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Walrasian Equilibria as Limits of Noncooperative Equilibria. Part I: Mixed Strategies*

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We examine the connection between Walrasian equilibria of a limit economy (with infinitesimal firms) and the noncooperative (Cournot) equilibria of approximating finite economies (with significant firms). Following earlier work of Novshek and Sonnenschein we allow for set-up cost and permit a minimal form of mixed strategies. We depart from them by requiring that the aggregate production set exhibits some degree (however small) of decreasing returns. Contrasting with their results, it is shown that a (regular) Walrasian equilibrium of a limit economy can always be approximated by a sequence of noncooperative equilibria for the tail of the approximating (finite) economies. Thus, there is a surprising qualitative discontinuity when one passes from the Novshek–Sonnenschein case of aggregate constant returns to scale of the decreasing returns case of this paper. *Journal of Economic Literature* Classification Numbers: 021, 022.

I. INTRODUCTION

In this paper we present a result that connects the Walrasian equilibria of a limit economy, in which firms are infinitesimal relative to demand, with the (quantity setting) noncooperative equilibria of approximating finite economies, in which firms are significant relative to demand. Such connections are explored to help us understand the significance of Walrasian equilibrium. The basic reference for this paper is Novshek and Sonnenschein [11] (see also [12]). As in their work, we allow for set-up costs and so forth, for nonconvex production sets. However, we depart from their analysis by assuming that constant returns to scale do not hold in the aggregate, i.e., the employment of a production set is associated with the use of a scarce factor.

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Still, the number of active firms in a market is determined by the possibility of profitable activity.

Recall that in [11], Novshek and Sonnenschein appealed to a minimal form of mixed strategies in order to prove the existence of equilibrium in the tail of the sequence of approximating economies. In addition, they showed that a condition on demand of the downward sloping variety (called DSD) was necessary and sufficient for a Walrasian equilibrium of the limit economy to be the limits of noncooperative equilibria of the approximating economy. The result of this paper estabishes that if (a) (p^*, y^*) is a nondegenerate Walrasian equilibrium of a limit economy ξ, and (b) Walrasian supply is single valued in a neighborhood of p^*, so that constant returns to scale in the aggregate do not hold, then (c) for every sequence of approximating economies, there is a sequence of noncooperative equilibria for the approximating economies that converges to (p^*, y^*). In order words, when returns to scale are not constant in the aggregate, the previous characterization in terms of the DSD condition fails. In fact, when the aggregate supply function is locally single valued, the limits of noncooperative equilibria include, in the regular situation, the entire set of Walrasian equilibria. This means that there is a surprising qualitative discontinuity in the connection between Walrasian equilibrium and noncooperative equilibrium (with mixed strategies) when one passes to the case of aggregate constant returns to scale.

Our conclusion from the result of this paper is that, as long as a minimal amount of mixed strategies is allowed, a positive solution to the approximability problem can be obtained under quite general hypotheses, and does not depend on the DSD condition, which, in a general-equilibrium context, is restrictive. The use of mixed strategies should be assessed in view of the following three facts: (i) the approximability problem is an existence question, (ii) the production sets of our model are nonconvex, and (iii) at the equilibrium, only a (vanishingly) small fraction of the total number of firms uses mixed strategies. In the simpler purely convex case, which is also formally covered by our result, no mixed strategies need to be considered. The convex case has been studied by Roberts [14]; see also Mas–Colell [10] for a state-of-the-art survey. Finally, we should mention that in the companion paper Novshek and Sonnenschein [13] have succeeded in establishing an intimate connection between the DSD condition and the positive resolution of the approximability problem in pure strategies.

This paper investigates conditions under which the set of Walrasian equilibria is contained in the set of limits of noncooperative equilibria. We do not discuss here the converse problem, i.e., the nature of the limits of noncooperative equilibria. This has already been extensively investigated (Novshek and Sonnenschein [11]; see also Gabzewicz and Vial [3], Hart [5, 7], Roberts [14], Mas–Colell [10]).

II. The Model

1. *Commodity Space*

The commodity space is R^l. Price vectors are $p \in R^l_+$.

2. *Individual Production Sets*

An individual production set $\hat{Y} \subset R^l$ is of the form $\hat{Y} = Y \cup \{0\}$, where

(i) There is a compact cube $K \subset R^l$ such that $Y \subset K$;

(ii) Y is closed and convex;

(iii) $Y \cap R^l_+ \subset \{0\}$;

(iv) The correspondence defined on R^l_{++} by

$$\xi_Y(p) = \{y \in Y: p \cdot y \geqslant p \cdot y' \text{ for all } y' \in Y\}$$

is a locally Lipschitzian function with Lipschitz constant $c(p)$ at p.

As usual, negative (resp. positive) entries of y denote inputs (resp. outputs). See Fig. 1.

Remarks. (i) Condition (iv) is a way to require, without having to assume smoothness, that the efficiency frontier of Y have some curvature. It eliminates (at the first-order level) the presence of flat segments. It holds if Y is, in the relevant region, strongly convex. See, for example, Vial [15] for this mild strengthening of the concept of strict convexity.

(ii) Except for closedness and boundedness above, hypotheses (i) and (ii) are in the nature of simplifications.

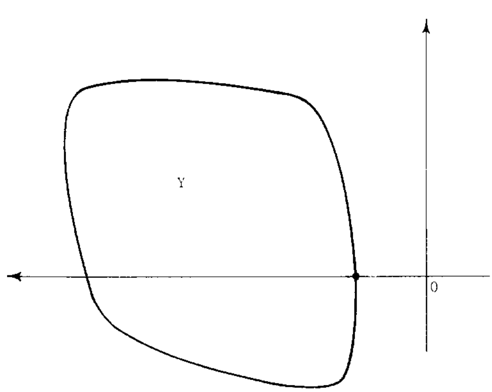

FIG. 1. Individual production set.

3. *Space of Production Sets*

Let Υ be the space of sets $Y \subset R^l$ which satisfy hypotheses (i)–(iv) of the previous section with respect to a priori fixed K and $c(p)$, $p \in R^l_{++}$. With the topology of the closed convergence (see Hildenbrand [9, pp. 18]) it becomes a compact, metric space.

4. *The Continuum Production Sector*

The set of firm names is the interval $I = [0,1]$ with Lebesgue measure λ. The production sector is described by a (Borelian) function $f: [0, 1] \to \Upsilon$. Each $\beta \in [0, 1]$ represents a firm with production set $\hat{f}(\beta) = f(\beta) \cup \{0\}$.

A *production* is a (Borelian) function $\tilde{y}: I \to R^l$. A production $\tilde{y}$ is *feasible* if $\tilde{y}(\beta) \in \hat{f}(\beta)$ for a.e. $\beta \in [0, 1]$. The aggregate production set $\mathbf{Y}$ is $\mathbf{Y} = \{\int \tilde{y}: \tilde{y}$ is a feasible production$\}$ which, by Richter's theorem on the integral of a correspondence (see Hildenbrand [9, pp. 62]) we know is a closed, convex set contained in the convex hull of K and $\{0\}$. Vectors in $\mathbf{Y}$ are called *feasible* aggregate productions.

The *aggregate supply correspondence* $S: R_{++} \to R^l$ is defined by

$$S(p) = \{y \in \mathbf{Y}: p \cdot y \geqslant p \cdot y' \text{ for all } y' \in \mathbf{Y}\}.$$

Remark. Observe that we take the total mass of the production sector to be bounded (i.e., $\lambda(I) = 1 < \infty$). Therefore, perfect free entry can be approximated, but not completely included, by our model. In other words, in our model, $S(p)$ can be a large set, but it is always bounded.

5. *The Demand Sector*

The demand sector is specified by a set $J \subset R^l_+$ and a correspondence $P: J \to R^l$. The interpretation is that J is the set of aggregate input–output vectors for which there is a nonempty set of market clearing prices $P(y)$. Thus, P is a general equilibrium version of the notion of indirect demand. We assume that P is given to us already normalized, i.e., if $p, p' \in P(y)$, then $p \neq \alpha p'$ for all $\alpha \in R$. Vectors in J are called *attainable* aggregate productions.

Suppose we had a consumption sector specified in the usual way by preferences, endowments, and shareholdings. Assume for simplicity that there is no limited responsibility and that consumers have the same share of profits (or losses) in all firms (although this share may vary across consumers). Then aggregate excess demand E depends only on prices p and aggregate profits π, i.e., we have a correspondence $E(p, \pi)$. Given p and aggregate production y, aggregate profits are $p \cdot y$. We can then view the graph of P as the collection of pairs (p, y) that satisfy the equation $y \in E(p, p \cdot y)$.

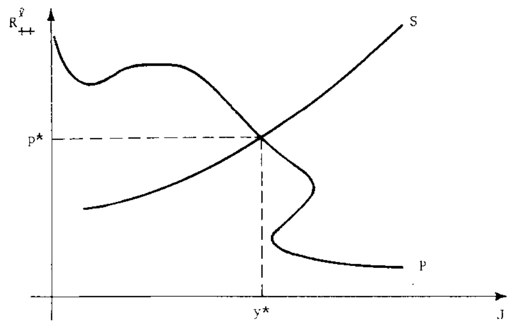

FIG. 2. Aggregate demand and supply graphs.

Remark. It would be natural to impose boundary conditions on P, e.g., that prices be uniformly bounded away from zero, but it in view of the strong hypothesis on the production sets (in particular, assumption (i)) we shall not need to do so.

6. *The Continuum of Firms Economy*

The continuum limit economy is specified by $\mathscr{E} = (f, P)$. This is to be understood as a normalized presentation. The two components f and P should not be thought of as independent. We have taken the production sector to have mass 1 and, implicitly, scaled the demand sector relative to it. In other words, aggregate quantities are being measured in per firm terms.

7. *The Walrasian Equilibrium of the Continuum Economy*

DEFINITION. $(p^*, y^*) \in R^l_{++} \times J$ is a Walrasian equilibrium for $\mathscr{E}$ if

$$y^* \in S(p^*) \qquad \text{and} \qquad p^* \in P(y^*).$$

In other words, (p^*, y^*) is an intersection point of the demand and supply schedules. Of course, y^* is both feasible and attainable. See Fig. 2.

8. *Regular Walrasian Equilibrium*

Our definition of regular Walrasian equilibrium will consist of two parts. First, we will require that, locally, both supply and indirect demand be (smooth) functions. Second, the intersection of demand and supply should be nondegenerate.

DEFINITION. The Walrasian equilibrium (p^*, y^*) of the economy $\mathscr{E}$ is regular if (see Fig. 3):

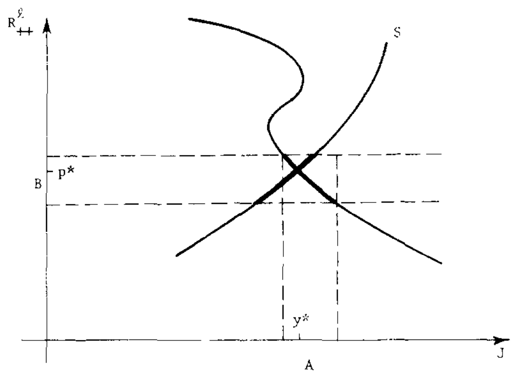

FIG. 3. Regular equilibrium.

(a) There are open (relative to R^l) sets $A \subset J$, $B \subset R^l$ with $y^* \in A$, $p^* \in B$ such that

(i) the graph of P restricted to $A \times B$ is the graph of a C^2 function $q: A \to B$;

(ii) the supply correspondence S is a C^1 function on B; and

(b) The C^1 function $q(S(p)) - p$, defined on a neighborhood of p^*, has a full rank derivative at $p = p^*$, i.e., rank $(Dq(y^*) \cdot DS(p^*) - I) = l$.

Remarks. (i) A *selection* from P is a function $g: P \to R^l$ such that $g(y) \in P(y)$ for all $y \in J$. Condition (i) of part (a) implies that any selection g which satisfies $g(y^*) = p^*$ and is continuous in a neighborhood of y^* will coincide locally with the function q.

(ii) The expression $q(S(p)) - p$ gives the difference between the demand and supply prices at $S(p)$, or, in other words, the difference between a price vector and the demand price vector of the induced supply. We could as well have looked at $S(q(y)) - y$. Note that rank $(Dq(y^*) \cdot DS(p^*) - I) = l$ if and only if rank $(DS(p^*) \cdot Dq(y^*) - I) = l$.

(iii) We shall not carry out the genericity analysis of our regularity definition. Except for the purely convex case, condition (ii) of part (a) requires some dispersion of individual production sets or, informally, some continuous gradation of efficiency scales.

Part (b) is equivalent to the regularity condition of Debreu [1] and Dierker [2] (i.e., maximal rank at equilibrium of the derivative of excess demand), and is well known to be generic. Condition (i) of part (a) is specific to the present theory, and its genericity in an appropriate sense can also be established.

(iv) The smoothness hypotheses are convenient, but not essential for our theory. What is crucial is that P be, locally, a continuous function, that S be locally an upper hemicontinuous compact-valued correspondence (which in our model is always guaranteed to be), and that (p^*, y^*) constitute a so-called essential intersection of the demand and supply graph.

9. *Finite Economies*

Let $I_n = \{1, ..., n\}$. A production sector is defined as in Section 4, except that I is replaced by I_n, i.e., we have a function $f: I_n \to Y$, and production $\tilde{y}: I_n \to R^l$. Given a production $\tilde{y}$ the *aggregate production vector* is $(1/n) \sum_{j \in I_n} \tilde{y}(j)$. Thus, as indicated before, aggregate quantities are averages per firm. The demand sector is as in Section 5. A vector $y \in R^l$ is j-feasible if $y \in \hat{f}(j)$, and attainable if $y \in J$. A finite economy $\mathscr{E}$ is specified by the pair (f, P).

10. *Pure Cournot Equilibrium of a Finite Economy*

DEFINITION. The production $\tilde{y}: I_n \to R^l$ is a Pure Cournot equilibrium production for the economy $\mathscr{E} = (f, P)$, $f: I_n \to Y$, with respect to the selection $g: J \to R^l$ from P if, for all $j \in I_n$, $\tilde{y}(j)$ maximizes $g((1/n)[\sum_{h \neq j} \tilde{y}(h) + y]) \cdot y$ subject to y being j-feasible and $(1/n)[\sum_{h \neq j} \tilde{y}(h) + y]$ attainable.

Remarks. (i) The selection g represents the prediction of which particular equilibrium price vector will prevail at each $y \in J$. It is a priori given. In one form or another, such a prediction device is needed if profits have to be evaluated at hypothetical aggregate productions.

(ii) We refer to Novshek–Sonnenschein [11], Hart [6], and Mas–Colell [10] for a discussion of the profit motive in this or similar contexts.

11. *Mixed Cournot Equilibrium*

Suppose a finite economy $\mathscr{E} = (f, P)$, $f: I_n \to Y$ is given. We will now allow the production plans of individual firms $j \in I_n$ to be random. To economize on notation, the symbol y will now stand for a random variable with values in R^l, and by j-feasible plan we will mean a random variable y with values in $\hat{f}(j)$. A production $\tilde{y}$ will be a collection of *independent* random variables $\tilde{y}(j)$, $j \in I_n$. An aggregate production y, also a random variable, is attainable if it takes values in J. The expectation operator is denoted E.

DEFINITION. The production $\tilde{y}: I_n \to R^l$ is a Mixed Cournot (or simply, Cournot) Equilibrium Production for the economy $\mathscr{E} = (f, P), f: I_n \to Y$, with respect to the selection $g: J \to R^l$ from P, if, for all $j \in I_n$, $\tilde{y}(j)$ maximizes

$Eg((1/n)[\sum_{h \neq j} \underset{\sim}{y}(h) + y]) \cdot y$, subject to y being j-feasible and $(1/n)[\sum_{h \neq j} \underset{\sim}{y}(h) + y]$ being attainable.

Remarks. (i) We are implicitly assuming the risk neutrality of firms. This is not crucial. General utility maximization would do. Also, and this cannot be made precise at this point, it will be a consequence of our results that, in the economies with which we will be concerned, there is very little randomness to worry about.

(ii) The concept of mixed strategies admits of flexible interpretations (see Harsanyi [8]).

12. *Sequences of Finite Economies*

Let $\mathscr{E}_n = (f_n, P)$, $f_n: I_n \to Y$, be a sequence of finite economies, and $\mathscr{E} = (f, P)$ a continuous economy. Note that the demand sector is the same for all economies. Let ν_n be the counting measure induced by f_n on Y, i.e., for each Borel $U \subset Y$, we put $\nu_n(B) = (1/n) \#\{j: f_n(j) \in U\}$. Finally, let ν be the distribution induced by f, i.e., $\nu = \lambda \cdot f^{-1}$. For the concepts of weak convergence of measures, support of a measure, and closed convergence of sets, see Hildenbrand [9, Part I].

We say that $\mathscr{E}_n \to \mathscr{E}$ if: (i) $\nu_n \to \nu$ in the weak convergence for measures, and (ii) $\text{supp}(\nu_n) \to \text{supp}(\nu)$ in closed convergence.

Let y be a nonrandom aggregate production, and y_n a sequence of random aggregate productions. By $y_n \to y$ we mean uniform convergence, i.e., given $\varepsilon > 0$ there is N such that for $n > N$, $\| y_n - y \| < \varepsilon$ with probability 1.

Remark. The procedure we use to generate a sequence $\mathscr{E}_n$ approximating a limit $\mathscr{E}$ is the simplest consistent with dispersion of production sets at the limit and, therefore, with a well-defined limit supply function. This rules out replication of firms. If we made explicit the population of consumers underlying P, then we could view $\mathscr{E}_n$ as generated from an n-size sampling from Y according to the probability ν (this would guarantee $\nu_n \to \nu$ and $\text{supp}(\nu_n) \to \text{supp}(\nu)$) coupled with an n-size replication of the set of consumers (which would guarantee the invariance of P across n).

III. The Result

THEOREM. *Given the continuum economy $\mathscr{E} = (f, P)$, let (p^*, y^*) be a Regular Walrasian equilibrium, and $g: J \to R^l$ a selection from P which satisfies $g(y^*) = p^*$, as is locally continuous at y^*. Suppose that $\mathscr{E}_n \to \mathscr{E}$. Then there is N, and for each $n > N$ a Mixed Cournot equilibrium production $\underset{\sim}{y}_n$ for $\mathscr{E}_n$ relative to g, such that:*

(i) $(1/n) \sum_{j \in I_n} \underset{\sim}{y}_n(j) \to y^*$, *and*

(ii) $\lim_n (1/n) \#\{j: 0 < \text{prob}(\underset{\sim}{y}_n(j) = 0) < 1\} = 0$ *and* $\underset{\sim}{y}_n(j)$ *is nondegenerate (i.e., nonconstant) if and only if* $0 < \text{prob}(\underset{\sim}{y}_n(j) = 0) < 1$. *Further, if* $\hat{f}_n(j)$ *is convex, then* $\underset{\sim}{y}(j)$ *is degenerate.*

The firms j with $0 < \text{prob}(\underset{\sim}{y}_n(j) = 0) < 1$ are to be interpreted as the marginal firms in the Cournot equilibrium $\underset{\sim}{y}_n$. They necessarily make zero profits.

Remarks. (i) The theorem guarantees that mixed Cournot equilibrium exists for the finites economies in the tail of every approximating sequence $\mathscr{E}_n$. H. Dierker and B. Grodal have recently shown by example that existence for every term of the sequence is not guaranteed.

(ii) The conclusion of the theorem is the same as the corresponding result of Novshek and Sonnenschein [11]. The hypotheses are different. We do not require the Downward Sloping Demand condition, but impose that the aggregate supply function be well defined (at least locally), thus eliminating aggregate constant returns (i.e., strict free entry). In the companion paper, Novshek and Sonnenschein [13] show that the Downward Sloping Demand condition is intimately related to the approximability of Walras equilibria by nonmixed Cournot equilibria.

(iii) Although technical, the proof of conclusion (ii) of the theorem is routine. The conclusion itself is not surprising. Because the only technological nonconvexity involves the entry decision, we conclude that in large economies only marginal firms randomize at equilibrium. That the fraction of marginal firms tends to zero is the logical consequence of the continuous gradation of efficiency scales implicit in the well definiteness of a supply function. This is, incidentally, the key contribution of this hypothesis to our theorem.

(iv) Focusing on the basic existence result (conclusion (i)), it will be useful to discuss a simple example. Let $l = 2$. Commodity 1 (resp. commodity 2) is an input (resp. output). Indirect demand is given by $P(y^1, y^2) = (1, 2y^2)$. Note that the price of the input is fixed at unity, and that the price of the output only depends on the quantity of output. The (upward sloping) indirect demand function for positive output is represented in Fig. 4. All the firms to be considered have production set $\hat{Y} = \{(-1, 1)\} \cup \{(0, 0)\}$. Thus, the only production decision is to produce or not to produce one unit at a cost of one.

In the continuum of firms case, the Walrasian supply correspondence is (fixing $p^1 = 1$): $S(p^2) = (-1, 1)$ if $p^2 \geqslant 1$, $S(p^2) = \{(-\alpha, \alpha): 0 \leqslant \alpha \leqslant 1\}$ if $p^2 = 1$, $S(p^2) = (0, 0)$ if $p^2 < 1$. See Fig. 4. Thus, we have three Walrasian equilibria corresponding to $(p^2, y^2) = (0, 0)$, $(p^2, y^2) = (1, \frac{1}{2})$ and $(p^2, y^2) = (2, 1)$. If we now look at a finite economy with $n > 1$ firms, we see that the

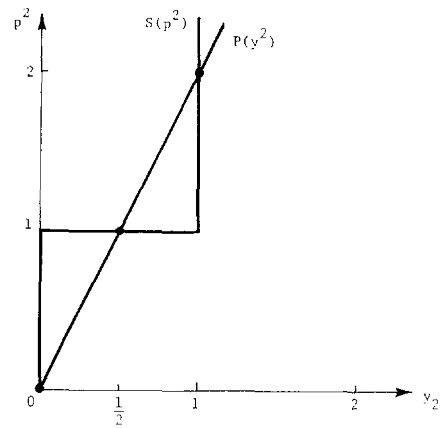

FIG. 4. Equilibria with upward sloping demand.

productions $y(j) = (0, 0)$ for all j and $y(j) = (-1, 1)$ for all j are both Cournot equilibria. Thus, the ("stable") Walrasian equilibria $(p^2, y^2) = (0, 0)$ and $(p^2, y^2) = (2, 1)$ can certainly be approximated by CN equilibria as $n \to \infty$. But what about $(p^2, y^2) = (1, \frac{1}{2})$? Observe that because the demand function is upward sloping, it should be the case that at a Cournot equilibrium, if any firm assigns a positive probability to $(-1, 1)$, then no firm can assign probability 1 to $(0, 0)$. Even more, at a CN equilibrium the probability assigned to $(-1, 1)$ must be the same for all firms. Call this probability π. If $\pi = 0$, or $\pi = 1$, we get the two nonrandom equilibria we already know. Is there an equilibrium with $0 < \pi < 1$? The answer is yes, and direct computation yields $\pi = (n-2)/(2(n-1))$. Of course, as $n \to \infty$ this probability approaches $\frac{1}{2}$ and provides us with the approximating sequence for the Walrasian equilibrium $(p^2, y^2) = (1, \frac{1}{2})$. Thus, all the Walrasian equilibria can be approximated by Cournot equilibria.

The example illustrates well why the conclusion fails in the strict free entry situation of Novshek and Sonnenschein (which in the continuum limit yields an unbounded set for $S(1)$). In their approach, we always have infinitely many potential firms even when the size of an individual firm is significant relative to aggregate production. Thus, at equilibrium there is always some inactive firm. But in the context of our example, this means that every firm is inactive with probability 1, hence $(1, \frac{1}{2})$ cannot be approached.

In the example, conclusion (ii) of the theorem fails. The only way to approach $(1, \frac{1}{2})$ is for every firm to use mixed strategies. This failure is a consequence of $S(p^2)$ not being a function at $p^2 = 1$. For conclusion (i), i.e., existence, what is crucial is the compactness of $S(1)$, but for conclusion (ii) what counts is that $S(1)$ be a singleton set. It can be surmised that the existence part of the theorem can be extended to include general nonconvex production sets. The hypothesis that should be retained is the uniform boundedness above of individual production sets.

(v) Of course, conclusion (i) and the local continuity of g imply that the random equilibrium price $g((1/n)\sum_{j\in I_n}\underset{\sim}{y}_n(j))$ also converges uniformly to p^*. Because $y^* \in S(p^*)$, this should in turn imply by the usual decentralizing principles that, except (possibly) for a vanishingly small fraction of firms and provided n is large enough, every $\underset{\sim}{y}_n(j)$ takes value "near" $\xi_{\tilde{f}_n(j)}(p^*)$. In fact, as the proof of Theorem 1 will make clear, there is uniformity, i.e., $\lim_n \max_j \min_{y\in \tilde{f}_n(j)(p^*)} \|\underset{\sim}{y}_n(j) - y\| = 0$ a.s.

(vi) In the purely convex case, i.e., $0 \in f_n(j)$ for all n and j, the equilibria of the theorem are not mixed. Thus, Theorem 1 includes Roberts' [14] existence results (see also, Mas–Colell [10] for a review of the convex case).

IV. Proof of the Theorem

The idea of the proof is to carry out a fixed point argument in a neighbourhood of the limit equilibrium. This is done in Section 3, which constitutes the heart of the proof. Section 1 contains the required ε and δ preliminaries, while Section 2 appeals to degree theoretic arguments to establish a general fixed point result, which is then applied in Section 3. Section 4 proves concusion (ii).

The proof would be somewhat simpler if we had succeeded at: (i) reducing the existence problem to finding a fixed point in a space of dimension r, where r is independent of n, and/or (ii) obtaining the approximating sequence via an Implicit Function Theorem. We have, however, accomplished neither.

1. *Preliminaries*

Without loss of generality, we can assume that the compact cube K contains the origin. Then $Y \cup \{0\} \subset K$ for all $Y \in \Upsilon$. Let $\mathscr{C}$ be the space of nonempty, closed subsets of K. We endow $\mathscr{C}$ with the topology of the closed convergence, which makes it compact. Generic elements are denoted Z. By convention, $\hat{Z} = Z \cup \{0\}$.

Denote by $C^1(K)$ the space of continuously differentiable functions on K endowed with the $\|\ \|_1$ norm. Generic elements are η. Without risk of confusion, we use the same symbol p for linear functions on R^l and for their gradient vectors. Let $C^1_\alpha(K)$ be the compact subset of $C^1(K)$ formed by the functions which gradients functions admit a Lipschitz constant α.

For given $\alpha \geqslant 0$, define the correspondence $\Psi_\alpha: \mathscr{C} \times C^1_\alpha(K) \times [0,1] \to K$ by $\Psi_\alpha(Z, \eta, \rho) = \{z \in Z: \eta'(z) \geqslant \eta'(z')$ for all $z' \in Z$ and some $\eta' \in C^1_\alpha(K)$ with $\|\eta - \eta'\|_1 \leqslant \rho\}$.

LEMMA 1. *Let $Y \in Y$, and c be a Lipschitz constant for the function ξ_Y on a δ neighborhood of $p \gg 0$. If $\alpha < 1/2c$ and $\|p-\eta\|_1 < \delta$, then $\Psi_\alpha(Y, \eta, 0)$ is a singleton.*

Proof. Suppose not, i.e., there are $y_1, y_2 \in \Psi_\alpha(Y, \eta, 0)$, $y_1 \neq y_2$. Call $p_1 = \partial\eta(y_1)$, $p_2 = \partial\eta(y_2)$. Then $\|p_1 - p_2\| \leqslant \alpha \|y_1 - y_2\| < 1/2c \|y_1 - y_2\|$. Hence, $\|y_1 - y_2\| > c\|p_1 - p_2\|$. However, $y_1 = \xi_Y(p_1)$ and $y_2 = \xi_Y(p_2)$. Also, $\|p-\eta\|_1 < \delta$ implies $\|p - p_1\| < \delta$ and $\|p - p_2\| < \delta$. So, $\|y_1 - y_2\| \leqslant c\|p_1 - p_2\|$ and we have a contradiction. ∎

LEMMA 2. *The correspondence Ψ_α, $\alpha \geqslant 0$, is upper hemicontinuous.*

Proof. Let $z_n \in \Psi_\alpha(Z_n, \eta_n, \rho_n)$, $Z_n \to Z$, $\eta_n \to \eta$, $\rho_n \to \rho$, and $z_n \to z$. For each n, pick $\eta'_n \in C^1_\alpha(K)$ such that $\|\eta_n - \eta'_n\| \leqslant \rho_n$ and $\eta'_n(z_n) \geqslant \eta'_n(z'_n)$ for all $z'_n \in Z_n$. Because $C^1_\alpha(K)$ is compact, we can assume that $\eta'_n \to \eta' \in C^1_\alpha(K)$. Of course, $\|\eta' - \eta\|_1 \leqslant \rho$. Consider now any $z' \in Z$. Since $Z_n \to Z$, there is $z'_n \to Z'$ such that $z'_n \in Z_n$. So, $\eta'_n(z_n) \geqslant \eta'_n(z_n) \geqslant \eta'_n(z'_n)$ which, letting $n \to \infty$, yields $\eta'(z) \geqslant \eta'(z')$. Therefore, $z \in \Psi_\alpha(Z, \eta, \rho)$. ∎

Now let $\mathscr{E}_n \to \mathscr{E}$ be as in the statement of the theorem.

LEMMA 3. *Let $\bar{B} \subset R^l_{++}$ be a compact price region where the supply correspondence S of the continuum economy is single-valued. Given $\varepsilon > 0$, there is N and $\delta > 0$ such that if $n > N$ and, for all $j \in I_n$, $y(j) \in \Psi_1(\hat{f}_n(j), p, \delta)$, $p \in \bar{B}$, then $\|S(p) - 1/n \sum_{j \in I_n} y(j)\| < \varepsilon$.*

Proof. Let ν_n, ν be the distributions on Y induced by $\mathscr{E}_n$, $\mathscr{E}$. By convention $\nu_0 = \nu$. The convex hull of a set $T \subset R^l$ is denoted co T. Observe that the values of Ψ_1 are always subsets of K.

For this proof, we shall make extensive use of the theory of the integral of a correspondence (see Hildenbrand [9, D. II, p. 53]). For $n \in \{1, ..., \infty\}$, $p \in \bar{B}$ and $\rho \in [0, 1]$, define $S(p, 1/n, \rho) = \int \text{co}\, \Psi_1(Z, p, \rho)\, d\nu_n$. Note that $S(p, 0, 0) = S(p)$. Therefore, what we want to establish is the existence of δ such that if $1/n < \delta$ and $\rho < \delta$, then $\|y - S(p, 0, 0)\| < \varepsilon$ for all $y \in S(p, 1/n, \rho)$ and $p \in \bar{B}$. Because $\bar{B}$ is compact, this is implied by the following continuity property: if $p_n \to p$, $1/n \to 0$, and $\rho_n \to 0$, then $S(p_n, 1/n, \rho_n) \to S(p, 0, 0)$.

By the Skorohod Theorem (Hildenbrand [9, p. 50]), there are measurable functions h_n, h: $[0, 1] \to Y$ such that $\nu_n = \lambda \circ h_n^{-1}$, $\nu = \lambda \circ h^{-1}$, and $h_n \to h$ a.e. We can change variables and put $S(p_n, 1/n, \rho_n) = \int \Psi_1(\hat{h}_n(t), p_n, \rho_n)\, dt$, $S(p, 0, 0) = \int \Psi_1(\hat{h}(t), p, 0)\, dt$ (Hildenbrand [9, Theorem 5, p. 67, and Theorem 4, p. 64]). Therefore, $S(p_n, 1/n, \rho_n) \to S(p, 0, 0)$ follows from Hildenbrand [9, Theorem 6, p. 68]). ∎

Let (p^*, y^*) be the regular Walrasian equilibrium for $\mathscr{E}$ under consideration. Let $y^* \in A$, $p^* \in B$ be neighbourhoods of y^*, p^*, respectively, such that: (i) there is a C^2 selection $g: \bar{A} \to R^l$ from P such that $g(y^*) = p^*$, and (ii) B is convex, S is C^1 on $\bar{B}$, $S(\bar{B}) \subset A$, and "$g(S(p)) - p = 0, p \in B$" implies $p = p^*$ (to guarantee all this, choose an A satisfying (i), and then take a very small convex B. The last property follows from the rank condition). Now choose $c > 1$ such that $c > c(\bar{B})$ (remember that $c(p)$ is a uniform local Lipschitz constant for $\xi_Y(p)$, $Y \in \mathcal{Y}$), and $\mu > 0$ such that (i) if $p \in B$ and $\|v\| < \mu$, then $S(p) + v \in A$, and (ii) $\|g(S(p)) - p\| > \mu$ for all $p \in \partial B$. Let then $\varepsilon < \mu/2$ be such that $p \in \bar{B}$ and $\|S(p) - z\| \leqslant \varepsilon$ implies $\|g(S(p)) - g(z)\| < \mu/2$. Obviously, this ε exists by continuity. Finally, let $N_1$and $\delta > 0$ be as in Lemma 3 with respect to this ε. We can also assume that $\|p' - p\| < \delta$ and $p \in \bar{B}$ implies $c(p') < c$.

By taking an arbitrary C^2 extension let us assume that g is defined on the whole of R^l. Pick a large enough r so that $\|v\|$, $\|g(v)\|$, $\|Dg(v)\|$, $\|D^2 g(v)\| < r$ for all $v \in K$. For a given n let $\bar{y}_1, ..., \bar{y}_n$ be an arbitrary collection of independent random variables with values in K. For a fixed $p \in \bar{B}$, consider the function $h_p^n: K \to R$ defined by

$$h_p^n(z) = E\left[\left(g\left(\frac{1}{n}\sum_{i=1}^{n-1} \bar{y}_i + \frac{1}{n} z\right) - Eg\left(\frac{1}{n}\sum_{i=1}^{n} \bar{y}_i\right) + p\right) \cdot z\right].$$

Immediate calculation yields $\|p \cdot z - h_p^n(z)\| \leqslant (3/n) r^3$, $\|p - Dh_p^n(z)\| \leqslant (3/n) r^2$ for all $z \in K$ and $\|Dh_p^n(z) - Dh_p^n(z')\| \leqslant 1/n\ (r + r^2) \|z - z'\|$ for all $z, z' \in K$. Obviously, the same holds for $h_p^1, ..., h_p^{n-1}$. Henceforth, by choosing $N > N_1$ sufficiently large, we can guarantee that if $n > N$ then

(i) $2r/n < \mu/2$, and for all $p \in \bar{B}$, $j \in I_n$, and random variables $\bar{y}_1, ..., \bar{y}_n$;

(ii) $h_p^j \in C^1_{1/3c}(K)$;

(iii) $\|h_p^j - p\|_1 < \delta$.

This N will turn out to be large enough to yield the conclusion of the theorem. At this point, it will be useful to collect in three lemmas some consequences of the constructions carried out so far.

Maintained hypotheses for the lemmas are that $n > N$ and $\bar{y}_1, ..., \bar{y}_n$ are a collection of independent random variables taking values in K. For $p \in \bar{B}$, $h_p^1, ..., h_p^n: K \to R$ are defined as above.

LEMMA 4. *For any $j \in I_n$ and $p \in \bar{B}$, the solution to "Maximize $h_p^j(z)$ s.t. $z \in f_n(j)$" is unique.*

Proof. Lemma 1 and properties (ii), (iii) above. ∎

LEMMA 5. *Given $p \in \bar{B}$, suppose that, for all $j \in I_n$, $\hat{z}_j$ solves "Maximize $h_p^j(z)$ s.t. $z \in \hat{f}_n(j)$." Then $\|(1/n) \sum_{j \in I_n} \hat{z}_j - S(p)\| < \varepsilon$ (which implies $\|g(S(p)) - g((1/n) \sum_{j \in I_n} \hat{z}_j)\| < \mu/2$).*

Proof. This follows from Lemma 3 and property (iii) above. ∎

LEMMA 6. *Given $p \in \bar{B}$, suppose that for all $j \in I_n$ every value taken by y_j solves "*Max $h_p^j(z)$ *s.t.* $z \in \hat{f}_n(j)$." *Then, for all j and $z \in K$, $\bar{y} = (1/n) \sum_{i=1}^{n} \bar{y}_i$ and $\bar{y} - (1/n)(\bar{y}_n - z)$ belong to A a.s.*

Proof. By Lemma 5, $\|\bar{y} - S(p)\| < \varepsilon < \mu/2$ *a.s. Hence*, $\bar{y} \in A$ *a.s. Also*, $\|\bar{y} - (1/n)(\bar{y}_n - z) - S(p)\| \leqslant (1/n)\|\bar{y}_n - z\| + \|\bar{y} - S(p)\| \leqslant 2r/n + \varepsilon \leqslant \mu/2 + \mu/2 = \mu$. (Here, property (i) above is being used.) Therefore, again, $\bar{y} - (1/n)(\bar{y}_n - z) \in A$ a.s. ∎

2. *A Fixed-point Theorem*

The existence result (part (i)) of Theorem 1 will follow from a general fixed-point theorem to be proved in this section by degree theoretic methods.

Let $M \subset R^n$ be homeomorphic to the $(n-1)$ sphere. The degree of a continuous function $h: M \to R^n \backslash \{0\}$, denoted $\deg h$, is defined to be the topological degree of the function $m \to (1/\|h(m)\|)\, h(m)$ which, via a homeomorphism of M, can be looked at as a function from the $n-1$ sphere into itself (intuitively, the degree measures how many times the sphere is wrapped around itself). See, for example, Guillemin and Pollack [4, Chap. 8] for the definition of topological degree and its basic properties. A fundamental fact is *Hopf's Theorem: A continuous function $h: M \to R^n \backslash \{0\}$ can be extended to a continuous function on the bounded region limited by M without taking the value* 0 *if and only if* $\deg h \neq 0$ (see Guillemin and Pollack [4, p. 145]).

Two continuous functions $h, h': M \to R^n \backslash \{0\}$ are homotopic if there is a continuous function $H: M \times [0, 1] \to R^n \backslash \{0\}$ such that $H(\cdot, 0) = h$ and $H(\cdot, 1) = h'$. The degree of a map is invariant under homotopy. Now let $h, h': M \to R^n \backslash \{0\}$ be upper hemicontinuous (u.h.c.), compact, convex-valued correspondences. Allowing for H to also be one such, we have a concept of h, h' being homotopic. Define the degree of an u.h.c., compact, convex-valued correspondence h to be the degree of any continuous function h' homotopic to h. It is easy to verify that this concept is well defined and remains a homotopy invariant. Most important, Hopf's theorem remains valid (i.e., same statement with "continuous" replaced by "u.h.c.," and "function" by "convex, compact-valued correspondence").

Let $M \subset R^n \times R^m$ be a nonempty set. A correspondence $\Phi: M \to R^n \times R^m$ is a *product correspondence* if for all $z \in M$, $\Phi(z)$ can be written in the form $\Phi(z) = \Phi_n(z) \times \Phi_m(z)$, where $\Phi_n(z) \in R^n$ and $\Phi_m(z) \in R^m$.

We can now state:

FIXED-POINT THEOREM. *Let $A \subset R^n$, $B \subset R^m$ be nonempty, full-dimensional, compact, convex sets.*

Let $\Phi: A \times B \to R^n \times B$ *be an u.h.c., compact, convex-valued product correspondence, i.e.,* $\Phi(a, b) = \Phi_A(a, b) \times \Phi_B(a, b)$.

Let $h: \partial A \to R^n$ *be a continuous function with* $\deg h \neq 0$ (∂ *stands for boundary*). *Put* $\operatorname{Min}_{z \in h(\partial A)} \|z\| = \varepsilon > 0$.

Suppose that whenever $a \in \partial A$ *and* $b \in \Phi_B(a, b)$, *we have* $\|z - a - h(a)\| < \varepsilon$ *for all* $z \in \Phi_A(a, b)$. *Then* Φ *has a fixed point, i.e., there is* $(a, b) \in A \times B$ *such that* $(a, b) \in \Phi(a, b)$.

Remark. (i) If Φ mapped into $A \times B$, then Kakutani's theorem would yield a fixed point with no other hypothesis than convex, compact-valuedness, and upper hemicontinuity. But the fact is that it maps into $R^n \times B$. The extra conditions (i.e., product correspondence, assumption involving h), make up for this by providing a sufficiently "good" behavior of Φ at the boundary $\partial A \times B$.

(ii) The fixed-point theorem is tailor made to our needs. It does not pretend to be most general in any sense. In particular, it may be surmised that a version treating A and B symmetrically should be available.

Proof of the Fixed-point Theorem. It proceeds in two steps.

Step 1. We show that without loss of generality we can assume that $\Phi_A(a, b) = a + h(a)$ whenever $a \in \partial A$. For any $a \in R^n$, let $\Pi(a)$ be the foot of a in A, i.e., $\Pi(a)$ is the point $z \in A$ which minimizes $\|a - z\|$. Let $\hat{A} = \{a \in R^n: \|a - \Pi(a)\| \leqslant 1\}$. The set $\hat{A}$ is compact, convex. We extend Φ from $A \times B$ to $\hat{A} \times B$ as follows:

$$\begin{aligned} \hat{\Phi}_A(a, b) = a &+ \|a - \Pi(a)\| \, (h(\Pi(a))) \\ &+ (1 - \|a - \Pi(a)\|)(\Phi_A(\Pi(a), b) - \Pi(a)), \\ \Phi_B(a, b) &= \Phi_B(\Pi(a), b). \end{aligned}$$

This is a genuine extension because $a \in A$ implies $\Pi(a) = a$. Also, for $a \in \partial \hat{A}$ we have $\hat{\Phi}_A(a, b) = a + h(\Pi(a))$ and, of course, $\deg(h \circ \Pi \mid \partial \hat{A}) = \deg(h \mid \partial A)$. Therefore, all we need to show is that in the extension no new fixed point is added. We proceed by contradiction. Suppose that $(a, b) \in \hat{\Phi}(a, b)$ for $(a, b) \in \hat{A} \backslash A \times B$. Then $b \in \hat{\Phi}_B(a, b)$, i.e., $b \in \Phi_B(\Pi(a), b)$. But $\Pi(a) \in \partial A$ which, by the hypothesis of the theorem, implies that if $z \in \Phi_A(\Pi(a), b)$, then $\|z - \Pi(a) - h(\Pi(a))\| < \varepsilon$. Also, $a \in \hat{\Phi}_A(a, b)$ which, denoting $\alpha = \|a - \Pi(a)\|$, yields that for some $z \in \Phi_A(\Pi(a), b)$ we have $\alpha h(\Pi(a)) + (1 - \alpha)(z - \Pi(a)) = 0$. Hence, from $h(\Pi(a)) = \alpha h(\Pi(a)) + (1 - \alpha)(z - \Pi(a)) - (1 - \alpha)(z - \Pi(a)) + (1 - \alpha) h(\Pi(a))$, we get $\|h(\Pi(a))\| < (1 - \alpha) \varepsilon < \varepsilon$, which is the desired contradiction (remember the definition of ε).

Step 2. We assume that $\Phi_A(a, b) = a + h(a)$ when $a \in \partial A$. Let I be the identity map in $A \times B$. Pick $\bar{b} \in \operatorname{Int} B$. Let $\eta: B \to R^m$ be $\eta(b) = \bar{b} - b$. Of course, $\deg \eta \mid \partial B = (-1)^m \neq 0$. Let $\Psi: A \times B \to R^n \times R^m$ be given by $\Psi(a, b) = (\Phi_A(a, b) - a, \bar{b} - b)$. Consider the homotopy $t\Psi + (1-t)(\Phi - I) \equiv \Lambda^t$. We claim that $0 \in \Lambda^t(a, b)$ is not possible for $(a, b) \in \partial(A \times B)$ and $t \in (0, 1]$. We argue by contradiction. Suppose $0 \in \Lambda^t(a, b)$, $t > 0$, $(a, b) \in \partial(A \times B)$. Then, either

(i) $a \in \partial A$, yielding $\Psi_A(a, b) = h(a)$ and $(\Phi_A - I_A)(a, b) = h(a)$, which implies $\Lambda^t_A(a, b) = h(a) \neq 0$, or

(ii) $b \in \partial B$. In this case, $t(\bar{b} - b) + (1-t)(z - b) = 0$ for some $t > 0$ and $z \in \Phi_B(a, b) \subset B$. But this is impossible because both vectors $\bar{b} - b$ and $z - b$ point inwards and $\bar{b} - b \neq 0$.

So, for $t > 0$, $\deg(\Lambda^t \mid \partial(A \times B)) = \deg(\Lambda^1 \mid \partial(A \times B)) = \deg(\Psi \mid \partial(A \times B))$. Because the A and B coordinates of Ψ depend only on their own variables, the degree of $\Psi \mid \partial(A \times B)$ is straightforwardly computed:

$$\deg \Psi \mid \partial(A \times B) = (\deg h \mid \partial A)(\deg \eta \mid \partial B) \neq 0.$$

Therefore, $\deg \Lambda^t \mid \partial(A \times B) \neq 0$ and, by Hopf's theorem, there is (a_t, b_t) such that $0 \in \Lambda^t(a_t, b_t)$. Let $(\bar{a}, \bar{b})$ be a limit point of (a_t, b_t) as $t \to 0$. Then, by the u.h.c. of Λ on $[0, 1] \times A \times B$ we have $0 \in \Lambda^0(\bar{a}, \bar{b})$, i.e., $(\bar{a}, \bar{b}) \in \Phi(\bar{a}, \bar{b})$. ∎

3. *The Fixed-point Map*

Let N be as at the end of Section 1, and $n > N$.

We shall define an u.h.c. compact, convex-valued product correspondence

$$\Phi: \bar{B} \times K^n \times [0, 1]^n \to R^l \times K^n \times [0, 1]^n,$$

such that (i) every fixed point of Φ yields an appropriate Cournot equilibrium production $\tilde{y}_n$, and (ii) the conditions of the fixed-point theorem of the previous section are met. Since A can be taken arbitrarily small, this will end the proof of Theorem 1.

Denote by $(p, y, \alpha) \in \bar{B} \times K^n \times [0, 1]^n$ the generic entry in the domain of Φ. For (p, y, α) given, let $\bar{y}_j$, $1 \leqslant j \leqslant n$, be independent random variables defined by $\operatorname{prob}(\bar{y}_j = y_j) = \alpha_j$, $\operatorname{prob}(\bar{y}_j = 0) = 1 - \alpha_j$. Put $\bar{y} = (1/n) \sum_{j=1}^n \bar{y}_j$. With g as in Section 1, let z_j be the unique (Lemma 4) solution in $f_n(j)$ to the maximization of $E[(g(\bar{y} - (1/n)\bar{y}_j + (1/n) z_j) - Eg(\bar{y}) + p) \cdot z_j]$, and let m_j be the maximum value. Let then $\bar{\alpha}_j = \{1\}$ if $m_j > 0$, $\bar{\alpha}_j = [0, 1]$ if $m_j = 0$, and $\bar{\alpha}_j = \{0\}$ if $m_j < 0$. Finally, put

$$\Phi(p, y, \alpha) = \{Eg(\bar{y})\} \times \prod_{j=1}^n \{z_j\} \times \prod_{j=1}^n \bar{\alpha}_j.$$

We claim:

LEMMA 7. *If (p, y, α) is a fixed point of Φ, then the mixed production $\underset{\sim}{y}_n$ given by $\underset{\sim}{y}_n(j) = \bar{y}_j$ is a Cournot equilibrium with respect to any selection which coincides with g on A. Also, $\underset{\sim}{y}$ has aggregate production-taking values in A.*

Proof. This is straightforward. Because $y_j = z_j$, every $\bar{y}_j$ takes values in $\{z_j, 0\}$. Since $p = Eg(\bar{y})$, and $\alpha_j \in \bar{\alpha}_j$, we conclude that every value taken by $\bar{y}_j$ solves the right problem, i.e., maximizes $E[g((1/n)\sum_{i \neq j} \bar{y}_i + (1/n)z) \cdot z]$ on $\hat{f}_n(j) = f_n(j) \cup \{0\}$. Lemma 5 insures that $\bar{y}$ takes values in A a.s., and Lemma 6 that the result does not depend on how the price equilibrium selection is made on $J \backslash A$. ∎

LEMMA 8. *With $A = \bar{B}$ and $B = K^n \times]0, 1[^n$, the conditions of the fixed-point theorem of Section 2 are satisfied.*

Proof. We need to find a function $h: \partial\bar{B} \to R^l$ with the desired properties. Let it be $h(p) = g(S(p)) - p$. Because rank $Dh(p^*) = l$ and $h(p) = 0$, $p \in \bar{B}$ implies $p = p^*$, we have deg $h \neq 0$. Let μ be as in Section 2. Then $\|z\| > \mu$ for all $z \in h(\partial\bar{B})$.

Let $(p, y, \alpha) \in \bar{B} \times K^n \times [0, 1]^n$ be such that $p \in \partial\bar{B}$, $y_j = z_j$, and $\alpha_j \in \bar{\alpha}_j$, for all j. To verify the conditions of the theorem, it suffices to show $\|Eg(\bar{y}) - p - h(p)\| < \mu$, i.e., $\|Eg(\bar{y}) - g(S(p))\| < \mu$. But this is precisely what Lemma 5, and $y_j = z_j$, $\alpha_j \in \bar{\alpha}_j$ for all j, guarantee. ∎

4. *Proof of Conclusion* (ii)

Let $\mathscr{E}_n \to \mathscr{E}$ and $\underset{\sim}{y}_n$ be the Cournot productions converging to the Walrasian equilibrium (p^*, y^*). Denote by $h_{nj}: K \to R$ the profit function of firm $j \in I_n$, i.e., $h_{nj}(z) = E(g((1/n)\sum_{i \notin j} \underset{\sim}{y}_n(i) + (1/n)z) \cdot z)$, and Π_{nj} the maximum value for $z \in f_n(j)$.

The space Y is closed in $\mathscr{K}$. For each $Y \in Y$ define $\Pi(Y) = \max p^*Y$. For each $m > 0$ let $Z = \{Y \in Y: 0 \in Y\}$, $Z_m = \{Y \in Y: \Pi(Y) \leqslant -1/m\}$, $Z_{\bar{m}} = \{Y \in Y: \Pi(Y) \geqslant 1/m\}$, $U_m = \mathscr{K} \backslash Z \cup Z_m \cup Z_{\bar{m}}$. The three sets Z, Z_m, and $Z_{\bar{m}}$ are closed in $\mathscr{K}$. Henceforth, U_m is open. Let $I^m = \{\beta \in [0, 1]: f(\beta) \in U_m\}$, $I^\infty = \{\beta \in [0, 1]: \Pi(f(\beta)) = 0, 0 \notin f(\beta)\}$. Because $S(p^*)$ is single-valued, we should have $\lambda(I^\infty) = 0$. Because $I^\infty = \bigcap_m I^m$, we get $\lambda(I^m) \to 0$, or $v(U_m) \to 0$. Since $v_n \to v$ weakly, $\lim_n v_n(U_m) = v(U_m)$ for every m.

Let $\varepsilon > 0$ be arbitrary. Pick m such that $v(U_m) < \varepsilon/2$, and N such that whenever $n > N$:

(i) $v_n(U_m) - v(U_m) < \varepsilon/2$, implying $v_n(U_m) < \varepsilon$;

(ii) for all $j \in I_n$, there is a unique maximizer of h_{nj} on $f_n(j)$ (Lemma 1);

(iii) for all $j \in I_n$, $|\Pi_{nj} - \Pi(f_n(j))| < 1/m$ (Lemma 2).

Consider any $n > N$. By (ii) above, $\underset{\sim}{y}_n(j)$ is nondegenerate if and only if $0 < \text{prob}(\underset{\sim}{y}_n(j) = 0) < 1$. We claim also that if $f_n(j) \notin U_m$, then $\underset{\sim}{y}_n(j)$ is degenerate. Indeed, three cases are possible: (a) $f_n(j) \in Z$. Then $\hat{f}_n(j) = f_n(j)$, and the degeneracy of $\underset{\sim}{y}_n(j)$ follows from (ii) above; (b) $f_n(j) \in Z_m$. Then by (iii) above, $\Pi_{nj} < 0$, and so $\underset{\sim}{y}_n(j) = 0$ a.s.; (c) $f_n(j) \in Z_{\bar{m}}$. Then by (iii) above, $\Pi_{nj} > 0$. Hence, the value 0 is not taken by $\underset{\sim}{y}_n(j)$, and degeneracy follows from (ii) above. So, we conclude that $\{j \in I_n : \underset{\sim}{y}_n(j)$ is nondegenerate$\} \subset \{j \in I_n : f_n(j) \in U_m\}$. The latter set has ν_n measure less than ε by (i) above.

In the convex production case, $f_n(I_n) \subset Z$. Then $f_n(I_n) \cap U_m = \varnothing$ and so, $\underset{\sim}{y}_n(j)$ is degenerate for all j. ∎

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