3 REVEALED PREFERENCE AFTER SAMUELSON

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The first and fundamental paper of P. A. Samuelson on revealed preference theory (1937) is now almost forty-five years old. He must have been barely twenty when it was written. One feels, nevertheless, that revealed preference has not just been a youthful interest, but has remained very close to his heart. He has written on it repeatedly (e.g., 1948, 1950), it features prominently in his Nobel Laureate lecture, and it was chosen by him as the subject of his 1973 Gibbs lecture at the American Mathematical Society meetings. Revealed preference is as foundational and purely theoretical a subject as one can find, and one cannot help thinking that this is part of its fascination. Indeed, of how many topics can it be said that, to paraphrase what Samuelson wrote in the Georgescu-Roegen *Festschrift*, people will be discussing them a hundred years from now? Certainly, the pure theory of rational choice is one, and I hope that it will be an appropriate appreciation of Samuelson's contributions if I devote these few pages to giving a nonscholarly and nonexhaustive account of some of the developments that his seminal revealed preference work inspired — to show, in a word, that if one starts the clock in 1937, his prediction has already become almost half fulfilled.

Revealed preference was born of the Pareto-Hicks program of purging consumer and demand theory of subjectivist components. Pareto and, very forcefully, Hicks in *Value and Capital*, discovered that for the purposes of the consumer theory of the day, it was sufficient if tastes were thought of as families of ordered indifference curves. Samuelson's insight was to push the idea one step further and enunciate the general methodological postulate that the basic axioms of a theory must be operational — that is, they must be refutable by observable data generated from feasible experiments.

Let's be more specific. In the language of binary relations, which became popular after the war, the Pareto-Hicks theory can be formulated as follows. There is a set X of commodity bundles. In X a typical consumer has defined a relation xRy to be read as "commodity bundle x is at least as good as commodity bundle y," which satisfies the rationality principles:

1. xRx (reflexivity);
2. Either xRy or yRx (completeness);
3. If xRy and yRz, then xRz (transitivity).

If xRy and yRx, we put xIy (x indifferent to y). If xRy but not yRx, we put xPy (x preferred to y). It must be noted at the outset that Pareto-Hicks theory passes in principle the operationality test. There is a trivial way to transform the indifference-preference framework into a choice-based theory: Just let the domain of choice experiments include every pair of commodity bundles. If we now interpret xRy to mean "when the decision agent is constrained to choose between x and y and is asked to eliminate a bundle if he does definitely not want it, he does not eliminate x," the three axioms have the obvious experimental meaning (and in fact only the third has any teeth). What Samuelson perceived is that these choice experiments may not be feasible in the sense of being observable in a given market environment. What he suggested was to stick to the given rather than hypothetical observables and to impose the rationality axioms on them. In the concrete case of competitive market behavior, he took the position that only the demand function would be observable and that this is therefore the entity to be axiomatized. He proceeded to propose a very basic axiom, the so-called Weak Axiom (WA) of Revealed Preference. Let us say that the commodity bundle x is revealed preferred to the commodity bundle y (written xSy) if for some competitive budget (i.e., for some given prices and income) x is chosen when y is affordable. The WA says then that whenever xSy, we cannot have ySx. This is a most intuitive and natural requirement, and Samuelson showed that, as we will describe later, it yields many of the implications of the Pareto-Hicks consumer theory. Many, but not all. Samuelson was aware that for some purposes the Weak Axiom needed extension. It was provided by Houthakker (1950) in the form of the Strong Axiom (SA) of Revealed Preference: If we have a finite domain of direct preference revelations $x_1Sx_2, \ldots, Sx_n$, then we say that x_1 is indirectly revealed preferred to x_n (written x_1Hx_n), and we rule out the possibility that x_nSx_1. Again, from the standpoint of the rationality of choice, this is a most appealing and natural extension.

For years now the old cardinal versus ordinal utility dispute has been obsolete, and cardinal utilities are being widely used. It may therefore be worth emphasizing before proceeding any further that the Pareto-Hicks-Samuelson drive to liberate consumer theory from mysticism was theoretically sound. It has also been successful and fundamental for further progress. If cardinal utility has been restored, it is only because the theory of choice under uncertainty of von Neumann and Morgenstern (1944) has succeeded in putting it on a solid operational basis, well within the Pareto-Hicks-Samuelson tradition, since, in principle, it is possible to derive the cardinal utilities from observable choice experiments involving lotteries.

It was clear from the beginning that the consumer theory based on the revealed preference axiom had to be closely related to the theory based on the preference hypothesis (i.e., on the ordered indifference classes of Pareto and Hicks). Indeed, if demand derives from preferences, then obviously the Strong Axiom should be satisfied. Research has concentrated on the reverse direction: To what extent is it possible to work backwards (this is Samuelson's expression) and go from "rational demand" to "rational preferences"? As it turns out, it can be done, and by now this is well understood. In the next section we review some of the results that establish the compatibility of demand functions satisfying the Strong Axiom with underlying preferences (this is an *existence* issue). Then we discuss the uniqueness question (i.e., can preferences be unambiguously recovered from demand?). In the following section we comment on the relationships of revealed preference theory with integrability theory, an alternative and older (Antonelli 1886) approach to an axiomatic treatment of "rational" demand. Finally, we dwell on the demand theory one gets from the Weak, rather than the Strong, Axiom of Revealed Preference. After all, this was the original Samuelson postulate.

The "working backwards" from demand to preferences is not merely a matter of purely logical interest or of aesthetic perfection (although it is that too). What explains its centrality in the research effort over the years is that it is in a sense vital to an economic theory based on feasible choice experiments. The sense is this: Economic science is to a large extent a normative and predictive theory. From what is observed (say, in our case, competitive budget choices), one wants to evaluate and predict what would happen with a given change in the environment (say, the imposition of a tax). The task of evaluation is to ask for preferences and utility (welfare economics without them would be severely limited). The task of prediction does not quite ask for it, but, given the power of the preference hypothesis, it is most expedient to postulate the existence of preferences underlying the observable choices. If preferences can be recovered via revealed preference analysis, then they can be used for predictive purposes. Note that the overall result is a strictly operational (i.e., refutable) theory.

It should be stressed that the revealed preference approach and, more concretely, the revealed preference axioms, are quite general. They apply as well to the recovery of preferences from undistorted, competitive demand functions (to be used, perhaps, to predict in a distorted, non-competitive situation) as to their recovery from a collection of observations generated by a distorted, imperfectly competitive economic world (and to be used, say, to predict the ideal competitive outcome). The essence of revealed preference theory is the realization that the observable choice data will be far from inclusive of all conceivable choice experiments (if they did, then the Strong Axiom would amount to nothing but a rephrasing of the transitivity axiom on preferences), and that therefore the task of recovering preferences will typically be far from trivial. We will say more on all this in the next two sections.

As we have already said, it will be a conclusion of revealed preference analysis that one can go from "rational" demand functions (i.e., satisfying the Strong Axiom) to Pareto-Hicks preferences. Given the broader scope of those (they are not tied to a particular institutional specification), this is to be welcomed; indeed, the preference hypothesis is today at the basis of economic theory. Revealed preference has only reinforced it. The reader may ask: To end up where we started, was the trip worth it? I do not hesitate to answer in the affirmative: The house is now on a firmer foundation, and, lest anyone think that the whole enterprise was without difficulties, it could be pointed out that the conceptually close relative of the preference theory we have been discussing — namely, the von Neumann-Morgenstern theory of choice under uncertainty — is still at the Pareto-Hicks stage. It is operational because binary preferences on lotteries can be identified from choices over all conceivable pairs of lotteries. But those are not typically the budget choices that will be observable in a market situation. To my knowledge no one has yet provided an analog of the revealed preference axioms and theorems for choice functions over some restricted and economically natural class of lottery budgets.

THE EXISTENCE OF UNDERLYING PREFERENCES

The Weak Axiom of Revealed Preference alone does not imply the existence of preferences underlying a given demand function. For a two-commodity world the Weak Axiom implies the Strong, and so, for a while, there was some uncertainty whether this may not be the general situation. But an example of Gale's (1960), inspired by an informal argument of Samuelson's, settled the matter: There are demand functions that satisfy the Weak but not the Strong Axiom. Hence they cannot be generated from preferences.

The question is, then, if the Strong Axiom is equivalent to the preference hypothesis. The answer is yes, and we shall call this the revealed preference theorem. It has been established under a variety of conditions on demand by a central tradition that begins with Little (1949) and Samuelson (1948), continues with Houthakker's fundamental paper, (1950), and culminates in Uzawa (1959) and Stigum (1973). The mathematical treatment that this tradition gave to the problem, however, was quite cumbersome. Their techniques (i.e., the so-called price-income sequences) would lead one to believe that the revealed preference theorem belongs to the area of differential equations, which is definitely not the case since the revealed preference tests allow from the beginning for noninfinitesimals comparisons.

It was M. Richter (1966, 1971), who recognized the revealed preference theorem for what it is — a theorem in set theory. He abstracted the logical problem of existence of underlying preferences from the nonessential features of the competitive budget sets specification and with this laid bare the basic structure. What one has is a set of alternatives, X; a given family of budgets, B (each budget B is a subset of X); and a choice function, $C(B)$, which assigns a choice from B to B. Note that the family of budgets is totally unrestricted. Nevertheless, the Strong Axiom still makes sense in this context. It tells us that if we define x to be preferred to y (written xPy) whenever x is directly or indirectly revealed preferred to y, then P is transitive (i.e., we cannot get cycles). Of course, only some pairs x,y will stand in the relation, P, which, for this reason, constitutes what is called a partial order. The preference hypothesis amounts to the existence of a complete and transitive R underlying C. In particular, any complete and transitive extension of P will generate C and will, therefore, solve our problem. But a basic principle (Zorn's lemma) of the usual set theory asserts that every partial order has a complete, transitive extension. And we are done! There is nothing more to it. No other hypothesis but the Strong Axiom is made. Richter's revealed preference work is one of the best examples in economics of the power that an abstract treatment may have (but, of course, doesn't need to have; there is no automatism here).

While Richter's result is very general, it may be felt that it gives little clue to the nature of underlying preferences. This is true. In particular, the preferences one gets directly from the precedent construction are somewhat peculiar. No two alternatives are indifferent, for example. All this can be fixed up, and we refer to Richter's papers to see how. We would, nevertheless, like to report on an alternative approach to the revealed preference theorem that is much more specific about the obtained preferences. We refer to the important and seminal contribution of Afriat (1967), which, while as direct as Richter's, proceeds not by abstracting the economic aspects but by emphasizing them.

Afriat's analysis is placed on the perfectly competitive demand function's economic environment in which revealed preference theory was born. He begins by recognizing that in any practical situation only a finite number of competitive budget sets will be observable as data. He then writes down a linear program such that if the data do not satisfy the Strong Axiom, then it has no solution, and if it does have a solution, then this takes the form of a preference relation that is compatible with the given observations. Further, the solution is given by a piecewise linear, concave function. The point of the whole thing is that the solution is very nice and, via the simplex method, easily computable. In fact, once in the track opened by Afriat, it is not hard to convince oneself that even a linear program is not necessary. The solution can be computed recursively. Also, if one's standard for "being nice" is differentiability rather than piecewise linear, this can be obtained too.

Summing up: The Strong Axiom implies the preference hypothesis and, in quite different spirits, Richter and Afriat have provided very elegant, direct, and comparatively clean ways to proceed from demand to preferences.

THE UNIQUENESS OF UNDERLYING PREFERENCES

We saw in the previous section that the Strong Axiom guarantees the existence of underlying preferences. For applications, however, it may be important that those preferences be, in some appropriate sense, unique. Only then can we speak of "recovering" the preferences. Further, uniqueness in itself would be useless without an actual technique to accomplish the recovery in practice.

A first observation is the obvious need to narrow down the meaning of uniqueness. Indeed, even with the nicest sort of demand function, if there is an underlying preference relation, then there are infinitely many. Just take an indifference class and order its commodity bundles in some arbitrary way, leaving unaltered the preference ordering with the bundles in other indifference classes. The demand function will remain unaffected. The problem is, of course, that the family of competitive budgets is not large enough for a complete identification of the preference relation. The remedy, however, is also clear. We should look for uniqueness inside a natural class, one that will be large enough for any purpose for which the preferences have to be used. One such class is the class of preferences that satisfies the regularity property of continuity (or, more conveniently for revealed preference analysis, upper semicontinuity).

Restricting the sense of uniqueness to the one described in the previous paragraph, it is well established by now that underlying a given demand function there is a unique continuous preference relation if, besides the Strong Axiom, the demand satisfies (uniformly) a certain weak regularity condition (i.e., the so-called income lipschitzian property). Locally, this is somewhat stronger than continuity, but much weaker than differentiability. Therefore, if the reader believes that a hypothesis of differentiability imposes no empirical restriction, we need not worry about the income Lipschitzian property (which, it may be worthwhile to point out, is not implied by the continuity of preferences alone). The property was first explicitly identified by Houthakker (1950) and has been used by him, Samuelson (1948), Uzawa (1959), Stigum (1973), and many others for a simultaneous attack on the existence and uniqueness questions via the already alluded to price-income sequences, a constructive technique patterned on the approximation algorithms for the solution of differential equations. Another, less constructive, method is to establish on general grounds that under the technical hypotheses there can be at most one underlying preference relation, the existence of one coming from the results reported in the previous section.

In the setup proposed by Afriat (1967), where only a finite number of budget choices are available, nonuniqueness is inherent. But it can be shown that, under an appropriate version of the income lipschitzian property, as the number of choice observations increases in a regular manner, the set of possible underlying preferences narrows down in a well-defined way. Since, besides, Afriat's method is entirely constructive, it provides what is perhaps the most practical approach to revealed preference in the narrow (i.e., competitive demand choices) sense. The uniqueness problem is treated in Mas-Colell (1978), from which the observations of this section are taken.

REVEALED PREFERENCE AND THE INTEGRABILITY PROBLEM

When Samuelson invented revealed preference analysis, the search for demand restrictions that would imply (and be implied by) the preference hypothesis already had a long history (see Samuelson 1950 for an account). The older tradition is the so-called integrability approach, which was begun by Antonelli (1886). Its distinctive feature is that it focuses on infinitesimal conditions on demand — that is, conditions on derivatives.

There are two (dual) versions of integrability. The oldest begins with Antonelli. It imposes restrictions on indirect demand functions (dependent variables are prices and income; independent variables are commodity bundles) and recovers direct preferences (the usual type). For a modern treatment, see Debreu (1972). A more recent version imposes the integrability conditions on demand functions and obtains indirect preferences (i.e., preferences are defined on budgets rather than commodity bundles). A definitive treatment has been provided by Hurwicz and Uzawa (1971).

An advantage of the more recent version is that the integrability conditions are quite familiar. They are simply the symmetry and negative definiteness of the Slutsky matrix (i.e., the matrix of price derivatives of compensated demand). The negative definiteness condition has a clear economic interpretation; it is simply the law of demand for compensated demand function — that is, the (compensated) demand of any commodity, simple or composite, must increase when the price declines. Whether this should be looked at as a rationality or a stability condition is a question we shall not get into. The point is that it has an obvious economic meaning. The same cannot be said of the symmetry condition. While to get it as a theorem is very interesting (since it is just beyond what one can deduce about the logic of maximizing behavior without the help of mathematics), to impose it as a primitive axiom of the theory is economically quite opaque. In fact, in the 1930s a number of researchers hesitated to do so and entertained a notion of nonintegrable preferences that, once all is said, led to a dead end.

Regularity conditions aside, integrability and revealed preference theory are equivalent since both the Strong Axiom and the two integrability conditions characterize the preference hypothesis by restrictions involving only the demand function. Integrability work has been important in the development of the box of tools of consumer theory. But as a theory that characterizes rational demand, one can only agree with Samuelson that revealed preference is superior. While in the physical sciences it is often the case that the primitives of a theory are laws of motion given in terms of differential equations, there is no reason why this should be so in consumer theory, where the postulate of a global optimizer does not seem objectionable. Integrability theory, much influenced from the beginning by the mathematical methods of the physical sciences, considers only infinitesimal comparisons and must depend, therefore, on an unintuitive condition, such as symmetry of the Slutsky matrix. Revealed preference theory makes global comparisons (something, we repeat, not unnatural in economics) and is able to accomplish the task with a very intuitive hypothesis, the Strong Axiom. At the level of mathematical techniques, revealed preference analysis was at the beginning still much influenced by the differential equations methods proper to the integrability approach. Not until the contributions of Richter and Afriat did it develop tools attuned to the new, global, finitistic point of view.

THE DEMAND THEORY OF THE WEAK AXIOM

With two commodities the Weak Axiom (the original Samuelson postulate) implies the Strong and, therefore, the preference hypothesis. This does not generalize to more than two commodities. The reason was perceived by Samuelson. He noted that with two commodities the Slutsky symmetry condition is automatically satisfied, but with more than two it is a substantive restriction. Take a demand function for three commodities that satisfies the integrability conditions (and therefore the Strong Axiom) and perturb it slightly. Then we will destroy the symmetry of the Slutsky matrix (and by implication the Strong Axiom), but we may still hope that the Weak Axiom holds. This reasoning is correct. An implementation yielding an example was given by Gale (1968), and a general argument can be found in Kihlstrom, Mas-Colell, and Sonnenschein (1976), from which the remarks of this section are taken.

What turns out to be the case is that with differentiable demand the Weak Axiom is essentially equivalent to the negative definiteness of the

Slutsky matrix (i.e., to the law of demand), but has no implications whatsoever for symmetry. This is something, incidentally, that Samuelson knew well. It would appear that if, in some sense, the Weak Axiom is satisfied but the Strong fails, this is akin to a failure of transitivity of underlying preferences. The sense, however, has not been made precise.

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