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Perfect Competition and the Core

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1. INTRODUCTION

It is part of the conventional wisdom of economics that a mass market, i.e. one with many relatively small participants, will tend to be perfectly competitive. One possible meaning of this is that it will behave *as if* prices were quoted and taken as given by the traders. The final allocation of goods and services is then called Walrasian.

There are at least two classical formal theories of perfect competition. They can be associated with the names of Cournot (1838) and Edgeworth (1881). Edgeworth's contribution has led to the modern theory of the Core of a market (see Debreu-Scarf (1963), Aumann (1964), and Hildenbrand (1974) and his references). Its central result, the Core Equivalence Theorem, states conditions under which, in a mass economy, competition in the sense of Edgeworth yields at equilibrium Walrasian allocations. The extension to general equilibrium analysis of Cournot's partial equilibrium model is currently being actively pursued (Gabzewicz-Vial (1972), Hart (1979), (1980), Shubik (1973), Novshek-Sonnenschein (1978). See also the April 1980 symposium issue of the *Journal of Economic Theory*). By now, a variety of Cournot Equivalence Theorems is available. There is substantial (but not complete) overlap between the set of conditions identified as important by the Edgeworth and the Cournot type analysis. By and large, both approaches have proved to be useful tools for testing the competitiveness of some theoretical market situations. For example, differentiated commodities models have been analyzed in Mas-Colell (1975) from the point of view of the Core and in Hart (1979) from the Cournot perspective. For yet another approach with a classical foundation to the theory of perfect competition, see J. Ostroy (1980).

It is probably fair to say that the economics profession at large thinks of perfect competition more in Cournot's than in Edgeworth's terms. The purpose of this paper is to make the case for the Core in the light of some of the issues raised by the recent Cournot analysis. We will try to argue that the Edgeworth notion of competition yields a quite sensible concept and a very flexible tool and that, not surprisingly, Edgeworth and Cournot competition are close relatives once properly viewed. By way of motivation, it will be useful to begin by examining and commenting on four points of real or apparent contrast between the two approaches:

1. *Cournot theory is non-cooperative, while the Core is cooperative*

Without an explicit and precise model of communication channels, this is a meaningless distinction. Perhaps the most attractive economic story behind the Core notion runs in "non-cooperative" terms. As Edgeworth himself did, one could look at final allocations of goods as the outcome of a complex system of contracts among traders. Imagine that, in addition, a number of potential entrepreneurs are competing among themselves for profitable deals. The concept of a blocking coalition can then be identified with that of a feasible set of contracts an entrepreneur can put together at his own gain and with the voluntary participation of everyone concerned. If there is free entry for entrepreneurs (which, of course, can double as usual traders), then an allocation that can be blocked will not fail to be so.

2. *Core theory is more parsimonious than Cournot's*

For the Cournot type analysis, one needs a specification of transaction institutions. This may be a cumbersome and non-trivial task. In contrast, the Core determines final allocations from the basic economic data: tastes, technologies, endowments. This is the feature of the Core which makes it so attractive, but the level of abstraction at which it moves has an unfortunate consequence. It is this: in the absence of an institutionally explicit model, the only sensible feasibility constraint for a blocking coalition is that the proposed reallocation affect can be attainable for *all* commodities simultaneously considered. This, of course, means that competition works by the setting of entirely separated subeconomies! This feature has, no doubt, militated against the wide acceptance of the Core notion. Its importance, however, has to be evaluated in the light of the following two observations: (i) the Core Equivalence Theorem is only reinforced by making blocking difficult, and (ii) if, in fact, we face an institutionally explicit model, then we can without difficulty relax the blocking feasibility constraint in realistic ways. We shall consider such a situation in Section 4 and do as indicated.

3. *Cournot competition is quantity setting while the Core's is more general*

This is true enough and accounts for some real differences such as the fact that in an economy with a finite number of traders the Walrasian equilibrium is in the Core while it is not in general a Cournot equilibrium. This is not, however, an aspect on which we will concentrate, partly because it is not difficult to envision Cournotian models with more complex strategy spaces.

4. *The Core competition concept requires more information than Cournot's*

This is a key difference and the one we will focus on. The working of the blocking concept requires the public availability of all the relevant information on tastes, technologies, trading sets, etc. In contrast, Cournot theory "only" demands the publicness of the strategies played by the traders. It is precisely this informational tightness of the Cournot approach which accounts for some of the competition failures identified in different varieties of Cournot models by Shubik (1973) and Hart (1980) (see also Makowski (1980)).

In this paper we shall concentrate on the informational aspect. Our programme is to reexamine the notion of blocking underlying competition in the sense of Edgeworth by being very explicit on information requirements. Section 2 introduces some basic concepts and generalities. Section 3 analyzes the traditional institution-free model of Core theory. For the exchange case (we also consider production) what we do can be summarized thus. We have a population of traders characterized by trading sets (allowing for private production) and preferences. Consider a given net trade allocation of goods. Our basic hypothesis is that the whole allocation is public knowledge, but that this exhausts the extent of public information. For example, the nature and characteristics of a particular commodity are publicly known only if the commodity is actually traded. We say that the allocation is arbitrage-free if no blocking coalition can be formed which gives to every blocking member more of each commodity. Arbitrage is the weakest notion of competition, since it does not use private information beyond the fact that preferences are monotone. Next, we define a hierarchy of blocking notions ordered by coordination complexity. We say that for an integer $m \geqq 1$, the allocation is m-blocked if it is possible to form a blocking coalition and allocation with the property that the blocking traders can be divided into two groups, which we call passive and active. The passive traders get at the new allocation the same net trade they were previously getting and so knowledge of their private characteristic (i.e. preferences, trade sets) is not necessary in order to form the coalition. The active traders are the ones that change their trading plans for the coalition to form, i.e. their preferences and trading sets are essentially involved. We require that there be no more than m distinct types of those active traders. See Section 3 for a discussion of this concept. Under the hypothesis of a continuum of traders we establish: (i) Arbitrage-free allocations are compatible with non-zero price systems (Propositions 1 and 5). (ii) If the allocation is not Walrasian relative to the set of active markets (we call those semi-Walrasian allocations) then it can be 1-blocked (Propositions 2 and 6). In other words: it is easy for competition to destabilize active markets which are not in equilibrium. (iii) In order to destabilize an allocation which is not fully Walrasian, we may need to m-block where m may be as large as but not larger than the number of non-active markets (Propositions 3 and 7). Different types of complementarities account for the worst possible cases. (iv) If, in a precise sense, there is enough substitutability among commodities, then an allocation which is not Walrasian can be 2-blocked (Propositions 4 and 8).

It is worth remarking that some of the phenomena arising in the Cournot treatment of competition reappear in our Edgeworth type approach. Thus, if our theory of competition was 1-blocking (this would be akin to the Shubik (1973) model), then Walrasian equilibrium is guaranteed only if all markets are active. If it was 2-blocking (this is akin to Hart (1979), and Novshek-Sonnenschein (1978)), then only if substitution prevails. If this is not the case (as in Hart (1980) and Makowski (1980)), then 2-blocking may not do essentially better than 1-blocking, i.e. to guarantee the equilibrium of active markets. The fundamental difference between active and non-active markets is easy to trace (see also Hart (1980)): if arbitrage is at work, then in every active market there is a price to be "taken"; the question of who sets the prices is irrelevant as price signals are embodied in actual transactions. On the contrary, in a non-active market there is no price signal of any kind; and this may lead to the prevalence of the quantity signal "no trade possible".

Section 4 analyzes a model with some institutional content. More specifically, trade in goods takes place against a numeraire commodity. We will relax the feasibility requirements of blocking in a natural manner, i.e. we will impose them only market by market. We shall then proceed to establish results entirely analogous to the ones in Section 3. The point of the exercise is that the model is now very close to the ones extensively studied via Cournot (e.g. Hart (1980) and Makowski (1980)) and so are our conclusions reached via the Core. Thus, the model serves to bridge the gap between the Cournot approach and the abstract Core analysis of Section 3.

Section 5 gathers the proofs. We should emphasize that this paper will contain no surprises for anyone familiar with the interiorities of Core theory. Our work is more in the nature of a reexamination of the latter. A concept (and result) related (but not the same) to our notion of m-blocking has been used by Grodal (1972). With different names, concepts analogous to "arbitrage-free" and "semi-Walrasian" have been used by Schmeidler and Vind (1972) (see also Vind (1977)) in the theory of fair net trades.

For expositional simplicity we will not, in this article, strive to prove the strongest possible results. For some of the latter, see Mas-Colell (1979*a*). There is one strengthening we would like to mention, however. As usual in Core theory all the blocking can be made by arbitrarily small coalitions. This was shown by Schmeidler (1972) and it holds in our case, too. It has significance in view of the fact that the natural analogue of an individual small trader in the Cournot approach is a small coalition in our Core treatment with a continuum of traders. For an analysis of how small is small in the finite approximations to the continuum (and an argument that it means "bounded above with independence of the size of the economy") see Mas-Colell (1979*b*).

2. THE BASIC MODEL OF AN ECONOMY

The commodity space is R^l. There is a continuum of traders represented by the interval $I=[0, 1]$ equipped with Lebesgue measure (denoted by λ).

There is given a closed, convex cone $Y \subset R^l$ representing the (mean) technological possibilities open to society. We take the knowledge of Y as being in the public domain, i.e. Y is available to every member of society. In this, we follow McKenzie (1959) and Debreu and Scarf (1963). We assume free disposal, i.e. $-R^l_+ \subset Y$.

Trader's characteristics are described by pairs $(X, \succ)$, where $X \subset R^l$ is a closed set of admissible net trades and $\succ \subset X \times X$ is a continuous (i.e. open graph) preference relation (i.e. $\succ$ is irreflexive and transitive). Observe that private production is allowed by the model. For simplicity, we assume that $\succ$ is monotone (i.e. if $x \in X$ and $x' \gg x$, then $x' \in X$ and $x' \succ x$) and also that $0 \in \operatorname{Int} X$. To facilitate exposition, our space of characteristics $\mathscr{A}$ will be composed of only a finite number of characteristics pairs, to be denoted types.

An economy is completely specified by Y and a measurable map $\mathscr{E}: I \to \mathscr{A}$ describing traders' characteristics. We denote $\mathscr{E}(t) = a_t = (X_t, \succ_t)$. Observe that production possibilities are encompassed both at the public (via Y) and the private (via X_t) level. In a more elaborated treatment a "semipublic" level could be incorporated (e.g. some technological possibilities may be public only within the confines of some restricted groups). If $Y = -R^l_+$, we have an exchange situation (with free disposal). If further $X_t = R^l_+ - \{\omega_t\}$ for all t, we have a pure exchange economy of the usual variety.

An allocation is an integrable map $\underset{\sim}{x}: I \to R^l$ such that $\int \underset{\sim}{x} \in Y$.

An allocation is feasible if $\underset{\sim}{x}(t) \in X_t$ for a.e. $t \in I$.

For a model without the strong hypothesis of this part (e.g. monotonicity, free disposal, $0 \in \operatorname{Int} X$), see Mas-Colell (1979*a*).

3. COMPETITION AND EQUILIBRIUM: A GENERAL APPROACH

We shall now present an abstract model of competition and equilibrium. Its main feature is that it is institution-free, i.e. no mechanisms of trade are made explicitly. For a more institutional model, see Section 4.

3.1. *Equilibrium concepts*

An economy Y, $\mathscr{E}: I \to \mathscr{A}$ is given.

Given a feasible allocation $\underset{\sim}{x}$, we can associate with it the smallest linear space $L_{\underset{\sim}{x}}$ such that $\underset{\sim}{x}(t) \in L_{\underset{\sim}{x}}$ for a.e. $t \in I$. We can interpret $L_{\underset{\sim}{x}}$ as an incomplete (if $\dim L_{\underset{\sim}{x}} < l-1$) budget set compatible with the observed allocation. It represents the maximum information about budgets that can be unambiguously extracted from transactions.

We will call a feasible allocation $\underset{\sim}{x}$ price compatible if there is $p \neq 0$ such that:

(i) $p \cdot \int \underset{\sim}{x} \geqq \sup p \cdot Y$,

and

(ii) $p \cdot L_{\underset{\sim}{x}} = 0$.

Because of free disposal, we always have $p \geqq 0$.

A feasible allocation x is semi-Walrasian if for a.e. $t \in I$, $\underset{\sim}{x}(t)$ is $\succ_t$-maximal on $X_t \cap (L_{\underset{\sim}{x}} + Y)$. Because of the monotonicity of preferences, semi-Walrasian allocations are price compatible. The notion of semi-Walrasian allocation is an abstract (i.e. coordinate-free) analogue of the concept of Walrasian equilibrium with respect to an incomplete set of markets. For example, if $L_{\underset{\sim}{x}} = \{x \in R: x^1 = \cdots = x^k = 0$ and for some $p \neq 0$, $p \cdot x = 0\}$, we can quite literally interpret $L_{\underset{\sim}{x}}$ as a usual budget set subject to a no-trade constraint in commodities 1 to k.

A feasible allocation $\underset{\sim}{x}$ is Walrasian if there is $p \in R^l$ such that: (i) $p \cdot \int \underset{\sim}{x} \geqq \sup p \cdot Y$ (profit maximization), and (ii) for a.e. $t \in I$, $\underset{\sim}{x}(t)$ is $\succ_t$-maximal on $X_t \cap \{x \in R^l: p \cdot x \leqq 0\}$.

Every Walrasian allocation is semi-Walrasian. Conversely, if the allocation x is semi-Walrasian *and* dim $L_x = l-1$, then it is Walrasian.

The concept of Walrasian allocation is, of course, standard. Concepts analogous to price-compatible and semi-Walrasian allocations have been used before by Schmeidler and Vind (1972).

3.2. *Arbitrage*

Let an economy Y, $\mathscr{E}: I \to \mathscr{A}$ and an allocation $\underset{\sim}{x}: I \to R^l$ be given.

Let us now postulate that Y, $\underset{\sim}{x}$ are publicly available information. Under those circumstances, the allocation $\underset{\sim}{x}$ will lack stability if it is possible to put together a deal (i.e. a coalition) which guarantees to everyone concerned at least as much of every good as at $\underset{\sim}{x}$ and to some traders strictly more of each commodity.

Definition 1. The feasible allocation $\underset{\sim}{x}$ is arbitrage-free if there is no $C \subset I$ and $\underset{\sim}{x}': C \to R^l$ such that:

(i) $\underset{\sim}{x}'(t) \gg \underset{\sim}{x}(t)$ for a.e. $t \in C$, and

(ii) $\int_C \underset{\sim}{x}' \in Y$.

The set of traders C constitutes, in the usual terminology, a blocking coalition. We could interpret in economic terms the notion of blocking underlying Definition 1 (and Definition 2 in the next section) as follows: the allocation $\underset{\sim}{x}$ is the outcome of a set of contracts among traders. It lacks stability if it is possible for a potential entrepreneur to propose to a set of traders new contracts which they find advantageous under the stringent test that none of the old contracts can be counted on.

It is to be emphasized that the point of Definition 1 is that in the formation of the blocking coalition no preference information whatsoever is used by the promoter beyond the fact that more is preferred to less.

We have:

Proposition 1. *If $\underset{\sim}{x}$ is arbitrage-free, then it is price compatible.*

In words: if $\underset{\sim}{x}$ is arbitrage-free, then there is a non-zero price vector p such that, first, aggregate production is profit maximizing and, second, transactions are compatible with p. While p may not be unique, it is obvious that the dimension n of the convex set of price vectors satisfying (i) and (ii) of Proposition 1 is related to $L_{\underset{\sim}{x}}$ by $n \leqq l - \dim L_{\underset{\sim}{x}}$. Proposition 1 is mathematically clear enough, and it is certainly well known. See, for example, Schmeidler and Vind (1972).

3.3. *The coordination requirements of competition*

It is evident that one cannot hope to go beyond Proposition 1 without involving preferences in an essential way. While it is inescapable that the theoretical treatment of perfect competition must contemplate more general blocking coalitions, it will still be useful to have a notion of the degree of complexity of a blocking coalition which is directly related to the variety of nonpublic information necessary to put it together. This motivates the following definition, which is a generalization of Definition 1.

Definition 2. Let the economy Y, $\mathscr{E}: I \to \mathscr{A}$ be given. For any integer $m \geqq 0$, we say that the feasible allocation $\underset{\sim}{x}: I \to R^l$ is m-blocked if there is $C \subset I$, $\underset{\sim}{x}': C \to R^l$ such that:

(i) $\# \{a(t): t \in C$ and not "$\underset{\sim}{x}'(t) \gg \underset{\sim}{x}(t)$"$\} \leqq m$

(ii) $\underset{\sim}{x}'(t) \succ_t \underset{\sim}{x}(t)$ for a.e. $t \in C$

(iii) there is $C' \subset I$ such that $C \cap C' = \emptyset$ and

$$\int_C x' + \int_{C'} x \in Y.$$

In the definition, we could call the members of C "active traders" and the members of C', who remain with the same net trade before and after blocking takes place, "passive traders". The definition is predicated on the idea that the possibility or likelihood of a blocking deal is directly related to the variety of sources of private information that must be brought together for it to take place. In contrast, we do not concern ourselves with the difficulties associated with coordinating identical people, or, in other words, only the number of distinct active types, not of distinct active traders, matters. An entrepreneur who promotes a blocking coalition will have to incur two kinds of costs: (i) the cost of finding out about the possibility of blocking, i.e. a purely informational "research" cost, and (ii) the cost of organizing the blocking coalition across individuals and types. In this paper we are focussing only on the first. The second is, of course, also important; but it is of a quite different nature. Thus, our informal interpretations should be taken in ceteris paribus terms: for the same size coalition and the same "potential gains", the more types of active traders involved, the more difficult blocking is going to be.

Arbitrage is 0-blocking. Arbitrage aside, the simplest form of blocking is 1-blocking. The interpretation of it is straightforward; an allocation x is 1-blocked if a group of identical traders, by using no more than the information embodied on Y and x (i.e. the arbitrage opportunities), can put together a deal profitable to themselves.

We have:

Proposition 2. *If a feasible x is not 1-blocked, then x is semi-Walrasian.*

Simple as it is, Proposition 2 is quite interesting. Indulging in fanciful language, it says that if we identify the working of the Invisible Hand with entrepreneurs (i.e. small groups of identical traders), exploiting arbitrage opportunities in markets, then in a mass economy, the Invisible Hand can be relied upon to guarantee the equilibrium of active markets. In the Cournotian approach analogous conclusions have been obtained in the class of models descended from Shubik (1973) (see Mas-Colell (1981) for an account). In fact, it can be argued that the notion of competition underlying those models is akin to 1-blocking.

In the very restricted sense of Proposition 2 (i.e. 1-blocking), the Invisible Hand cannot be counted on to equilibrate inactive markets. Suppose that a market is inactive (remember that this implies that the existence of the potential commodity is not public information), but that there are potential gains from trade, i.e. at some terms of trade (involving say, commodities in some active markets), some trader would be willing to change his trade and buy some amount of the commodity and some other trader is willing to produce and sell (i.e. to innovate). So, in order to block, both should become active. Typically, they are going to be of different types (this would always be the case if preferences were convex and we refined the definition of type to include as characteristic of t the net trade $x(t)$). Thus, at least 2-blocking will need to be considered. In the next section we will examine a case where this suffices. As the following example shows, however, the situation can be much worse (the example violates the condition $0 \in \text{Int}\, X$ but this is not essential both because it can be fixed up and because a more general treatment does not impose the condition. See Mas-Colell (1979*a*) for the latter).

Example. There are three goods: labour, intermediate good, and consumption good. The technology is $Y = -R^3_+$. Traders, of which there are three types, care only about the consumption good. Type 1 can only supply labour. Type 2 can transform labour into intermediate goods, and type 3 can transform intermedaite goods and labour into consumption goods. Types 2 and 3 cannot supply labour. Clearly, the three types are needed in order to break away from the null allocation $\underset{\sim}{x} \equiv 0$ where no market is active. This "intermediate good" effect seems to be quite typical. What it does is to give rise to a (technological) complementarity among goods. Complementarities have already been made responsible by O. Hart (1980) and L. Makowski (1980) for the different performance in active and nonactive markets of a Cournotian variety of perfect competition models. The intermediate good example has been given independently by Makowski (1980).

Bad as the previous example is, one may derive some comfort from knowing that it is the worst possible.

Given a feasible $\underset{\sim}{x}$ let $Y_{\underset{\sim}{x}}$ be the smallest face of the convex set Y containing $\int \underset{\sim}{x}$.

We have:

Proposition 3. *Let $\underset{\sim}{x}$ be a feasible allocation. Then if $\underset{\sim}{x}$ is not (l-dim $(L_{\underset{\sim}{x}} + Y_{\underset{\sim}{x}})$) blocked, it is Walrasian.*

In words, Proposition 3 roughly says that if $\underset{\sim}{x}$ is a semi-Walrasian but not Walrasian, then in order to activate the markets that should be active, it will not be necessary to involve in an essential way (i.e. by changing their trade) more types of traders than the number of inactive markets. This, of course, could be a very large number and the example shows that the upper bound can be attained. In the next section we will present a case where things are much better behaved.

It could be argued that if the failures of the invisible hand are associated with the inactivity of markets then approximate efficiency would be guaranteed by a public policy oriented to keeping every market at a minimal level of activity. The argument has a grain of truth, but it is basically fallacious. The grain of truth is that competitiveness can only be helped by a public policy promoting the dissemination of information and the transparency of markets. It is fallacious because in many interesting cases even the possibility of the new market may be embodied in the private production possibilities X_t. In other words, if a public agent is introduced, then one should be very explicit about the extent and sources of information available to him.

Stimulated by the work of Hart [1980], we can extract an interesting corollary from Proposition 3. Denote by J the set of commodity indexes. So, $\# J = l$ and we identify the commodity space with R^J. For an allocation $\underset{\sim}{x}$ being Walrasian with respect to $J' \subset J$ we mean that $\underset{\sim}{x}$ is Walrasian when the commodity space is restricted to be $R^{J'}$ (in particular $\underset{\sim}{x}$ should be feasible in $R^{J'}$). Now let $\underset{\sim}{x}$ be a feasible allocation such that, for some $J' \subset J$, $L_{\underset{\sim}{x}} \subset R^{J'}$ and $\dim L_{\underset{\sim}{x}} \geqq \# J' - 1$. So, for this allocation J' represents the active markets. Then we have: if $\underset{\sim}{x}$ is not 2-blocked, then for any $k \in J$, $\underset{\sim}{x}$ is Walrasian relative to $J' \cup \{k\}$.

It should be emphasized that the interpretation of Proposition 3 in terms of complexity of blocking groups cannot be taken too literally, since the result depends on the a priori given specification of commodities. Indeed, if the primitively given commodity space was thought of as infinite dimensional, the sort of dimension counting of Proposition 3 would be meaningless. The issue would have to be handled in a different way. How, it is not clear, and constitutes an interesting research topic.

Proposition 3 is closely related to a result of Grodal (1972) asserting that, for the usual notion of blocking, every allocation which can be blocked can be blocked by a coalition involving at most l types. As the reader no doubt has already perceived, both Grodal's result and ours are derived from Caratheodory's theorem. One could read Proposition 3 as saying that in forming a blocking allocation for $\underset{\sim}{x}$ one could restrict $\dim(L_{\underset{\sim}{x}} + Y_{\underset{\sim}{x}})$ of the types involved in a Grodal blocking coalition to receive the same net trade as at $\underset{\sim}{x}$.

3.4. *A special case: enough substitutability*

We shall now discuss a case where the intuition about buyers and sellers (i.e. two types) sufficing to activate markets (that should be activated) is correct. Since the culprits of non-sufficiency are complementarities it is clear that the solution will rest with enough substitutability prevailing. Enough substitutability will mean that, at the given allocation, there are some traders whose set of preferred net trades has no (convex) kinks at the assigned net trade, i.e. they are as in Figures 1a, 1b, and 1c and not as in Figure 1d. More formally:

Proposition 4. *Let Y and $\mathscr{E}: I \to \mathscr{A}$ be given. Suppose that at the feasible allocation $\underset{\sim}{x}: I \to R^l$ there is a set $C \subset I$ with $\lambda(C) > 0$ such that for all $t \in C$, there is a $p_t \in R^l$, $\|p_t\| = 1$, with the property that whenever $p_t \cdot v > 0$ then for some $\bar{\alpha} > 0$ and $y \in Y$ we have $\underset{\sim}{x}(t) + \alpha v + y >_t \underset{\sim}{x}(t)$ for all $0 < \alpha < \bar{\alpha}$. Then if $\underset{\sim}{x}$ is not 2-blocked it is Walrasian.*

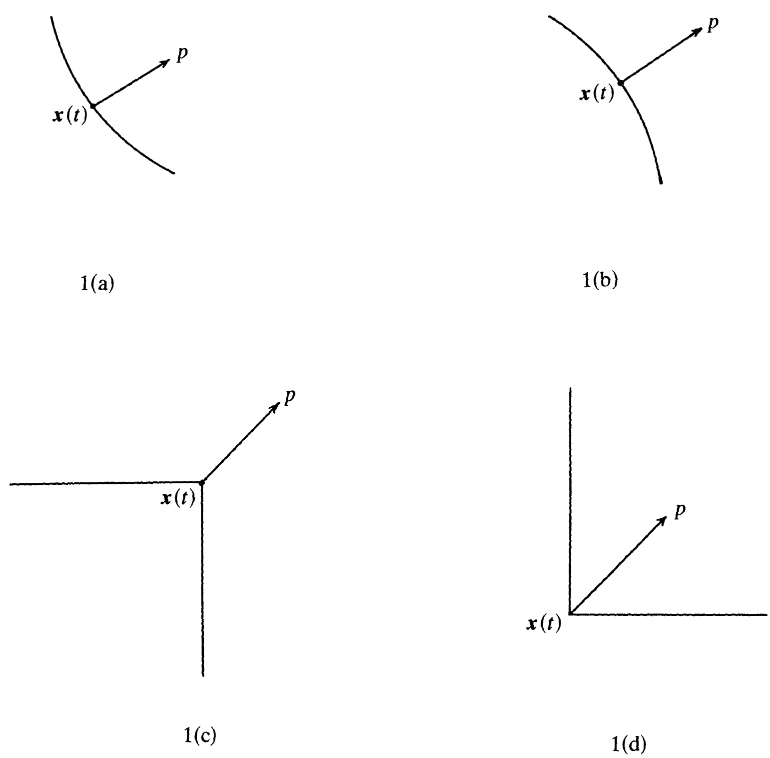

FIGURE 1

The hypotheses of the Proposition are by no means harmless. In the exchange case (i.e. $Y = -R^l_+$) they will be satisfied if for some traders: (i) preferences are representable by a C^1 utility function (smoothness), and (ii) the net trade allocated is not at the boundary of the trading set (interiority). The private production example of the previous section illustrates well the role of (ii). In a production model, interiority, even if only for some traders, is an unreasonably strong hypothesis. It may be the case that if there is enough smoothness and one could find some way to rule out formally the intermediate good effect, then a conclusion such as the one of Proposition 4 could be rescued without eliminating sensible forms of production. But we have not investigated this point. Observe that convexity of preferences is not assumed.

In the Cournotian approach to perfect competition it could be argued that a concept of competition parallel to 2-blocking (the two types being one firm and consumers) underlies the models of Gabszewicz-Vial (1972), Novshek-Sonnenschein (1978) and Hart (1979). They also have results parallel to Proposition 4. But the frameworks are sufficiently different for the hypotheses to be difficult to compare.

It is to be noted that the conclusion of Proposition 4 as well as some of the basic arguments going into the proof have an air of similarity with Rader's results on pairwise Pareto optimality (1968). The differences should be clear, however. Rader's theorem concludes that pairwise Pareto optimality of an allocation implies over-all Pareto optimality, while we could phrase Proposition 4 as concluding that a type of pairwise non-blockability of an allocation implies overall non-blockability (and therefore, via the usual core equivalence theorem, the Walrasian character of the allocation).

4. COMPETITION AND EQUILIBRIUM: A MODEL WITH MONEY

In this part we will redo the analysis of Section 3 in a context which is very specific about the institutional arrangements for trade. This is in the nature of an exercise. We hope that it is an instructive one. We will assume that there is a "money" commodity (i.e. a universally valued commodity) and that trade is organized in markets where single commodities are exchanged against money. Obviously this is a rather rigid and extreme set-up, but it is well suited to our present purposes, which are to highlight parallels and differences of institutionally explicit models with the abstract treatment of Section 3.

4.1. *The model*

Commodity 1 will be money. Let $e^1=(1,0,\dots,0)$. In addition to the hypothesis of Section 2 we assume that preferences are strictly monotone in the money commodity.

There are $(l-1)$ markets where commodities $j=2,\dots,l$ are exchanged for money. A trade in market j is specified by $z^j=(z^{j1},z^{j2})\in R^2$ where z^{j1} are the money payments (if $z^{j1}<0$) or receipts (if $z^{j1}>0$) and z^{j2} the amount of good j bought (if $z^{j2}>0$) or sold (if $z^{j2}<0$) for that amount of money.

A trade will be a vector $z=(z^{21},z^{22},\dots,z^{l1},z^{l2})\in R^{2(l-1)}$. Any trade z induces a net trade $\hat{z}=(\sum_{j=2}^{l}z^{j1},z^{22},\dots,z^{l2})\in R^l$.

An assignment will be an integrable map $\underset{\sim}{z}:I\to R^{2(l-1)}$ such that, letting $z=\int \underset{\sim}{z}$, we have $\hat{z}\in Y$. An assignment $\underset{\sim}{z}$ is feasible if $\hat{\underset{\sim}{z}}(t)\in X_t$ for a.e. $t\in I$. Note that every assignment $\underset{\sim}{z}$ induces an allocation $\hat{\underset{\sim}{z}}$.

In the spirit of a "monetary" model, we fix the price of money to be equal to one, i.e. $p^1=1$. A price vector is then any $p\in R^l$ with $p^1=1$.

A feasible assignment $\underset{\sim}{z}$ is price compatible if there is a price vector p such that:

(i) $p\cdot\int \underset{\sim}{z}\leqq \sup p\cdot Y=0$,

and

(ii) $\underset{\sim}{z}^{j1}(t)+p^j\underset{\sim}{z}^{j2}(t)=0$

for a.e. $t\in I$ and all $j=2,\dots,l$. The expression "p is compatible with $\underset{\sim}{z}$" has the obvious meaning.

An assignment $\underset{\sim}{z}$ is Walrasian if it is feasible and there is a compatible price vector p such that for a.e. $t\in I$, $\hat{\underset{\sim}{z}}(t)$ is $\succ_t$-maximal on $\{x\in R^l: p\cdot x\leqq 0\}\cap X_t$. If assignments are Walrasian or price compatible, the induced allocations are so in the sense of Section 3.

It will be useful to introduce the following notation. The set of markets is J and its generic index $j\in J$. Of course we identify markets with the commodities 2 through

l. In the obvious way we view a trade z as an element of the product space R^{2J}. Similarly, given a subset of markets $J' \subset J$, we can consider trades $z \in R^{2J'}$. In fact, we shall always identify $z \in R^{2J'}$ with the trade in R^{2J} obtained by putting $(z^{j1}, z^{j2}) = 0$ whenever $j \notin J'$. With this convention the concept of assignment $\underset{\sim}{z}: I \to R^{2J'}$ (and feasible assignment $\underset{\sim}{z}: I \to R^{2J'}$ relative to J') is well defined. If J' is not explicitly mentioned, it is understood that $J' = J$.

Given an assignment $\underset{\sim}{z}$, market $j \in J$ is active if $\int |\underset{\sim}{z}^{j2}| \neq 0$. The set of active markets is denoted $J(\underset{\sim}{z})$. Observe that if $\underset{\sim}{z}$ is price compatible, then prices are uniquely determined on $J(\underset{\sim}{z})$.

We will call an assignment semi-Walrasian if it is Walrasian with respect to the active markets. Formally, if $J(\underset{\sim}{z}) \subset J' \subset J$ we say that $\underset{\sim}{z}$ is Walrasian with respect to J' if it is feasible and there is a compatible price vector p such that $p \cdot \int \underset{\sim}{z} \geqq \sup p \cdot Y$ and $z(t)$ is $\succ_t$-maximal on $\{\hat{z} \in R^l : p \cdot z \leqq 0,\ z^j = 0 \text{ for all } j \notin J'\} \cap X_t$ for a.e. $t \in I$. The assignment $\underset{\sim}{z}$ is semi-Walrasian if it is Walrasian with respect to $J(\underset{\sim}{z})$.

4.2. *Arbitrage*

We shall proceed quite analogously to Section 3. Suppose Y and $\underset{\sim}{z}: I \to R^{2J}$ are publicly available information. A first requirement of a competitive outcome is that it should not be possible to extract profitable deals from Y and $\underset{\sim}{z}$. More specifically, it should not be possible for a potential entrepreneur to "enter" the different markets and while giving to trading partners the same trade they get at $\underset{\sim}{z}$, end up with a surplus of money.

There will, however, be an important difference from Section 3. The institutional explicitness of the model allows us to relax in a natural (within the present framework) manner the feasibility requirements for "blocking", i.e. for the formation of a profitable deal. Indeed, we will not require that the potential entrepreneur coordinate trading partners across different markets. Every market can be entered separately and we postulate that, irrespective of what is happening in other markets, traders in a market are indifferent (or ignorant) about trading partners provided the same trade is guaranteed. In slightly different words, suppose an assignment is viewed as the outcome of a system of contracts, then our present approach amounts to the hypothesis that when traders are offered a new contract in a market, they behave as if all the old contracts affecting other markets were still available.

Formally,

Definition 3. The feasible assignment $\underset{\sim}{z}$ is arbitrage-free if there are no $C_j \subset I$, $2 \leqq j \leqq l$, and $y \in Y$ such that:

$$\left(\Sigma_j \int_{C_j} \underset{\sim}{z}^{j1}, \int_{C_2} \underset{\sim}{z}^{j2}, \ldots, \int_{C_l} \underset{\sim}{z}^{l2}\right) = y - \delta e^1 \quad \text{for some } \delta > 0.$$

In the special exchange case, i.e. $Y = -R^l_+$, this definition amounts to the assertion that $\underset{\sim}{z}$ is arbitrage-free if and only if every $\underset{\sim}{z}^j$ is so (i.e. every market is arbitrage-free) when considered in isolation. Without any loss in the subsequent results we could add to Definition 3 that the sets C_j be arbitrarily small and disjoint.

Proposition 5 is a result analogous to Proposition 1. Its interpretation is the same.

Proposition 5. *If the feasible $\underset{\sim}{z}$ is arbitrage-free, then it is price compatible.*

Since, given $\underset{\sim}{z}$, $\hat{\underset{\sim}{z}}: I \to R$ is a feasible allocation in the sense of Section 3, we could apply to it the arbitrage theory developed there. If $\hat{\underset{\sim}{z}}$ is arbitrage-free according to the definition of this section, then $\hat{\underset{\sim}{z}}$ is arbitrage-free according to the definition of Section 3. The converse, however, is not true. (Examples are trivial to come by. They are, however, somewhat singular.) This, of course, is due to the fact that blocking is now easier. The theoretical payoffs of the feasibility relaxation are two: (i) arbitrage-freeness may determine prices more sharply than in Section 3 (the arbitrage prices of Proposition 5 are also arbitrage prices for Proposition 1, but not necessarily conversely), (ii) the amount of public information needed for arbitrage-type blocking is smaller than in Section 3. Given z, in Section 3 one needs to regard as public the distribution on R^l induced by z while here only the marginal distribution induced by $\hat{z}$ on every R^{2j} needs to be known. These comments also apply to the results of the next subsection. It should be remarked that Propsition 1 is both mathematically and economically deeper than Proposition 5 and, precisely because it is institution-free, of considerably more theoretical interest and usefulness.

4.3. *The coordination requirements of competition*

We will now reformulate the blocking notions of Section 3 in the present context. As in the previous section for arbitrage, the only conceptual modification will be that the feasibility (and, consequently, information) conditions for blocking will be relaxed by allowing recontracting market by market.

Definition 4. For any integer $m \geqq 0$ we say that a feasible assignment $z : I \to R^{2J}$ is m-blocked if there is $C \subset I$, $z' : C \to R^{2J}$ such that:

(i) $\#\{a(t)\colon t \in C$ and no "$\hat{z}(t) \geqq \hat{z}(t)$, $\hat{z}'^1(t) > \hat{z}^1(t)$"$\} \leqq m$

(ii) $\hat{z}'(t) \in X_t$ and $\hat{z}'(t) \succ_t z(t)$ for a.e. $t \in C$

(iii) there is $y \in Y$ and $C_j \subset I$, $j \in J$, such that $C \cap C_j = \emptyset$, and

$$\textstyle\sum_{j \in J'} \left(\int_{C_j} z^{j1} + \int_C z'^{j1} \right) - y^1 = 0, \qquad \int_{C_j} z^{j2} + \int_C z'^{j2} - y^j = 0$$

for all $j \in J$.

The interpretation of the definition is clear: z can be m-blocked if there is a (small) group of m types of traders that can put together a profitable deal by coordinating their actions and exploiting the arbitrage opportunities embodied in the publicly available information. An assignment z is not 0-blocked if and only if it is arbitrage-free. Results entirely analogous to Proposition 2 and 3 are available:

Proposition 6. *If the feasible assignment z is not 1-blocked, then z is semi-Walrasian.*

Proposition 7. *If the feasible assignment z is not $(l$-$\#J(z))$-blocked and $J(z) \neq \emptyset$, then z is Walrasian.*

The role of the hypothesis $J(z) \neq \emptyset$ is to ensure the positivity of the price of money. As in the previous section, we note that the price vector obtained from Proposition 6 may be determined more sharply than the one obtained from Proposition 2. In fact, it may even be the case that an assignment z for which the corresponding allocation $\hat{z}$ is semi-Walrasian (in the sense of Section 3) is not semi-Walrasian at all in the context of this section. Exactly the same comment applies to Proposition 7 and the Walrasian concept. Note that we always have $\#J(z) \geqq \dim L_{\hat{z}}$. It can be shown that if equality holds, then a price compatible z is semi-Walrasian (resp. Walrasian) if and only if $\hat{z}$ is. Hence discrepancies can only occur if $\#J(z) > \dim L_{\hat{z}}$. If traders' characteristics are dispersed, the occurrence of this at equilibrium would seem rather exceptional. This is a point, however, which has not been investigated in any depth.

4.4. *Smoothness hypothesis*

Given our particular institutional set-up, we will try now to obtain conclusions similar to Proposition 4 with a somewhat different hypothesis.

We restrict ourselves to the pure exchange case, i.e. $Y=-R^l_+$ and $X_t=R^l_+-\{\omega_t\}$ for a.e. $t\in I$. Further, we shall assume that, for a.e. $t\in I$, $>_t$ is a convex (i.e. $\{x'\in X_t: x'>x\}$ is convex for all $x\in X_t$) preference relation representable by a C^1 utility function $u_t: X_t\rightarrow R$ with $\partial_1 u_t(x)>0$ for all $x\in X_t$. This is natural in view of our hypothesis of strict monotonicity with respect to money.

Proposition 8. *Let z be a feasible assignment such that $J(z)\neq\emptyset$ and for a.e. $t\in I$, $\hat{z}^1(t)-\delta_t e^1\in X_t$ for some $\delta_t>0$. Then if z is not 2-blocked, it is Walrasian.*

Four comments are in order:

1. For the pure exchange case, the main conceptual difference between Propositions 8 and 4 is that it is now possible for the allocation z to yield a boundary net trade for all $t\in I$. The interiority requirement for *some* traders has been replaced by the condition that at $\hat{z}$ *all* traders are not at the boundary with respect only to the money commodity. It is essential to the validity of Proposition 8 that the constraints defining trading sets be of the coordinate type (i.e. $x^j\geqq b$). Therefore, production is not accommodated.
2. Under the hypotheses of Proposition 8 (including the pure exchange assumption), Propositions 6 and 8 could be refined further by restricting the formation of blocking assignments to affect a single market. The same conclusions obtain. This prompts a more general comment on the model of this section. It should be possible to develop an entirely parallel "dual" theory where m, the "degree of complexity", would stand for the number of markets, not the number of types, involved in the formation of a blocking deal, or, perhaps, for both.
3. Like Proposition 4, Proposition 8 has a parallel in the theory of pairwise Pareto optimality, namely the result established by Feldman (1973). The analysis of this part and especially Proposition 8 has a family resemblance with some of the models developed in Malinvaud-Younès (1977) (see especially Section 5, 6, and Proposition 3).
4. The validity of Proposition 8 in the context of Section 3 (i.e. with allocations instead of assignments and blocking in the sense of Definition 2) is an open question. A counterexample may exist for $l>3$.

5. PROOFS

5.1. *Preliminaries*

The purpose of this subsection is to recast the arbitrage-free and blocking definitions in equivalent ways so as to focus and simplify the forthcoming proofs.

We begin by observing that it is an immediate consequence of Lyapunov's theorem on the range of an atomless vector measure (see Hildenbrand (1974), p. 45) that if an allocation (or assignment) is blocked in any of our senses then it can be blocked in the same sense by coalitions arbitrarily small in measure. This point was made by Schmeidler (1972) for the usual notion of the core and mutatis-mutandis it applies to our case as well.

We concentrate first on Section 3. Given a feasible allocation x we say that $v\in R^l$ is an attainable direction if for some $C\subset I$, $y\in Y$ and $\alpha>0$, we have $\alpha v=y-\int_C x$. The cone of attainable directions is denoted $F_x\subset R^l$. Also, Y_x is the minimal face of Y containing $\int x$ and $L(Y_x)$ the linear space spanned by Y_x.

A simple but important fact is:

Lemma 1. *Given a feasible x, $F_x=L_x+Y=L_x+L(Y_x)+Y$.*

Proof. Let $D=\{\int_C x: C\subset I\}$. Clearly $0\in D$ and, by Lyapunov's theorem, D is compact and convex, its linearity space being precisely L_x. By definition F_x is the cone spanned by $Y-D$. Therefore $F_x\subset L_x+Y\subset L_x+L(Y_x)+Y$. We need to show $L_x+L(Y_x)+Y\subset F_x$, or simply $L_x+L(Y_x)\subset F_x$. Let $\hat{D}$ be the cone spanned by D. We shall show $L_x+L(Y_x)\subset Y_x-\hat{D}$. Note that $Y_x-\hat{D}$ is a convex cone contained in $L_x+L(Y_x)$. So, we can take $p\in L_x+L(Y_x)$ such that $p\cdot v\geqq 0$ for all $v\in Y_x-\hat{D}$. If we show that this imples $p=0$ we are done. Since $\int x\in Y_x$ and $-\int x\in -\hat{D}$ we have $p\cdot\int x=0$. Now let $v\in -\hat{D}$, $v\neq 0$, i.e. $\alpha v=-\int_C x$ for some $\alpha>0$ and $C\subset I$. From $-\int_C x-\int_{I\setminus C}x=-\int x$ we obtain $\alpha p\cdot v+p\cdot(-\int_{I\setminus C}x)=0$. Since $-\int_{I\setminus C}x\in -\hat{D}$ we have $p\cdot v=0$. Let $v\in Y_x$, then (remember Y_x is the face of minimal dimension containing $\int x$), we can find $v'\in Y_x$ such that $v+v'=\alpha\int x$ for some $\alpha>0$. But this gives $p\cdot v=0$ once again. Therefore $p\cdot v=0$ for all $v\in Y_x-\hat{D}$. But $p\in L_x+L(Y_x)$ which means that $p=\sum_{j=1}^k \alpha_j v_j$ for some $\alpha_j\in R$ and $v_j\in Y_x-\hat{D}$. So, $\|p\|^2=0$, i.e. $p=0$. ||

One implication of Lemma 1 is that, unless $x\equiv 0$, the cone of attainable directions is never pointed, i.e. it always contains some non-trivial linear subspace.

For any $C\subset I$ let $F_{x|I\setminus C}$ have the obvious meaning as the cone of attainable directions when x is restricted to $I\setminus C$. If $C_n\subset I$ is a sequence $C_{n+1}\subset C_n$ with $\lambda(C_n)\to 0$ then $L_{x|I\setminus C_n}$ is an increasing collection of subspaces contained in L_x and, because $\lambda(C_n)\to 0$, converging to L_x. Hence, there should exist N such that $L_{x|I\setminus C_n}=L_x$ for $n>N$. By Lemma 1 this means that $F_{x|I\setminus C_n}=F_x$ for $n>N$. Rephrasing: if C is sufficiently small in measure $F_{x|I\setminus C}=F_x$.

We conclude that it suffices that we prove Propositions 1 to 4 with the following modifications is the definitions:

Definition 1′. The feasible allocation x is arbitrage-free if

$$F_x\cap R^l_{++}=\emptyset.$$

Definition 2′. Identical to Definition 2 except that condition (iii) is dropped and the requirement "$\int_C x'$ is an attainable direction" is added.

Let us now go to Section 4 and proceed similarly. Given a feasible assignment z we say that $v\in R^l$ is an attainable direction if there is $\alpha>0$, $y\in Y$ and $C_j\subset I$ for all $2\leqq j\leqq l$ such that

$$\alpha v=y-\left(\textstyle\sum_{j=2}^l \displaystyle\int_{C_j} z^{j1}, \int_{C_2} z^{22},\ldots,\int_{C_l} z^{l2}\right).$$

The cone of attainable directions is denoted F_z. Let $L_z=\sum_{j\in J}L_{\hat{z}^j}\subset R^l$ where we should remember that z^j is looked at as an assignment with range in R^{2J} by putting $z^{j'}=0$ for $j'\neq j$. Clearly, $L_{\hat{z}}\subset L_{\hat{z}}$ and $\dim L_z\cap R^{J(z)}=\#J(z)$.
We have:

Lemma 2. $F_z=L_z+Y=L_z+L(Y_{\hat{z}})+Y.$

Proof. Identical to Lemma 1. Of course, now $D=\sum_{j\in J}\{\int_C \hat{z}^j: C\subset I\}$. As in Lemma 1, D has the crucial property that if $v\in D$ then $v+v'\in Y_{\hat{z}}$ for some $v'\in D$. As there, in going from v to v' we only need to systematically replace C_j by $I\setminus C_j$. ||

Again, and with the obvious meaning of the symbols, if C is sufficiently small in measure, then $F_{z|I\setminus C}=F_z$. Therefore, we can simply state part (iii) of Definition 4 as: $\int_C \hat{z}'\in F_z$. Suppose that for z given we have x': $C\to R^l$ with $\lambda(C)>0$ and $\int x'\in F_z$. There are many ways that we can go from x' to z': $C\to R^l$ in such a way that $\hat{z}'=x'$.

For example, let $C_j \subset I$, $j \in J$, and $y \in Y$ be such that

$$\int_C x'^1 = y^1 - \sum_{j\in J}\int_{C_j} z^{j1} \quad \text{and} \quad \int_C x'^j = y^j - \int_{C_j} z^{j2}$$

for $2 \leqq j \leqq l$. Then put $z''^{j2}(t) = x''^j(t)$ and

$$z''^{j1}(t) = \frac{1}{l-1}\left[x'^1(t) - \int_C x'^1 - y^1\right] - \int_{C_j} z^{j1}.$$

Summing up: it suffices that we prove Propositions 5 to 8 with the following modifications in the definitions:

Definition 3′. The feasible assignment z is arbitrage-free if

$$F_z \cap \{\lambda e^1 : \lambda \in R_{++}\} = \emptyset.$$

Definition 4′. Identical to Definition 4 except that "z'" is replaced by $x' : C \to R^l$, condition (iii) is dropped, and the requirement "$\int_C x' \in F_x$" is added.

Note that Definition 4′ has become very similar to Definition 2′.

5.2. *Proofs.*

Proofs of Propositions 1 *and* 5. To prove Proposition 1 let x be the given feasible allocation and suppose that $F_x \cap R^l_{++} = \emptyset$. By the separating hyperplane theorem there is $p \neq 0$ such that $p \cdot v \leqq 0$ for all $v \in F_x$ and $p \cdot v > 0$ for all $v \in R^l_{++}$. Of course, this means that $p \geqq 0$. It is clear then that x is price compatible with p since: (i) $L_x \subset F_x$ is a linear space, hence $p \cdot L_x = 0$, and (ii) $Y \subset F_x$, $p \cdot (\int x) = 0$.

To prove Proposition 5, proceed in the same manner with F_x replaced by F_z. By the separating hyperplane theorem we get a p with $p^1 > 0$. Price compatibility follows as above. Note in particular that for all j $L_{\hat{z}_j} \subset F_z$ implying $p \cdot L_{z_j} = 0$ and the uniqueness of p^j/p^1 for all $j \in J(z)$. ||

Proofs of Propositions 2 *and* 6. We deal first with Proposition 2. Let x be a feasible allocation and suppose that it is not semi-Walrasian, i.e. there is $C \subset I$ with $\lambda(C) > 0$ such that for a.e. $t \in C$, $x'(t) \succ_t x(t)$ for some $x'(t) \in X_t \cap (L_x + Y)$. Because there is only a finite number of characteristics in $\mathscr{A}$ we can assume that all $t \in C$ are of the same type. Also, $x' : C \to R^l$ can be chosen measurable. Since $\int_C x' \in L_x + Y = F_x$ we conclude that x is 1-blocked. Contradiction.

The proof of Proposition 6 is identical once we replace F_x by F_z. Since $\dim L_z \cap R^{J(z)} = \# J(z)$ a (up to normalization) unique compatible price vector is determined in the active markets. For the positivity of the price of money note that if $L_{\hat{z}^j} \subset R^2$ is 1-dimensional and $j \in J(z)$ then $L_{\hat{z}^j}$ cannot coincide with the money axis. ||

Proofs of Propositions 3 *and* 6. We deal first with Proposition 3.

Let $x : I \to R^l$ be the given feasible allocation.

Denote by $N \subset R^l$ the linear space which is the orthogonal complement of $L_x + L(Y_x)$. The perpendicular projection map on N is denoted $\Pi : R^l \to N$. Of course, $\dim N = l\text{-}\dim(L_x + Y_x)$.

Consider the function $\xi : I \to \mathscr{A} \times R^l$ defined by $\xi(t) = (a_t, x(t))$ and let $\mu = \lambda \circ \xi^{-1}$ be the distribution on $\mathscr{A} \times R^l$ induced by ξ. Put $V = \text{co}\left(\bigcup_{(a,x)\in \text{supp}(\mu)} \{\Pi x' : x' \in X_a, x' \succ_a x\}\right)$ and $\hat{Y} = \Pi(Y)$. Since Π is linear, $\hat{Y}$ is a convex cone. If $\hat{Y} \cap \text{Int}_N V = \emptyset$ then x is Walrasian. Indeed, let $p \in N$ and $s \in R$ be such that $p \cdot v \geqq s$ for $v \in V$ and $p \cdot v \leqq s$ for $v \in \hat{Y}$. Because $\hat{Y}$ is a cone we can take $s = 0$. Let $y \in Y$. Then $p \cdot y = p \cdot \Pi y \leqq 0$. On the other hand $\int \underset{\sim}{x} \in L_{\underset{\sim}{x}} + L(Y_{\underset{\sim}{x}})$. So, $p \cdot \int \underset{\sim}{x} = 0$. Hence, profit maximization holds. Take any $(a, x) \in \text{supp}(\mu)$. If $x' \in X_a$ and $x' \succ x$ then $\Pi x' \in V$ which yields $p \cdot x' = p \cdot \Pi x' \geqq 0$. Hence, for a.e. $t \in I$, $x \succ_t \underset{\sim}{x}(t)$ implies $p \cdot x \geqq 0$. Because $0 \in \text{Int}\, X_t$ cost minimization yields utility maximization and so, p is a Walrasian price vector.

Let us assume by way of contradiction that $\hat{Y} \cap \text{Int}_N V \neq \emptyset$. Then we can find finite collections $\{(a_j, x_j)\}_{j=1}^k \subset \text{supp}(\mu)$, $\{v_j\}_{j=1}^k \subset R^l$, and $y \in Y$ such that $v_j \succ_{a_j} X_j$ and $\Pi y \in \text{Int}_N \text{co}\{\Pi v_1, \ldots, \Pi v_k\}$, i.e. $\Pi y = \sum_{j=1}^k \alpha_j \Pi v_j$ for some $\alpha_j \geqq 0$, $\sum_{j=1}^k \alpha_j = 1$. By Caratheodory's theorem we can take $k = \dim N + 1$, but in fact we can do better. Since $\underset{\sim}{x}$ cannot be 0-blocked, it is price compatible. Let $p > 0$ be an admissible price. We have that $-p \in Y$ and $-p \in N$. Therefore, $y - \beta p \in Y$ and $\Pi(y - \beta p) = \Pi y - \beta p$ for all $\beta \geqq 0$. Henceforth, by replacing y by some $y - \beta p$ we can take $k = \dim N$ in the above expression.

Let $\{I_j\}_{j=1}^k$ by a collection of measurable disjoint sets $I_j \subset I$ such that, for each j: (i) $\lambda(I_j) > 0$, (ii) $a(t) = a(t')$ for $t, t' \in I_j$, and (iii) $v_j \succ_t \underset{\sim}{x}(t)$ for a.e. $t \in I_j$. Such a collection is easily seen to exist. Put $v = \max_j \lambda(I_j)$. By Lyapunov's theorem we can pick $C_j \subset I_j$ with $\lambda(C_j) = \alpha_j r$. Put $C = \bigcup_{j=1}^k C_j$ and define $\underset{\sim}{x}': C \to R^l$ by $\underset{\sim}{x}'(t) = v_j$ if $t \in C_j$. Then $\int_C \underset{\sim}{x}' = r \sum_{j=1}^k \alpha_j v_j$ and so, $r\Pi y = \Pi(\int_C \underset{\sim}{x}')$ or, in other words, $\int_C \underset{\sim}{x}' = ry + w$ for some $w \in L_{\underset{\sim}{x}} + L(Y_{\underset{\sim}{x}})$ which, of course, means that $\int_C \underset{\sim}{x}' \in F_{\underset{\sim}{x}}$ (Lemma 1) and proves $\underset{\sim}{x}$ can be dim N-blocked. Contradiction.

Proposition 7, which corresponds to case (ii) of Proposition 3, is proved in exactly the same manner. One only has to replace throughout $L_{\underset{\sim}{x}}$ with $L_{\underset{\sim}{z}}$ and $\underset{\sim}{x}$ with $\underset{\sim}{z}$ as the second argument of the function ξ. Since the Walrasian price p that is obtained is perpendicular to $L_{\underset{\sim}{z}}$ and $L_{\xi^j} \subset L_{\underset{\sim}{z}}$ for all $\underset{\sim}{z}$ there is price compatibility in the sense of Section 4. The positivity of the price of money is established as in the proof of Proposition 6 using the fact that $J(\underset{\sim}{z}) \neq \emptyset$. ||

Proof of Proposition 4. Let $\underset{\sim}{x}$ be the feasible allocation and $C \subset I$ the set of traders given in the statement of the Proposition. Define the (measurable) function $\xi: I \to \mathscr{A} \times R^l$ by $\xi(t) = (a_t, \underset{\sim}{x}(t))$ and let μ be the induced measure on $\mathscr{A} \times R^l$. Similarly, let μ_C be the measure induced by $\xi | C$.

Pick any $(\bar{a}, \bar{x}) \in \text{supp}(\mu_C)$. Note that $\bar{x} \in F_{\underset{\sim}{x}}$ and let p be as in the statement of the Proposition. We will show that $\underset{\sim}{x}$ is Walrasian with respect to this p. Suppose, by way of contradiction, that it is not. We argue first that there are $v \in R^l$, $y \in Y$ and $E \subset I$ with the properties that $\bar{x} - \lambda(E)v + y \succ_{\bar{a}} \bar{x}$, $a(t) = a(t')$ for all $t, t' \in E$, and $v \succ_t \underset{\sim}{x}(t)$ for a.e. $t \in E$. Indeed, we have that either (i) there is some $(a, x) \in \text{supp}(\mu)$ and $v \in X_a$ such that $p \cdot v < 0$ and $v \succ_a x$, or (ii) $p \cdot y > 0$ for some $y \in Y$. In case (i), $p(-v) > 0$ and so, for some $\bar{\alpha} > 0$ and $y \in Y$, $\bar{x} + \alpha(-v) + y \succ_{\bar{a}} \bar{x}$ for $\alpha < \bar{\alpha}$. Therefore, it suffices to let $E \subset I$ be such that $0 < \lambda(E) < \bar{\alpha}$, $a(t) = a(t') = a$ for all $t, t' \in E$ and $v \succ_t \underset{\sim}{x}(t)$ for all $t \in E$. In case (ii), take $E = \emptyset$. Since $p \cdot y > 0$ we can find $\alpha > 0$ and $y' \in Y$ such that $\bar{x} + \alpha y + y' \succ_{\bar{a}} \bar{x}$. Because Y is a convex cone this suffices. Furthermore, E can be chosen sufficiently small for $(\bar{a}, \bar{x}) \in \text{supp}(\mu_{I \setminus E})$ to be true. This means that we can choose $H \subset C$ such that $H \cap E = \emptyset$, and, for all $t \in H$, $a(t) = \bar{a}$ and $\bar{x} - \lambda(E)v + y \succ_t \underset{\sim}{x}(t)$. Pick $E' \subset E$ such that $\lambda(E') = \lambda(H)\lambda(E)$. Finally, define $\underset{\sim}{x}': H \to R^l$ by $\underset{\sim}{x}'(t) = \bar{x} - \lambda(E)v + y$ and $\underset{\sim}{x}'': E' \to R^l$ by $\underset{\sim}{x}'' = v$. Then

$$\int_H \underset{\sim}{x}' + \int_{E'} \underset{\sim}{x}'' = \lambda(H)\bar{x} - \lambda(H)\lambda(E)v + \lambda(H)y + \lambda(H)\lambda(E)v$$

$$= \lambda(H)(\bar{x} + y) \in F_{\underset{\sim}{x}}$$

which implies that $\underset{\sim}{x}$ is 2-blocked and concludes the proof. ||

Proof of Proposition 8. This will be very simple. Say that $2 \in J(\underset{\sim}{z})$. For every $j \in J \setminus \{2\}$ consider the restriction of $\mathscr{E}$ to the markets 2 and j. This is a 3-commodity economy with at least one active market. Since the assignment is not 2-blocked, there are, by Proposition 7, Walrasian prices p^2, p^j. So, using the hypothesis of the Proposition, we have, for a.e. $t \in I$ and $h = 2, j$, $\partial_h u_t(\hat{z}(t))/\partial_1 u_t(\hat{z}(t)) \leqq p_h$ with equality if $\hat{z}^h(t) > -\omega_t^h$. Because, for $h = 2$, equality must hold for a set of t's of positive measure, we have that p^2 is in fact independent of j. So, if we let the price vector $\bar{p}$ be given by $\bar{p}^2 = p^2$ and $\bar{p}^j = p^j$ for all $j \in J \backslash \{2\}$ we get: (i) z is price compatible with $\bar{p}$, and (ii) for a.e. $t \in I$, $\hat{z}(t)$ is u_t maximal on $\{x \in R : \bar{p} \cdot x \leqq 0\} \cap X_t$. ||

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