum_{i \in C} (w_i + \Pi_i),$$

and

$$\sum_{i \in C} (w_i - m_i') - c(x') \geq 0.$$

So, if $C = I$, we have

$$\sum_{i \in I} \Pi_i = \sum_{i \in I} v_i(x) - c(x) > \sum_{i \in I} \Pi_i + \sum_{i \in I} (w_i - m_i') - c(x') \geq \sum_{i \in I} \Pi_i,$$

a contradiction that takes care of (1). If $\Pi_i = 0$ and $v_i \geq 0$ for all i, then

$$0 = \sum_{i \in I} v_i(x) - c(x) \geq \sum_{i \in I} v_i(x') - c(x')$$
$$\geq \sum_{i \in C} v_i(x') - c(x') > \sum_{i \in C} (w_i - m_i') - c(x') \geq 0,$$

a contradiction that proves the only if part of (3). The only if part of (2) is an obvious consequence of the above arguments for (1) and the utility maximization hypothesis (b).

We now verify the "if" part. Let $(\bar{x},\bar{m})$ be a feasible state. For every $i \in I$, and $x \in K$, let $g_i(x) = w_i - z$ if $(x,z) \sim_i (\bar{x},\bar{m}_i)$ and $g_i(x) = -\infty$ if no such z exists. By (A.1) if z exists, it is uniquely defined and the function $g_i:K \rightarrow [-\infty,\infty)$ is continuous. We have $g_i(\bar{x}) = w_i - \bar{m}_i$ and therefore

$$\sum_{i \in I} g_i(\bar{x}) \geq c(\bar{x}).$$

Suppose now that $(\bar{x},\bar{m})$ is P.o. Let us have, for some $x \in K$,

$$a = \sum_{i \in I} g_i(x) - c(x) > 0;$$

then $g_i(x) > -\infty$ for all i. Consider the state (x,m') defined by $m_i' = w_i - g_i(x) + a/n$; since for all i, $(\bar{x},\bar{m}_i) \sim_i (x,w_i - g_i(x))$ we have $(x,m_i') \succ_i (\bar{x},\bar{m}_i)$ for all i, which is impossible because $(\bar{x},\bar{m})$ is P.o. and (x,m') is feasible. We therefore conclude that

$$\sum_{i \in I} g_i(x) \leq c(x)$$

for all $x \in K$. Thus, if we put $v_i = g_i$ and $\Pi_i = 0$, the "if" part of (1) is proved.

Let there be a status quo $0 \in K$ and let $(\bar{x},\bar{m})$ be maximally Pareto improving. Since $(\bar{x},\bar{m}_i) \succsim_i (0,w_i)$ for all $i \in I$, $g_i(0) \leq 0$ for all $i \in I$. Hence, if, as above, we have $v_i = g_i$, we are finished for the "if" part of (2).

Now let $(\bar{x},\bar{m})$ belong to the core. By definition of the core and the functions g_i, we must have

$$\sum_{i \in C} g_i(z) \leq c(x)$$

for all $C \subset I$, $C \neq \emptyset$ and $x \in K$. Then define $v_i:K \rightarrow [0,\infty)$ by $v_i(x) = \max\{0,g_i(x)\}$ and note that v_i is then nonnegative and continuous. For all $x \in K$,

$$\sum_{i \in I} v_i(x) \leq c(x),$$

since, letting $C = \{i \in I:g_i(x) \geq 0\}$,

$$\sum_{i \in I} v_i(x) = \sum_{i \in C} g_i(x) \quad \text{if } C \neq 0$$

and

$$\sum_{i \in I} v_i(x) = 0 \leq c(x) \quad \text{if } C = \emptyset.$$

On the other hand,

$$\sum_{i \in I} v_i(\bar{x}) \geq \sum_{i \in I} g_i(\bar{x}) \geq c(\bar{x}).$$

So,

$$\sum_{i \in I} v_i(\bar{x}) = c(\bar{x}),$$

and $\bar{x}$ maximizes profits. Also, for all $i \in I$, if $v_i(x) + m_i \leq w_i$, then $(\bar{x},\bar{m}_i) \succsim_i (x,m_i)$ because $v_i(x) \geq g_i(\bar{x})$ and so, the utility maximization hypothesis is satisfied.

Q.E.D.

Remark 3. Part (3) of Proposition 1 provides a price-like characterization of the core for the case where (A.1)–(A.2) are satisfied and c is nonnegative. An analogous characterization for general c could also be obtained. Of course, while maximal Pareto-improving states are easily seen to exist under the assumptions made, the core may well be empty.

III. Voluntary Financing Devices

N.B. In the present section we assume that there exists a status quo point $0 \in K$.

Our point of view will now be changed. Instead of looking at valuation systems as equilibrating parameters controlled by a hypothetical coordinating center, we shall regard them as willingness-to-pay functions under the direct control of the agents.

The voluntary financing public decision method to be described is not new; it was first proposed by Steinhaus [1949] and has recently been extended and discussed by Dubins [1977]. It is built on the old notion of the maximization of social surplus, and it is intimately related to the ideas of Malinvaud [1971] and Dréze-de la Vallée Poussin [1971] concerning dynamic procedures for the allocation of public goods (see Section VI). The name "device" is taken from Dubins.

DEFINITION 9. A *proposal of agent* i is a function $v_i{:}K \rightarrow [-\infty,w_i]$ such that $v_i(0) = 0$.

A proposal is to be interpreted as, for each $x \in K$, the amount $v_i(x)$ that agent i *volunteers* to pay if project x is implemented.

DEFINITION 10. A vector $\delta = (\delta_1, \ldots, \delta_i)$ with $\delta_i \geq 0$ and

$$\sum_{i \in I} \delta_i = 1$$

is called a *distribution vector.*

Given a vector of proposals $v = (v_1, \ldots, v_n)$, let K_v be the set of profit-maximizing projects, i.e.,

$$K_v = \left\{ x \in K: \sum_{i \in I} v_i(x) - c(x) \geq \sum_{i \in I} v_i(x') - c(x') \qquad \text{for all } x' \in K \right\}.$$

Let Π_v be maximum profits; since

$$\sum_{i \in I} v_i(0) - c(0) = 0,$$

we have $\overline{\Pi}_v \geq 0$.

DEFINITION 11. Given the proposals $v = (v_1, \ldots, v_n)$ and distribution vector $\delta = (\delta_1, \ldots, \delta_n)$, *an outcome* is any state (x,m) where
(i) $x \in K_v$,
(ii) $x \in K_v \backslash \{0\}$ if $K_v \backslash \{0\} \neq \emptyset$,
(iii) $m_i = w_i - v_i(x) + \delta_i \overline{\Pi}_v$.
So, given the stated willingness to pay of the agents, some profit-maximizing state is chosen. Unless there is no alternative, the status quo is not chosen.

N.B. From now on, a fixed distribution vector δ is given and if a triple (v,x,m) is considered, it is to be understood that (x,m) is an outcome for v and δ.

DEFINITION 12. A triple (v,x,m) is *stable* if no $i \in I$ can insure for himself a better outcome by changing his proposal, i.e., if v' is such that $v_j' = v_j$ for $j \neq i$, then for every outcome (x',m'), we have $(x,m_i) \succsim_i (x',m_i')$.

The word "insure" in the previous definition is justified. Indeed, suppose that for some outcome (x',m'), one had $(x',m_i') \succ_i (x,m_i)$. By replacing v_i' by a v_i'' with $v_i''(0) = 0$, $v_i''(x') = v_i'(x')$, and $v''(x'') < v'(x'')$ for all $x'' \neq 0,x'$, (x',m') becomes the only possible outcome (i.e., $K_v = \{0\} \cup \{x'\}$).

If the voluntary financing device is viewed as a game, then the notion of stability is akin to the concept of Cournot-Nash equilibrium.

DEFINITION 13. Let (v,x,m) be given. The proposals are called *self-protective* if, for all $x' \notin 0$ and i,

$$(x',w_i - v_i(x') + \delta_i\overline{\Pi}_v) \succsim_i (x,m_i).$$

The notion of self-protectiveness is akin to the maximin decision rule. For the proposal of agent i to be self-protective with respect to an outcome x, it means that the agent can guarantee himself, independently of the proposals of the other agents, the level of utility corresponding to x and the payment proposed.

PROPOSITION 2. (4) No (v,x,m) with $\overline{\Pi}_v > 0$ can be stable.

(5) If given (v,x,m), v are self-protective and (x,m) is maximally Pareto improving, then (v,x,m) is stable.

(6) Given (v,x,m), if $\overline{\Pi}_v = 0$ and the proposals v constitute a valuation system for (x,m), then (v,x,m) is stable.

Proof of Proposition 2. Claim (4) is quite clear. Take an i with $\delta_i < 1$. Replace v_i by v_i', where $v_i'(0) = 0$, $v_i'(x) = v_i(x) - \overline{\Pi}_v$, $v_i'(x') < v_i(x') - \overline{\Pi}_v$ for $x' \neq 0,x$. The function v' can be chosen to be continuous if so desired. Then (x,m') is an outcome for v' such that $\overline{\Pi}_{v'} = 0$, $m_j' = m_j$ for $j \neq i$ and

$$m_i' = w_i - v_i'(x) = w_i + \overline{\Pi}_v - v_i(x) > w_i + \delta_i\overline{\Pi}_v - v_i(x) = m_i,$$

i.e., i is better off.

For claim (5), suppose that it was not true. Then there would be an agent i who could insure himself a better outcome. But since proposals are self-protective, none of the other agents can be worse off with the new outcome. By hypothesis (A.2), the outcome could not be maximally Pareto improving.

To prove (6), consider any agent i. The outcomes that i can possibly enforce are of the form $(x',w_i - m_i')$, where

$$m_i' \geq c(x') - \sum_{j \neq i} v_j(x').$$

But

$$\sum_{j \in I} v_j(x') \leq c(x')$$

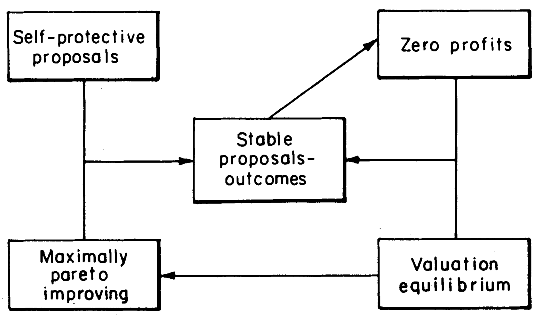

FIGURE I

and so $m_i' \geq v_i(x')$. Since v is a valuation equilibrium for (x,m), $(x,w_i - v_i(x)) \succsim_i (x',w_i - v_i(x')) \succsim_i (x',w_i - m_i')$, i.e., agent i cannot change his proposal to advantage.

Q.E.D.

Observe that if, with respect to (x,m), proposals v are self-protective and have the valuation property, then for all i and $x' \neq 0$, either $v_i(x') = w_i - z$, where $(x',z) \sim_i (x,m_i)$, or $v_i(x') = -\infty$ if no such z exists.

Figure I illustrates the interrelationships among the different concepts.

It is only for simplicity that in the definition of stability (Definition 12) we allowed departures only by individuals. Every result remains valid if, in addition we had required stability against departures by whole groups of individuals.

Now suppose that a voluntary financing device is put into operation in order to decide among a set of projects.

It is natural to surmise that prior to the emergence of the definitive proposals, there will be a more or less structured "negotiation period" where proposals are tentatively put forward by agents and then possibly revised. The analysis of negotiation behavior falls into the domain of game theory, and clear-cut deterministic rules cannot be expected from it. The position we shall take here is that, through some mixture of conflict and cooperation, bargaining will eventually lead to proposals yielding maximally Pareto-improving outcomes. In other words, *for the interpretation at hand* (not many projects and not many individuals) we accept the principle that if there are, for all, obvious and easily reachable gains from cooperation, bargaining will not get definitely stuck at proposals with very inefficient outcomes (someone will, sooner or later, break a deadlock by volunteering a larger contribution).

A different matter is the stability of the final agreement. If the final proposals are not stable, i.e., mutually reinforcing, there is an incentive on the part of some agents to break the agreement, i.e., "free riders" will appear and the agreed-on outcome will decompose or perhaps, simply, it will never quite be reached. The gains from cooperation cannot be consolidated. The plausibility and resoluteness of cooperative bargaining will therefore be particularly good if it leads to proposals, which besides giving maximally Pareto-improving outcomes, do not introduce enforcement problems, i.e, which are stable.

In the next section we present a very natural cooperative iteration procedure of the voluntary financing device that is utility monotone and leads to proposals and outcomes that are self-protective and have the valuation property. So, the proposals are stable, and the outcomes are maximally Pareto efficient. Thus, the voluntary financing device has the potential for quite a good performance by the standards of the two previous paragraphs. The procedure is the analog in our context of the mechanisms of Malinvaud and Drèze-de la Vallée Poussin.

There is another, more demanding, approach to the incentive problem that is associated with the names of Hurwicz [1972] and Groves-Ledyard [1977]. They formalize allocation procedures as *games* with messages as strategies. Further, the games are viewed as *noncooperative* and as a solution the notion of *Nash equilibrium* is appealed to. A procedure is good, incentive-wise, if *all* the Nash equilibria of the noncooperative game are Pareto optimal ("all" because there are no grounds to discriminate among them). While this may be an appropriate standard of adequacy in particular cases, one may wonder whether its imposition as a general rule is not too rigid a requirement. In the first place it can be questioned whether incentive problems are always best modeled as games, with its implied sophisticated and foresighted behavior on the part of agents. In the second place, one would feel that bargaining has as many elements of cooperation as of pure conflict so that a strictly noncooperative individualistic setup is quite limited. Rigorously speaking, a game is noncooperative if there are no preplay communication possibilities whatsoever. This is a very specific restriction for so general a problem as the one we are discussing. In the third place, the presumption that the process of noncooperative bargaining will come to rest at a Nash equilibrium is justified only in situations with a large number of agents; otherwise the Cournot-Nash conjecture is unrealistic.

At any rate, we should mention that the voluntary financing device does not have the potential to meet the desiderata of the last paragraph, as it is quite clear there may be plenty of stable proposals with outcomes that are very far from being Pareto optimal.

IV. A VOLUNTARY FINANCING PROCESS

The problem of this section is how, by means of a voluntary financing device, can we reach triples (v,x,m) where v are stable proposals and (x,m) is maximally Pareto efficient. We adopt the most extreme cooperative outlook and except for the stability requirement on v, incentive problems are put aside. See the end of the previous section for a discussion of those.

If individual utility functions were separable and linear in money, i.e., of the form $u_i(x,m_i) = \hat{u}_i(x) + k_i m_i$, then an obvious device is the maximization of surplus, i.e., we put $v_i = \hat{u}_i$. The result is a maximally Pareto-optimal outcome and self-protective, hence stable, proposals. Thus, we have a one-shot functioning device. This suggests that for the general case (i.e., when "income effects" exist), we try an iteration process where each step amounts to a maximization of surplus.

IV.1. Definition and Convergence Properties of the Process

N.B. In this subsection the distribution vector δ is given.

Without loss of generality we can assume that the range of the utility functions u_i (see Section I) is $[0,\infty)$.

For every i and $r \geq 0$ define $g_i^r{:}K \rightarrow [w_i, -\infty]$ by the solution to $r = u_i(x, w_i - g_i^r(x))$ if it exists. Otherwise $g_i^r(x) = -\infty$.

Define an iterative process $(v_0,x_0,m_0), \ldots, (v_t,x_t,m_t), \ldots$ as follows:

1. Take $x_0 = 0$, $v_{i0}(x) = 0$ all $x \in K, m_{i0} = w_i$.

2. Let (v_t,x_t,m_t) be given and suppose that $u_t \geq u_{t-1} \geq \cdots \geq 0$, where $u_{it} = u_i(x_t,m_{it})$. Then put $v_{i,t+1}(x) = g_i^{u_{it}}(x)$ if $x \neq 0$ and $v_{i,t+1}(0) = 0$. If $K_{v_{t+1}} = \{0\}$, take $x_{t+1} = 0$; if $K_{v_{t+1}} \backslash \{0\} \neq \emptyset$, take for x_{t+1} any project in the set $K_{v_{t+1}} \backslash \{0\}$. Of course, we let $m_{i,t+1} = w_i + \delta_i \Pi_{v_{t+1}} - v_{i,t+1}(x_{t+1})$. Every $v_{i,t+1}$ is upper semicontinuous, hence $K_{v_{t+1}} \neq \emptyset$, and the process is well defined. By construction the proposals v_{t+1} are self-protective for $(x_{,t},m_t)$ so that, indeed, $u_{t+1} \geq u_t$.

Let J be the (nonempty) set of limit points of $\{x_t\}$ and Π_t total profits at stage t.

PROPOSITION 3. (7) $\Pi_t \to 0$, the sequence (v_t,m_t) converges to a limit $(\bar{v},\bar{m})$ and for any $\bar{x} \in J$, $\bar{v}$ is a valuation system supporting $(\bar{x},\bar{m})$. Hence $(\bar{x},\bar{m})$ is maximally Pareto improving, and $(\bar{v},\bar{x},\bar{m})$ is stable.

Proof of Proposition 3. Note that the sequence u_t is monotone increasing and bounded above. Hence $u_t \to \bar{u}$ for some $\bar{u} \in R^n$. This, the compactness of K, and the strict monotonicity of preferences with respect to money imply that $\Pi_t \to 0$. By the continuity of preferences and the definition of v_t we must have $v_t \to \bar{v}$ for some proposals $\bar{v}$ and, therefore, $m_t \to \bar{m}$ for some $\bar{m} \in R^n$. For any t and i, (x_t,m_{it}) maximizes $\succsim_i$ on

$$\{(x',m'):v_{i,t+1}(x') + m' \leq w_i + \delta_i\Pi_t\}.$$

The profit maximization condition is also satisfied at every t. Hence, by continuity, $\bar{v}$ supports $(\bar{x},\bar{m})$ for any $\bar{x} \in J$.

Q.E.D.

Remark 4. If K is infinite, J may well not be unique. If K is finite, then there is a finite number of steps after which $u_t = \bar{u}$ and, therefore, $v_t = \bar{v}$ for all t. Indeed, suppose that $u_{t+1} > u_t$ infinitely often, then $x_{t+h} = x_t$, and $u_{t+h} > u_t$ for some t and h, and so $m_{t+h} > m_t$. Hence,

$$c(x_{t+h}) - \sum_{i \in I} w_i = \sum_{i \in I} m_{i,t+h} > \sum_{i \in I} m_{i,t} = c(x_t) - \sum_{i \in I} w_i,$$

which is a consideration. Summing up: the process can be stopped after a finite number of iterations.

Remark 5. The process just presented is nothing but a global version of the procedures proposed by Malinvaud [1971] and Dréze-de la Vallèe Poussin [1971] for the allocation of public goods. Naturally, the convergence and incentive properties are also the same. Let $K \subset R^m$ be some large convex cube and suppose that the utility functions are smooth. Then at step t instead of going to the global maximum of the profit function (which, incidentally, would require whole functions as proposals) we could simply follow the direction of the gradient. The process obtained thus would belong to the Malinvaud-Dréze-de la Vallée Poussin family, and, overall, would be much more economical in information using and processing.

IV.2. Neutrality (unbiasedness) Properties of the Process

Neutrality is Champsaur's term [1976], unbiasedness, Hurwicz's [1959]. Both are coined to denote the mechanism that, possibly de-

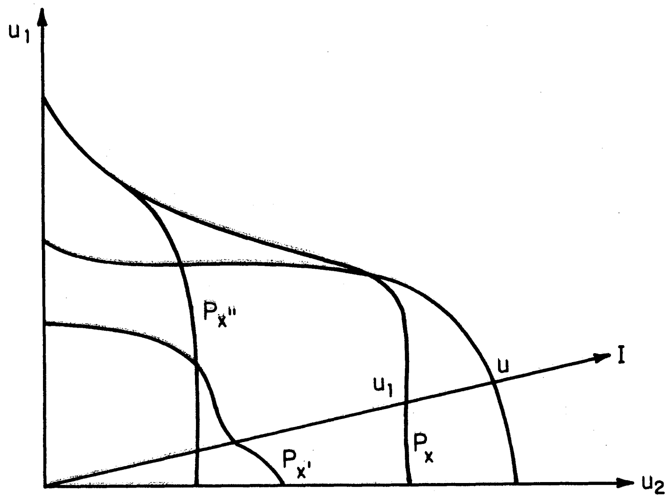

FIGURE II

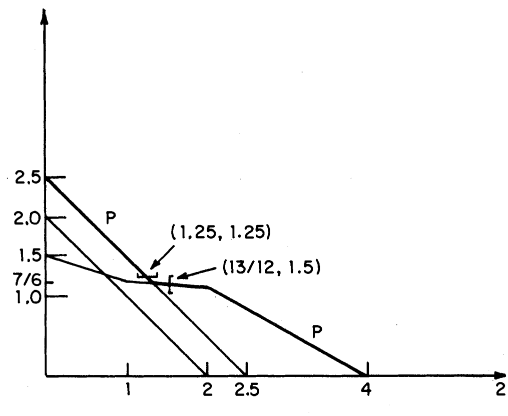

FIGURE III

pending on the specification of some parameter, can reach every Pareto-optimal point (or perhaps, every maximally Pareto-improving one).

Let $P \subset R^n_+$ be the Pareto set in nonnegative utility space. Remember that 0 is the utility of the status quo point.

For every project $x \in K$, let $P_x \subset P$ be the utility vectors in P that are feasible with the constraint that project x is in fact implemented. So, if x is at all feasible,

$$P_x = \left\{(u_1(x,m_1), \dots, u_n(x,m_n)) \in P\colon \sum_{i \in I} m_i = \sum_{i \in I} w_i - c(x) \geq 0\right\},$$

and P is the upper frontier of $\cup_{x \in K} P_x$. Because of (A.2) every P_x and, therefore, also P has the property that it intersects any nonnegative ray through the origin of R^n at exactly one point. So, P_x,P are, up to homeomorphism, simplices. See Figure II.

Consider now the voluntary financing process of the previous subsection. Given the distribution vector δ, let $P(\delta)$ be the utility vectors to which the process can lead. Given the possible indeterminacy if there is more than one profit-maximizing project, $P(\delta)$ need not be a singleton. Let $V = \cup_{\delta \in \Delta} P(\delta)$, where Δ is the $n - 1$ unit simplex.

For the Malinvaud-Drêze-de la Vallée Poussin process, Champsaur [1976] showed that $V = P$. This is the property of neutrality, or unbiasedness. Given the nonconvexities of our problem, it should not be surprising that

(8) *$V = P$ may not hold.*

Example. $K = \{0,b,c,d\}$, and there are two agents with $w_1 = w_2 = 1$. The cost is zero for every project. The utility functions are

$$u_1 = \begin{cases} u_1(0,m) = 1 - m \\ u_1(b,m) = m \\ u_1(c,m) = 2m - 1 \\ u_1(d,m) = \begin{cases} 2m - \dfrac{1}{2} & m \leq \dfrac{3}{4} \\ \dfrac{2}{3}m + \dfrac{1}{2} & m \geq \dfrac{3}{4} \end{cases} \end{cases}$$

$$u_2 = \begin{cases} u_2(0,m) = 1 - m \\ u_2(b,m) = m \\ u_2(c,m) = 2m - \dfrac{1}{2} \\ u_2(d,m) = \begin{cases} 2m - 1 & m \leq 1 \\ 4m - 3 & m \geq 1. \end{cases} \end{cases}$$

The Pareto set is graphed in Figure III. By a little tedious computing, it can be verified that when $\delta_1 > \delta_2$, agent 1 gets utility larger than 1.25, while if $\delta_2 > \delta_1$, agent 2 gets more than 1.5. At $\delta_1 = \delta_2$ two final outcomes are possible: $(u_1,u_2) = (1.25,1.25)$, or $(u_1,u_2) =$

(13/12,1.5). The "open segment" in P ((1.25,1.25),(13/12,1.5)) is therefore never reached.

Remark 6. It is not hard to verify that an example of nonneutrality must involve at least three projects, not counting the status quo.

Suppose that we drop from the specification of the voluntary financing procedure the requirement that at every step profits are redistributed according to a fixed predetermined vector δ, and let us leave this distribution indeterminate, i.e., any redistribution is admissible. Denote then by V' the utility vectors to which the process can lead. Trivially, we have

PROPOSITION 4.

(9) $V' = P$.

Proof of Proposition 4. This is very clear. Pick any $u \in P$ and let I be a half ray through u. Since the process converges to Pareto optimality and the iteration rules are the same at every step, it suffices to show that at step 1 profits can be redistributed so that $u_1 \in I$. Let x_1 be profit-maximizing at step 1. Then Px_1 intersects I between 0 and u. Let u_1 be the intersection point (see Figure II). Distribute profits at step 1 so as to reach u_1.

Q.E.D.

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