ON A THEOREM OF SCHMEIDLER*

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Over a decade ago D. Schmeidler (1973) introduced a concept of non-cooperative equilibrium for games with a continuum of agents and, under a restriction on the payoff functions, established the existence of an equilibrium in pure strategies. The purpose of this note is to reformalize the model and the equilibrium notion of Schmeidler in terms of distributions rather than measurable functions. We shall see how once the definitions are available we get (pure strategy) equilibrium existence theorems quite effortlessly and under general conditions. A number of remarks contain applications to, among others, incomplete information games.

1. Introduction

Over a decade ago D. Schmeidler (1973) introduced a concept of non-cooperative equilibrium for games with a continuum of agents an, under a restriction of the payoff functions, established the existence of an equilibrium in pure strategies. This result is of considerable interest both because the Cournot–Nash equilibrium concept is especially appealing when individual players are macroscopically negligible, and because it obviates the need to consider mixed strategies.

The purpose of this note is to reformalize the model and the equilibrium notion of Schmeidler in terms of distributions rather than measurable functions. In this we take our inspiration from Hildenbrand (1974), and Hart–Hildenbrand–Kohlberg (1974) who first proposed and convincingly established the great simplifying power of this approach in the context of the analysis of markets with a continuum of agents. [Mas-Colell (1975) constitutes another illustration.] The same is true for non-cooperative game theory. We shall see how once the definitions are available we get a (pure strategy) equilibrium existence theorem quite effortlessly and under general conditions. A number of remarks contain applications to, among others, incomplete information games.

Recent work on non-cooperative equilibrium with a continuum of players relevant to this note include E. Green (1984) and Ali Khan (1982).

2. The model

Let A be a non-empty, compact metric space of *actions* and $\mathcal{M}$ the set of (Borel) probability measures on A endowed with the weak convergence topology. The space $\mathcal{M}$ is compact and metrizable.

A *player* is characterized by a continuous utility function $u: A \times \mathcal{M} \to R$. Given the action $a \in A$ taken by a player with characteristics u and the action distribution of all traders $v, u(a, v)$ represents the utility enjoyed by the player. We should remark that a treatment via preference relations rather than utility functions could be carried out without difficulty.

Let $\mathcal{U}_A$ be the space of continuous utility functions $u: A \times \mathcal{M} \to R$ endowed with the supremum norm. It will be denoted as the space of *players characteristics*. Mathematically, it is a metric, separable and complete space. A game with a continuum of players is then characterized by a (Borel) measure μ on $\mathcal{U}_A$.

Definition. Given a game μ, a (Borel) measure τ on $\mathcal{U}_A \times A$ is a *Cournot–Nash (CN) equilibrium distribution* if, denoting by $\tau_{\mathcal{U}}, \tau_A$ the marginals of τ on $\mathcal{U}_A$ and A respectively, we have

(i) $\tau_{\mathcal{U}} = \mu$, and
(ii) $\tau(\{(u, a): u(a, \tau_A) \geqq (A, \tau_A)\}) = 1$.

The concept of CN equilibrium distribution is modelled after Hart, Hildenbrand and Kohlberg (1974).

3. Two existence theorems

The main result is:

Theorem 1. Given a game μ on $\mathcal{U}_A$ there exists a CN equilibrium distribution.

Proof. The proof is a straightforward application of the Schrauder fixed point theorem. Denote by $\mathcal{T}$ the set of probability measures on $\mathcal{U}_A \times A$ with the property that $\tau_u = \mu$. Under the weak convergence for measures $\mathcal{T}$ is a compact, convex set [see Hildenbrand (1974, pp. 49–50)].

Given $\tau \in \mathcal{T}$ denote $B_\tau = \{(u, a): u(a, \tau_A) \geqq u(A, \tau_A)\} \subset \mathcal{U}_A \times A$. The set B_τ is closed. Define now a correspondence $\Phi: \mathcal{T} \to \mathcal{T}$ by $\Phi(\tau) = \{\tau' \in \mathcal{T}: \tau'(B_\tau) = 1\}$. Obviously, $\Phi(\tau)$ is convex for all τ. It is also easily verified that the correspondence is upper hemicontinuous and compact valued. Therefore, the conditions of the Ky Fan fixed point theorem [see, for example, Berge (1963

p. 251)] are satisfied. Hence, there is $\tau \in \mathcal{T}$ such that $\tau \in \Phi(\tau)$, i.e., such that $\tau(B_\tau) = 1$.   Q.E.D.

Given a CN distribution τ for a game μ on $\mathcal{U}_A$ we can always find measurable functions $G:[0,1] \rightarrow \mathcal{U}_A$ and $f:[0,1] \rightarrow A$ such that τ is the distribution induced by (G, f) on $\mathcal{U}_A \times A$, i.e., $\tau = \lambda \circ (G, f)^{-1}$ where λ is the Lebesgue measure [see Hildenbrand (1974, p. 50)]. Thus, we can always represent the game and the equilibrium in terms of measurable functions on a space of players' names. What cannot be done in general is, given a $G:[0,1] \rightarrow \mathcal{U}_A$ and a CN distribution τ for $\mu = \lambda \circ G^{-1}$, to find an f such that $\tau = \lambda \circ (G, f)^{-1}$. Because of (insubstantial) measurability problems an a priori given representation of the game is thus a sort of straightjacket. The approach via distributions frees one from it. This accounts for its comparative ease.

Definition.   A CN distribution τ for μ is *symmetric* if there is a measurable function $h:\mathcal{U}_A \rightarrow A$ such that $\tau(\text{graph } h) = 1$. i.e., players with the same characteristics play the same action.

The next theorem establishes the existence of symmetric CN distributions wherever μ is atomless and A is finite. It can be shown by example (trivially for the atomlessness assumption) that neither of the two hypotheses can be relaxed to the general ones of Theorem 1.

Theorem 2.   A symmetric CN equilibrium distribution for a game μ on $\mathcal{U}_A$ exists whenever A is finite and μ atomless.

Proof.   Denote $A = \{a_1, \ldots, a_n\}$. We can identify $\mathcal{M}$ with the $n-1$ simplex. For any $1 \leq i \leq n$ let $e_i \in R^n$ be the vector with $e_i^i = 1$ and $e_i^j = 0$ for $i \neq j$. Denote $\mathcal{F} = \text{supp } \mu$ and define the correspondence $\Phi:\mathcal{F} \times \mathcal{M} \rightarrow R^n$ by

$$\Phi(u, v) = \{e_i : u(a_i, v) \geq u(A, v)\}.$$

It is non-empty valued and upper hemicontinuous. Now let $\Phi:\mathcal{M} \rightarrow \mathcal{M}$ be defined by

$$\Phi(v) = \int \Phi(u, v) \, d\mu(u).$$

Since μ is atomless, Φ is non-empty convex valued and upper hemicontinuous [see Hildenbrand (1974, pp. 64 and 73)]. By the Kakutani Fixed Point Theorem there is $\bar{v} \in \Phi(\bar{v})$, i.e., there is a measurable function $h:\mathcal{F} \rightarrow A$ such that $u(h(u), \bar{v}) \geq u(A, \bar{v})$ for $\mu - a.e.$ $u \in \mathcal{F}$ and $\bar{v} = \mu \circ h^{-1}$. So, define τ by letting $\tau(V) = \mu\{u \in \mathcal{F} : (u, h(u)) \in V\}$ for every Borel set V. This is our symmetric CN distribution.   Q.E.D.

It is of interest to point out that, in addition to a fixed point theorem, the proof of Theorem 2 relies on the Lyapunov theorem on the convexity of the range of a vector measure. Perhaps surprisingly, no such theorem is required for the main result, i.e. Theorem 1.

4. Remarks

Remark 1.   In our games all players have the same strategy set and only the distribution of actions matter, i.e., who plays what is irrelevant. Thus these are anonymous games. The degree of anonymity is, however, not so high as it may appear at first sight. Differences in players' strategy sets, even possibly dependent on the distribution of actions, can be embodied into the definition of the pay-off functions by making sure that non-available strategies can never be best responses. Also, the strategy set of a particular player can be made to carry its 'signature', i.e. to disclose some information on the player. However, the requirement that A be compact imposes a limit on how far one can go in this direction, and, therefore, the qualification as anonymous game has content.

Remark 2.   By using the trick alluded to in Remark 1 it is a simple matter to derive from Theorem 1 the standard mixed strategies equilibrium existence theorem for games with a finite number of players.

Remark 3.   The setting of games with a continuum of players can accommodate games with incomplete information [Harsanyi (1967)] played by a finite number of participants receiving independent signals, as in Radner and Rosenthal (1982). Take the two player case for specificity. For $i=1,2$, let S_i, A_i, $v_i(a_1,a_2,s_i)\geqq 0,\mu_i$ be, respectively, a measure space of signals, a compact metric space of actions, the utility finction (continuous on a_1, a_2 and measurable on s_i) and the distribution of signals [normalized so that $\mu_i(S_i)=\frac{1}{2}$]. The signal may well convey information to player i about an underlying state of the world but it does not provide information on the signal received by the other player.

   To put the above in our setting let $S=S_1\cup S_2$, $A=A_1\cup A_2$ be *disjoint* unions. The distributions μ_i are naturally defined on S and so, we can put $\mu=\mu_1+\mu_2$. Given a distribution v on A we let $v_i=v\,|\,A_i$. Finally, for each $s_1\in S_1$ (and similarly for $s_2\in S_2$) we define a continuous utility function u_{s_1} on $A\times\mathcal{M}$ by $u_{s_1}(\bar{a},v)=-1$ if $\bar{a}\notin S_1$ and $u_{s_1}(\bar{a},v)=\int v_1(\bar{a},a_2,s_1)\,\mathrm{d}v_2(a_2)$ if $\bar{a}\in S_1$. Then, we can view $S_1\cup S_2$ as a subset of $\mathcal{U}_A$. Let now τ be a CN distribution. It is concentrated on $(S_1\times A_1)\times(S_2\times A_2)$ and $\tau|(S_1\times A_1),\tau|(S_2\times A_2)$ constitute in the obvious way a pair of equilibrium strategies for the original incomplete information game, given to us in distribution form [see Milgrom and Weber (1980)]. In general the interpretation of these distributions is in

terms of mixed strategies but if the CN distribution is symmetric then the incomplete information game equilibrium is pure. By Theorem 2, symmetric equilibria exists for μ atomless and A finite. This, of course, agrees with the results of Radner and Rosenthal (1982).

Remark 4. For the sake of illustration we show, somewhat informally, how Theorem 1 can be made to yield a CN equilibrium existence theorem for a fairly general class of incomplete information games with a continuum of players.

Let $(S, \mathscr{S}, \eta)$ be an abstract, finite, measure space of states of the world and $D \subset R^m$ a non-empty, compact space of decisions. We first fix some notation. Denote by $\mathscr{D}$ the space of distributions on D and by A the space of $\mathscr{S}$-measurable functions $a: S \to D$. With the weak-star topology on such functions, A becomes a compact, metric space. To any distribution v on A there corresponds a function $\zeta_v: S \to \mathscr{D}$ which to every $s \in S$ associates the distribution on D induced by v through the evaluation map $a \mapsto a(s)$. As in the general model the space of distributions on A is denoted by $\mathscr{M}$ and the space on continuous functions on $A \times \mathscr{M}$ by $\mathscr{U}_A$. We shall also consider distributions on $\mathscr{U}_A$. Admittedly, so many layers of distributions may be at first a bit disorienting but it is, actually, all quite simple.

We now describe the primitive characteristics of an individual player. They consist of two ingredients: (i) an information structure π, i.e., a σ-algebra measurable with respect to $\mathscr{S}$, and (ii) a utility function $v: D \times \mathscr{D} \times S \to R$. Note that, in the spirit of anonymous games, v depends on the decisions of other players only through the distribution of decisions. We also assume that v is continuous (respectively π — measurable) with respect to the first and second (respectively third) argument. Denote by A_π the subset of A formed by the π-measurable functions. An action, or a strategy, of a player with information structure π is then any element of $A_\pi \subset A$.

Given a population of players a distribution of actions is then a measure v on A. Given v the utility derived by a player with primitive characteristics (π, v) from using strategy $a \in A_\pi \times \mathscr{M} \to R$ summarizes everything of interest about continuous function $u: A_\pi \times \mathscr{M} \to R$ summarizes everything of interest about the player and so we shall assume that the latter is describe to us in this form. In fact, and without loss of generality (see Remark 1), we shall take u to be defined on the entire $A \times \mathscr{M}$. Indeed, we can always extend u in some irrelevant manner, e.g., $u(a, v) = u(f(a), v) - \rho(a, f(a))$ where ρ is a metric on A and $f: A \to A_\pi$ is continuous and the identity on A_π.

It is clear now that we can describe an incomplete information game with a continuum of players as a measure μ on $\mathscr{U}_A$. We are then in the setting of our general model and Theorem 1 yields the existence of a CN distribution τ. The interpretation of τ as the usual concept of Bayesian equilibrium for incomplete information games is straightforward. Note that this a pure

strategy existence result which is entirely driven by the continuum of players hypothesis and not, as in the situation of Remark 3, by the incomplete information itself. In fact, we have made no hypothesis whatsoever (atomlessness, independence,...) on the signals received by the players.

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