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On Marginal Cost Pricing with Given Tax-Subsidy Rules*

PAULINA BEATO

Universidad Complutense, Madrid, Spain

AND

ANDREU MAS-COLELL

Harvard University, Cambridge, Massachusetts

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We provide an existence proof for Generalized Marginal Cost Pricing with given tax-subsidy rules. The novelty is that our approach allows for economic environments where all the generalized marginal cost pricing equilibria are inefficient in the aggregate. An example is recalled in order to suggest how non-pathological these environments are. *Journal of Economic Literature* Classification Numbers: 021, 022, 024. © 1985 Academic Press, Inc.

I. INTRODUCTION

Consider an economy where some firms exhibit increasing returns to scale and where tax-subsidy rules are given a priori. In particular, profits and losses of the production sector cannot be allocated in any convenient lump-sum way but must follow the stipulations of the inherited fiscal code. In spite of this realistic second-best feature, and perhaps under the influence of some long deceased economist, the managers of the firms, which we assume to be obedient civil servants, are instructed to follow a generalized marginal cost pricing rule. More precisely, they are asked to strive for the satisfaction of the first-order Kuhn–Tucker condition for profit maximization when prices are taken as given.

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The above defines a general equilibrium situation that departs from the standard variety only in the behavior rule postulated of firms. The resulting theory has been studied by Guesnerie [12], Beato [2, 3], Beato and Mas-Colell [4], Mantel [13], Brown and Heal [5, 6], Cornet [10], Brown, Heal, Khan, and Vohra [7], and, in a somewhat different context, Smale [15]. As usual, two issues are of paramount importance: the existence and the optimality of equilibrium.

A first solution to the existence problem has been offered by Beato [2, 3], Mantel [13], and Brown and Heal [6]. Their approaches are very similar and apply to the case where the aggregate production frontier is smooth. They obtain an equilibrium which is production efficient in the aggregate. It is a consequence of examples exhibited by Guesnerie [12], and then Brown and Heal [5], that this cannot be improved to Pareto optimality. Unless there is a representative consumer it can well happen that *every* equilibrium is not an optimum. Thus, the first fundamental theorem fails dramatically in the extended price equilibrium theory.

In Beato and Mas-Colell [4] we presented an example where the welfare limitations of the marginal cost pricing rule appear in a particularly strong manner. We described a situation with two goods, two consumers, two firms (only one with increasing returns to scale) and a natural tax-subsidy rule, where there are three marginal cost pricing equilibria for none of which aggregate production efficiency, let alone Pareto optimality, obtains. The example is remarkably simple and it has no pathological feature. For the sake of completeness we shall quickly summarize it in Section II.

We seem at this point to have a paradox. The above-mentioned research on existence gave us an aggregately production efficient equilibrium while we now claim that the nonexistence of any such equilibrium is non-pathological. The resolution of the puzzle lies in the hypothesis, required for the existence result we are now discussing, that the aggregate efficient frontier be smooth (as in Fig. 1(a)). This is far from an innocuous assumption in a non-convex world. In fact if there is more than one firm (or at least, more than one firm "per sector") and some have increasing returns to scale then the aggregate efficiency frontier will, but for degenerate cases, exhibit inward kinks. See Fig. 1(b), where the aggregate is obtained as the sum of two identical non-convex production sets or Figs. 2 and 4, where one of the two firms is of constant returns type. The inward kinks correspond to productions where keeping aggregate efficiency in their neighborhood requires drastically different individual production plans. The need of these global reorganizations if aggregate production efficiency is to be maintained is, of course, a well-known effect of increasing returns. It accounts for the difficulty of the theory and it can hardly be smoothed out.

The implication of the previous observation is that the scope of the

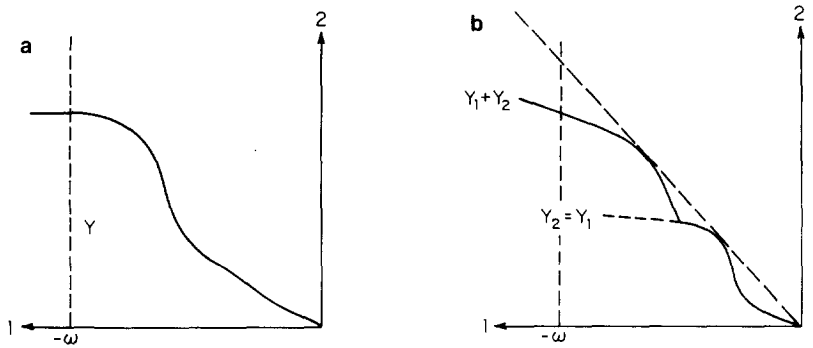

FIGURE 1

existence approach of Beato [2, 3], Mantel [13] and Brown and Heal [6] is more limited that it may look at first sight. In practical terms it only covers the one-firm case. It is of interest to mention, parenthetically, in the one-firm context the smoothness hypothesis can be much weakened (see Cornet [10]). This is important because it gives us a marginal cost pricing result for the case where the entire economy (which may be composed of many subunits and therefore exhibit inward kinks in the aggregate production frontier) is under a single coordinated management.

The main purpose of this note is to reexamine the existence problem in the several firms context. In Section III we provide a general existence theorem. Thus, we should not rush to conclude from the previous paragraph that in the disaggregated case there is a breakdown of existence. It is only the particular proof and its strong efficiency implication which fails.

Simultaneously and independently from us a marginal cost pricing existence theorem allowing for non-efficient equilibrium has been obtained by Brown, Heal, Khan, and Vohra [7].

II. Brief Description of the Example

There is a single input (denoted by x or z) and a single output (denoted by y). The production functions of the two firms and the preferences-endowments of the two consumers are represented in Figs. 2 and 3, respectively. Note that only one of the firms exhibits increasing returns. The income distribution rules are very simple. The first consumer, that is to say the consumer that neither has nor cares about the input good, absorbs all profits or losses, while the second consumer derives income only from selling input.

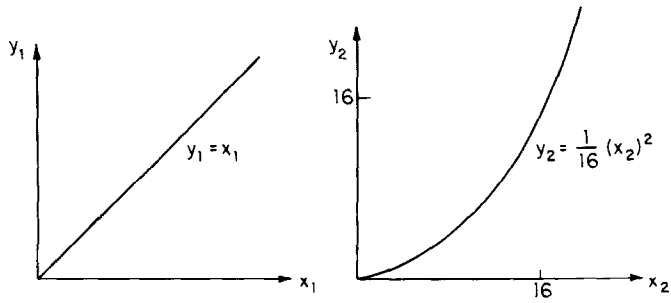

FIGURE 2

Assuming price-taking, equilibrium is defined in the usual way, except that the increasing returns firm is only required to satisfy the first order Kuhn–Tucker condition for profit maximization (i.e., $p_x > \frac{1}{8}x_2 p_y$, with equality if $x_2 > 0$). Without loss of generality we normalize to $p_y = 1$. The economy has then three equilibria which are described in Table I. In the table x_1 and x_2 are the actual input uses while $\bar{x}_1, \bar{x}_2$ are the input allocations that maximize total output subject to $\bar{x}_1 + \bar{x}_2 = x_1 + x_2$. Note that in the three cases the actual input allocations are inefficient. The aggregate production function and the aggregate production plans of the three equilibria are represented in Fig. 4.

To convince oneself that Table I describes all the equilibria it suffices to take a glimpse at the input supply and "demand" schedules of Fig. 5. In the figure S is the usual consumers inputs supply graph while D is the locus of $(p_x, x_1 + x_2)$ combinations at which the two firms are in marginal cost pricing equilibrium. The heavy dotted part of D are the aggregately efficient combinations.

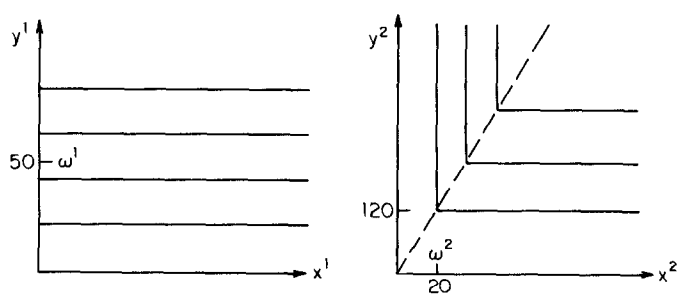

FIGURE 3

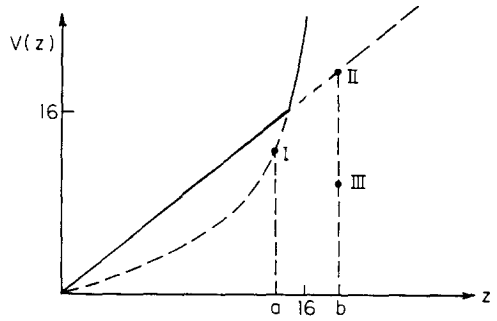

FIG. 4. $V(z) = \mathrm{Max}_{x_1 + x_2 = z}\, x_1 + \frac{1}{16}(x_2)^2$; $a = 8(2\sqrt{6} - 3)$; $b = \frac{120}{7}$.

Remark 1. It should be obvious that the preferences of the consumers can be smoothed out without changing the qualitative features of the example. The unboundedness of the production functions is also unimportant.

Remark 2. It is essential for our example that the aggregate demand behavior of the two consumers cannot be generated from a representative consumer. Otherwise, at least one equilibrium would be aggregately efficient. To obtain it we would simply maximize the preferences of the representative consumer on the aggregate attainable production set, as in Guesnerie [12] or Brown and Heal [5]. The fact that, in our case, the aggregate production frontier is not smooth does not pose any difficulty to this argument.

Because the preferences of our consumers are homothetic the above accounts for the need to have a distribution of final income which is price dependent (see, e.g., Chipman [9]). It also explains why, as it is clear from Fig. 5, what makes the example work is the backward bending supply function for the input. There is nothing pathological in this feature but it is, nevertheless, worth pointing out that if S was upward sloping then an aggregately efficient equilibrium would always exist. Because the "jumps" in the efficient part of D are always in the upward direction, an upward

TABLE I

	p_x	x_1	x_2	$\bar{x}_1$	$\bar{x}_2$
I	$2\sqrt{6} - 3$	0	$8(2\sqrt{6} - 3)$	$8(2\sqrt{6} - 3)$	0
II	1	$\frac{120}{7}$	0	0	$\frac{120}{7}$
III	1	$\frac{120}{7} - 8$	8	0	$\frac{120}{7}$

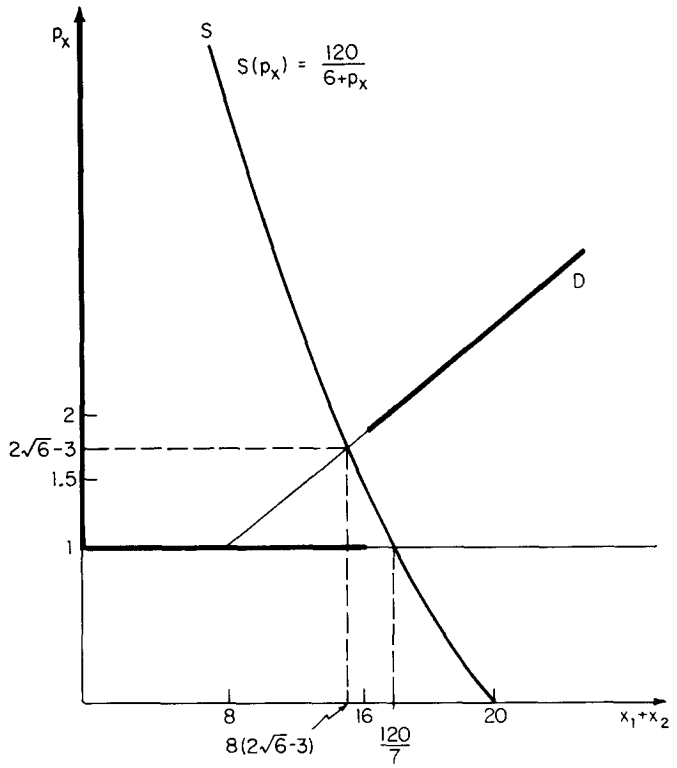

FIGURE 5

sloping S cannot pass through a gap in the graph of D while not hitting it. To understand this in terms of the representative consumer result it is enough to observe that an upward sloping S implies the fulfillment of the weak axiom of revealed preference by aggregate consumer demand. In a two-commodity world this is equivalent to the existence of a representative consumer.

Remark 3. For more than two commodities the existence of a representative consumer is a stronger hypothesis than the fulfillment of the weak axiom by the aggregate consumer demand. It is not known to us if the weaker hypothesis suffices to guarantee the existence of an aggregatively production efficient equilibrium.

III. A General Existence Theorem

In this section we provide an existence theorem for (generalized) cost pricing equilibrium in economies with several firms. Because of the

previous example those equilibria may not be aggregately efficient. Hence, one may legitimately wonder if it is worth it to take them seriously enough to worry about their existence. Of course, we grant that if distribution rules are given, then marginal cost pricing has no salient welfare properties (this is, after all, the point of our example). Nevertheless, we still think it worthwhile to clarify the existence question for two reasons. The first is making sure that in going from one firm to several the logical consistency of the model is not altered, although the added coordination complexity has destructive effects on the efficiency of the equilibria. The second is that marginal cost pricing is but one of many equilibrium theories proposed for a world with increasing returns firms. Thus, it is to be hoped that the problems and issues that it raises are of more general significance. More on this later in this section and the conclusion.

We assume that the production possibilities of the economy under consideration are described by m production sets $Y_j \subset R^l$, $j = 1,..., m$, satisfying:

(H.1) $Y_j = K_j - R^l_+$ *where* K_j *is compact.*

Some of the Y_j may represent initial endowments. With $e = (1,..., 1)$, let $r > 0$ be such that $K_j > -re$.

We shall further assume that the boundary of every Y_j, denoted ∂Y_j, is regular enough for there being defined at every point a convex set of tangents. At any rate this will only be a matter of interpretation because, with Δ the closed $l-1$ unit simplex, we shall simply assume that, for every j, we are given a correspondence g_j: $\partial Y_j \to \Delta$ satisfying:

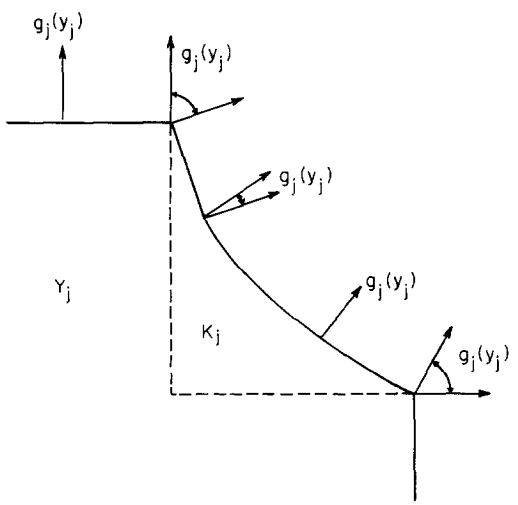

FIGURE 6

(H.2) *g_j is upperhemicontinuous and convex valued. Moreover, if $y_{jh} < -r$ and $p \in g_j(y_j)$ then $p_h = 0$.* See Fig. 6.

Denote $Y = Y_1 \times \cdots \times Y_m$. A pair $(y, p) \in Y \times \Delta$ is a *production equilibrium* if $y_j \in \partial Y_j$ and $p \in g_j(y_j)$ for every j. It is a *feasible production equilibrium* if, in addition, $\sum_{j=1}^m y_j \geqslant 0$. We assume:

(H.3) *At every production equilibrium (y, p) we have $M(y, p) \equiv p \cdot (\sum_{j=1}^m y_j) > 0$.*

This is a key survival hypothesis. To the extent that it is not directly predicated on the individual production sets (as it can be done in the purely convex case) it is not fully satisfactory. It will be satisfied if the convex part of the economy is sufficiently large to guarantee that any possible amount of losses arising at equilibrium can be financed. The convex part of the economy is the sum of the production sets which are convex. Remember that those include the initial endowments.

Remark 4. The need of (H.3), or something taking its place, seems to be related to the fact (pointed out to us by Cornet) that with (H.1) alone there is no guarantee that the set of feasible productions $\{y \in Y: \sum_{j=1}^m y_j \geqslant 0\}$ is homeomorphically convex or even connected. This appears to be a basic difficulty for any theory which allows nonconvex production sets (e.g., monopolistic competition. See Arrow and Hahn [1], especially pages 155–157 where this is recognized. See also Silvestre [14] for a discussion). We should add, however, that we have not studied the precise relationship between (H.3) and the topological nature of the set of feasible productions.

The consumption side of the economy shall be described in reduced form by means of a continuous function $f: Y \times \Delta \to R^l_+$ satisfying $p \cdot f(y, p) = M(y, p)$ whenever $M(y, p) \geqslant 0$. The underlying idea is that the consumption set of every consumer is R^l_+ and that, if at all possible, total wealth is distributed so as to give everyone a positive amount of it. For technical convenience the aggregate demand function is then extended (arbitrarily but continuously) so as to be defined for all price vectors in the closed price simplex and for all productions.

A pair $(y, p) \in Y \times \Delta$ is an *equilibrium* if it is a production equilibrium and $\sum_{j=1}^m y_j \geqslant f(p, y)$. Note that, because of (H.3), Walras law holds at equilibrium and so commodities in excess supply have zero price.

THEOREM. *Under* (H1), (H2), *and* (H3) *an equilibrium exists.*

Proof. The demonstration is directly inspired in Beato [3]. The difference is that the fixed point argument will now take place in the "disaggegated" rather than the aggregated production frontier.

Without loss of generality we can assume that every g_j is a function (standard approximation techniques and be legitimately appealed to in order to replace the correspondence by a function).

Because of (H.1) for every j the simplex Δ is homeomorphic to the set $\partial Y_j \cap (R^l_+ - re)$. Let η_j be a homeomorphism preserving the natural association of faces.

We now define a continuous map $\Phi = \Delta^{m+1} \to \Delta^{m+1}$ by associating with $(x, p) \in \Delta^{m+1}$ the values:

$$\Phi_{jh}(x, p) = \lambda_j(x_{jh} + \max\{0, p_h - g_{jh}(Y_j)\})$$
$$\text{for} \quad j \leqslant m \text{ and } 1 \leqslant h \leqslant l,$$

$$\Phi_{m+1,h}(x, p) = \lambda_{m+1}\left(p_h + \max\left\{0, f_h(p, y) - \sum_{j=1}^{m} y_{jh}\right\}\right) \qquad \text{for} \quad h \leqslant l,$$

where $y_j = \eta_j(x_j)$ and the real numbers λ_j, λ_{m+1} are chosen so as to guarantee that the values of Φ_j and Φ_{m+1} belong to the simplex.

By Brouwer's fixed point theorem, Φ has a fixed point $(\bar{x}, \bar{p})$. We claim that the corresponding $(\bar{y}, \bar{p})$ is an equilibrium. Indeed, because of the boundary condition in (H.2), $\bar{x}_j = \Phi_j(\bar{x}, \bar{p})$ is equivalent to $\bar{p} = g_j(\bar{y}_j)$. Hence, $(\bar{y}, \bar{p})$ is a production equilibrium and so, by (H.3), $\bar{p} \cdot f(\bar{p}, \bar{y}) = \bar{p} \cdot (\sum_{j=1}^{m} \bar{y}_j)$. This and $\bar{p} \in \Phi_{m+1}(\bar{y}, \bar{p})$ then yields $\sum_{j=1}^{m} \bar{y}_j \geqslant f(\bar{p}, \bar{y})$ and completes the proof. ∎

Remark 5. While, undoubtedly, hypothesis (H.3) cannot be dispensed with we have not been able to generate a counterexample to the theorem with two commodities *and* individual consumption sets equal to the non-negative orthant. So, it may be that for this special case it is dispensable.

Remark 6. Our treatment is quite different from Brown, Heal, Khan, and Vohra [7] but some of the hypothesis are similar. In particular, the key survival assumption (H.3) is also proposed by them.

IV. Concluding Remarks

This and our earlier note have pointed out some difficulties with the efficiency of marginal cost pricing equilibria when no lump-sum taxation parameters are available. We have also investigated the conditions required for existence. As we indicated in the previous section our discussion has to be evaluated in a broader context. Marginal cost pricing is not the only equilibrium theory devised to give an account of economies with increasing returns. There are many others. Some have a normative (e.g., Ramsey pricing) and some a positive (e.g., monopolistic competition) origin. All of them, however, encounter similar problems and difficulties.

Concerning the efficiency of aggregate production an important reference is Calsamiglia [8] where, roughly speaking, it is shown that no decentralized (in a certain precise sense) mechanism can cope with increasing returns, if by "coping" it is meant that all of its equilibria be production efficient. It seems plausible that this *impossibility theorem* could be strengthened to include mechanisms for which at least one equilibrium is efficient. From the present standpoint, this extension would be desirable and make clear that the efficiency failure of the example of Section II is a fundamental one.

Concerning the existence issue for general mechanisms, the relevant reference is Dierker, Guesnerie, and Neuefeind [11]. Our frameworks are not strictly comparable, but it is, nevertheless, worth pointing out that hypotheses similar to (H.1)–(H.3) do also play a key role in their, broader in scope, treatment.

References

1. K. Arrow and F. Hahn, "General Competitive Analysis," Holden-Day, San Francisco, 1971.
2. P. Beato, "Marginal Cost Pricing Equilibria with Increasing Returns," Ph.D. dissertation, University of Minnesota, 1976.
3. P. Beato, The existence of marginal cost pricing with increasing returns, *Quart. J. Econ.* **97** (1982), 669–689.
4. P. Beato and A. Mas-Colell, Gestion au Cout Marginal et Efficacité de la Production Agregée: Un Example, *An. INSEE* **51** (July–September 1983), 39–46.
5. D. Brown and G. Heal, Equity, efficiency and increasing returns, *Rev. Econ. Stud.* (1978), 571–585.
6. D. Brown and G. Heal, "Existence, Local-Uniqueness and Optimality of a Marginal Cost Pricing Equilibrium in an Economy with Increasing Returns," Cal. Tech. Social Science Working Paper No. 415, 1982.
7. D. Brown, G. M. Heal, M. Ali Khan, and R. Vohra, "On a General Existence Theorem for Marginal Cost Pricing Equilibria," Cowles Foundation Working Paper No. 724, 1985.
8. X. Calsamiglia, Decentralized resource allocation and increasing returns, *J. Econ. Theory* **14** (1977), 262–285.
9. J. Chipman, Homothetic preferences and aggregation, *J. Econ. Theory* **8** (1974), 26–38.
10. B. Cornet, "Existence of Equilibria in Economies with Increasing Returns," Berkeley Working Paper No. 311, 1982.
11. E. Dierker, R. Guesnerie, and W. Neuefeind, General equilibrium when some firms follow special pricing rules, *Econometrica*, in press.
12. R. Guesnerie, Pareto optimality in non-convex economies, *Econometrica* **43** (1975), 1–29.
13. R. Mantel, Equilibria con Rendimientos Crecientes a Escala, *An. Asoc. Argentina Econ. Polit.* **1** (1979), 271–282.
14. J. Silvestre, Increasing returns in general non-competitive analysis, *Econometrica* **46** (1978), 297–402.
15. S. Smale, Global analysis and economics. IV. Finiteness and stability of equilibria with general consumption sets and production, *J. Math. Econ.* **1** (1974), 119–129.