AN INTRODUCTION TO THE DIFFERENTIABLE APPROACH IN THE THEORY OF ECONOMIC EQUILIBRIUM*

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1. INTRODUCTION

The purpose of this article is to give a synthetic treatment of the modern mathematical approach, characterized by the heavy use of

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differential methods, to the theory of general economic equilibrium. The latter was founded slightly over a hundred years ago by Walras [51] and its modern versions have crystallized in works such as Debreu's *Theory of Value* [7] or Arrow and Hahn's *General Competitive Analysis* [2]. Although our exposition will be self-contained, there is no doubt that to develop a good intuition for the economic meaning of concepts and theorems, as well as for historical background, the reader will benefit from the consultation of these books.

Trained as they were in the nineteenth century, the classical mathematical economists (Walras [51], Pareto [39], Fisher [22]) who put together general equilibrium theory used calculus as their basic tool. Hicks' *Value and Capital* [25] and Samuelson's *Foundations of Economic Analysis* [41] represent the culmination of this classical line of work. After World War II there was a reversal. Under the combined impact of input-output analysis, linear programming and two-person zero-sum games, the differential methods were deemphasized and even fell into disrepute, in favour of topology and convexity theory. There were, undoubtedly, many valid reasons for this radical change of mathematical instruments. As one, especially relevant here, we could mention that the classical writers were quite often found guilty of settling the question of existence of solutions to the system of equations modelling an economy by a mechanical counting of equation and unknowns.

In the last ten years the differentiability techniques have come back in force. (It goes without saying that they were never completely lost; they were, for example, strongly relied upon in stability analysis, see Arrow and Hurwicz [1].) This renaissance has been due on the one hand to a purely mathematical development: the new vitality in the last two decades of the differential approach to topology, and on the other hand to the growing realization among general equilibrium theorists that equilibrium existence questions, while basic, did not exhaust the field of the determinateness of equilibrium problems (which would include matters such as uniqueness, local uniqueness, sensitivity, stability, etc.) and that in the broader context of this problem perhaps there was something valid in the counting of equations and unknowns of the old days. The seminal paper was by Debreu [8]. He and Smale (see, for example, his recent article [48]) have been key figures in the new developments. The influence of Milnor's book *Topology from the Differentiable Viewpoint* [38] has also been considerable.

Since our primary aim is to illustrate how the new mathematical techniques are put to use, we do not strive to present the most general economic model but settle for one that, hopefully, strikes a good balance between ease of manipulation and conceptual richness. Thus, we proceed by treating the consumption side of the economy very succinctly by means of a so-called aggregated excess demand function and the production side by means of Koopmans' linear activity model.

Our presentation in Parts 2 to 5 builds towards the statement in Part 4 of a fundamental result: a global index theorem for the equilibrium set of an economy. Part 7 gives the proof which, with an unknown degree of success, has been devised to be an instructive one. The references are at the ends of parts and sections.

For the mathematics used, the books of Guillemin and Pollack [24] and Hirsch [29] are recommended; but to read this article one needs only some familiarity with the implicit function theorem and with the notion of the derivative as a linear map. Survey papers on the economic theory are Debreu [13] and Dierker [16]. The article by Simon, Chapter 2, also testifies to the current interest in calculus tools in economics.

2. DESCRIPTION OF AN ECONOMY. EQUILIBRIUM

We will deal with the situation where there are $l \geqslant 1$ perfectly divisible *commodities*. In this context, a *price system* p is a vector $(p^1, \ldots, p^l)$. We will only consider strictly positive price vectors, i.e., $p \gg 0$ (meaning $p^i > 0$ for all i).

For the purposes at hand an *economy* shall be described by two objects: the *excess demand function* and the *production activities*. The excess demand function summarizes the (price-taking) behaviour of the consumption side of the economy while the production activities describe the technological possibilities of transforming commodities into commodities, i.e., the feasible input-output combinations.

2.1. Excess demand functions. Let $R^l_{++} = \{q \in R^l\colon\ q \gg 0\}$, $R^l_+ = \{q \in R^l\colon\ q \geqslant 0\}$. Formally, an excess demand function f is a *continuous* map from R^l_{++}, interpreted as the domain of prices, to R^l, interpreted as a set of net demands (positive components) and supplies (negative components), which satisfies:

H.1: (Homogeneity): For every $p \in R^l_{++}$ and $\lambda > 0$, $f(\lambda p) = f(p)$, i.e., only relative prices matter.

H.2: (Walras Law): For every $p \in R^l_{++}$, $p \cdot f(p) = 0$, i.e., intended expenditures in buying commodities equal the expected receipts of selling commodities.

H.3: (Boundedness below): There is a $\lambda \in R$ such that for all $p \in R^l_{++}$ and $1 \leqslant i \leqslant l$, $f^i(p) > \lambda$, i.e., it is not possible to supply an arbitrarily large amount of a commodity.

H.4: (Desirability): If $p_n \to p$, $p_n \gg 0$, $p \neq 0$ and $p^i = 0$ for some i, then $\|f(p_n)\| \to \infty$, where $\|\ \ \|$ is the usual Euclidean norm, i.e., if some, but not all, prices go to zero then the demand of some of the commodities with prices going to zero becomes infinite (assume that H.3 holds).

The concept of excess demand function as well as the set of hypotheses H.1–H.4 is entirely standard in economics (see, for example, Arrow and Hahn [2] Chs. 2, 4). An excess demand function fulfilling H.1–H.4 could, for example, be originated from more basic concepts as follows.

EXAMPLE 1. There are n consumers; each one of them, say the jth, has some initial endowments of commodities $\omega_j \in R^l_{++}$ and a continuous, strictly quasi-concave utility function $u_j\colon R^l_+ \to R$ such that if $x \geqslant y$, $x \neq y$, then $u_j(x) > u_j(y)$ (strong monotonicity). The individual excess demand function f_j is then defined by letting $f_j(p)$ be the vector obtained by subtracting ω_j (supply) for the unique maximizer of u_j on $\{x \in R^l_+\colon\ p\ x \leqslant p \cdot \omega_j\}$ (demand). Figure 1 illustrates $f_j(p)$ for $l = 2$. It is easily verified that f_j satisfies H.1–H.4. The excess demand function of the economy is

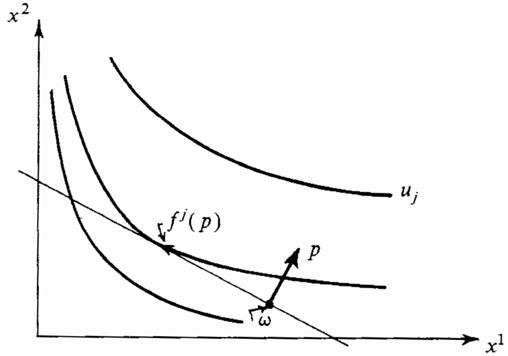

FIGURE 1

then $f = \sum_{j=1}^{n} f_j$ which, clearly, also satisfies H.1–H.4. It can be shown that essentially any excess demand function can be generated as in this example (see Sonnenschein [49], Mantel [34], Debreu [12], Mas-Colell [37]).

While hypotheses H.1–H.3 are in the nature of the concept of excess demand function, H.4 is quite restrictive in a production context, since it implies that every commodity will, at some prices, be demanded, i.e., every commodity is to some extent desirable and affordable for consumption purposes. Thus, in Example 1, it is because of H.4 that we have to impose the stringent and unnatural requirement of strong monotonicity of utility functions and strict monotonicity of initial endowments. Although the theory and results to be developed in this paper do not depend crucially on H.4, they become slightly more cumbersome to present without H.4. Since our purposes are mainly expository, we shall stick to H.4.

By H.1 only relative prices matter. Hence from now on we will regard $S = \{p \in R^l_{++}: \|p\|^2 = (p^1)^2 + \cdots + (p^l)^2 = 1\}$ as the price domain of the excess demand function. The advantage of $\|p\| = 1$ over other normalizations is that f becomes then a *tangent vector field* on S since, by H.2, for every $p \in S$, $f(p) \in T_p(S) = \{v \in R^l: p \cdot v = 0\}$, the tangent space to S at p; see Figure 2.

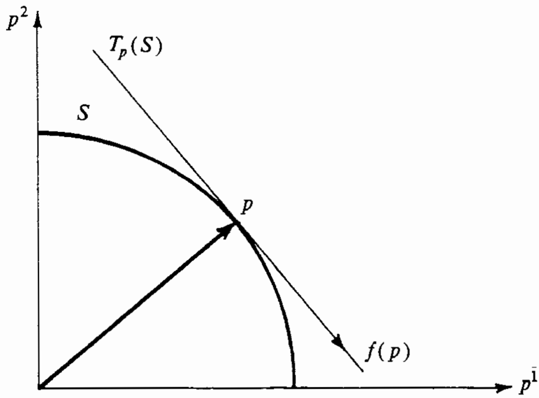

FIGURE 2

REFERENCES. For the concepts of this section see Arrow and Hahn [2] Chs. 2, 4, Debreu [7] Ch. 4, and Dierker [15].

2.2. Production activities. An *activity* is a vector $a \in R^l$. With the convention of giving outputs a positive sign and inputs a negative sign, any positive multiple of $a \in R^l$, designated as an activity, is interpreted as a feasible input-output combination. In other words, given any $\alpha > 0$ it is assumed that if, for every i, there was available the amount of commodity $\max\{0, -\alpha a^i\}$ it would be technically possible to produce the amounts $\max\{0, \alpha a^i\}$, $1 \leqslant i \leqslant l$, as outputs. The number α is called the *level of operation* of the activity a. For example, the activity vector $(-1, -2, 4)$ means that it is possible to produce 4α units of commodity 3 with α units of commodity 1 and 2α units of commodity 2.

We are given a *finite* set $\mathscr{A} \subset R^l$ of activities. The set $Y(\mathscr{A}) \subset R^l$ of feasible input-output combinations, called the *production set*, is then

$$Y(\mathscr{A}) = \left\{ \sum_{a \in \mathscr{A}} \alpha_a a \colon \alpha_a \geqslant 0 \quad \text{all } a \in \mathscr{A} \right\}.$$

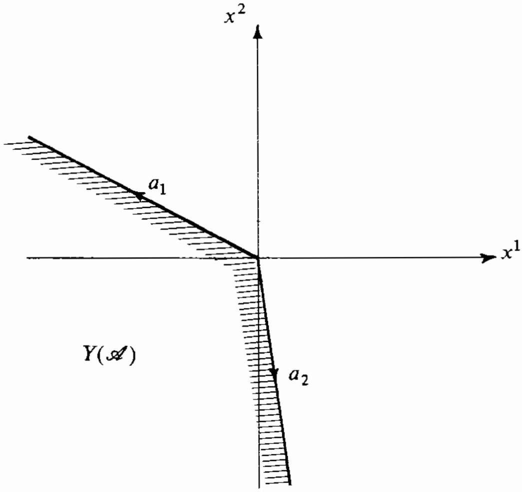

FIGURE 3

We assume

H.5: $Y(\mathscr{A})$ is a pointed cone containing the negative orthant. See Figure 3.

By definition $Y(\mathscr{A})$ is also convex and polyhedrical; H.5 embodies the economic assumptions of *constant returns to scale* (if $x \in Y(\mathscr{A})$ then $\alpha x \in Y(\mathscr{A})$ for all $\alpha \geqslant 0$), *impossibility of free production* ($Y(\mathscr{A}) \cap R^l_+ = \{0\}$), *free disposal* ($-R^l_+ \subset Y(\mathscr{A})$) and *irreversibility* (if $x \in Y(\mathscr{A})$ and $-x \in Y(\mathscr{A})$ then $x = 0$).

EXAMPLE 2. $\mathscr{A} = \{e_i : 1 \leqslant i \leqslant l\}$, where $e_i^j = 0$ if $j \neq i$ and $e_i^i = -1$. Then $Y(\mathscr{A}) = -R^l_+$. That is to say, the only conceivable productive activity is the disposal of already existing commodities.

REFERENCES. The production model just described is Koopmans' linear activity analysis [33]. See, also, Debreu [7] Ch. 3.

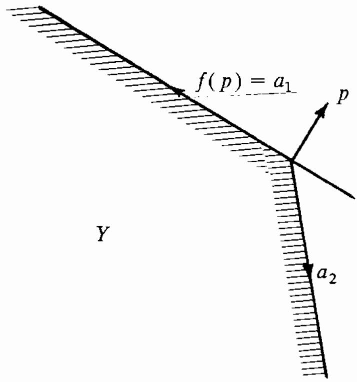

FIGURE 4

2.3. Equilibrium. An *economy* is, simply, the pair $(\mathscr{A}, f)$.

A price system $p \in S$ and activity levels $\{\bar{\alpha}_a\colon a \in \mathscr{A}\}, 0 \leqslant \bar{\alpha}_a < \infty$, are in *production equilibrium* if, for every $a \in \mathscr{A}$, the levels $\bar{\alpha}_a$ maximizes the profits of operation of the activity. Since profits at level α_a are $\alpha_a(p \cdot a)$, $\bar{\alpha}_a$ should maximize $\alpha_a(p \cdot a)$ on $\alpha_a \geqslant 0$. Clearly, if there exists such an $\bar{\alpha}_a$, $p \cdot a$ cannot be positive. In other words, $\{\bar{\alpha}_a\colon a \in \mathscr{A}\}$, $0 \leqslant \bar{\alpha}_A < \infty$, are in production equilibrium at prices p if and only if $p \cdot a \leqslant 0$ for all a and $p \cdot a = 0$ whenever $\alpha_a > 0$.

A price system p and activity levels $\{\bar{\alpha}_a\colon a \in \mathscr{A}\}$ constitute *an equilibrium for the economy* $(\mathscr{A}, f)$ if:

(i) there is production equilibrium,

(ii) $\sum_{a \in \mathscr{A}} \bar{\alpha}_a a = f(p)$, i.e., the net input-output vector equals the vector of supplies and demands.

We also say that a price vector p is an equilibrium price if there is a set of activity levels satisfying (i) and (ii) above. See Figure 4, where $\bar{\alpha}_2 = 0$ and $\bar{\alpha}_1 = 1$.

We will now introduce some concepts and reformulate the equilibrium condition in a manner convenient for later application.

Given any $A \subset \mathscr{A}$, let L_A, L_A^+ be, respectively, the subspace and the positive cone spanned by A (by convention, $L(\varnothing) = \{0\}$); $L_A^\perp$ is the orthogonal complement of L_A. Denote $S_A = S \cap L_A^\perp$ i.e., S_A is the set of prices at which the activities in A just break even, and let Π_A be the perpendicular projection map of R^l onto $L_A^\perp$. Define $f^A\colon S_A \to R^l$ by $f^A(p) = \Pi_A \circ f(p)$. Note that $p \cdot f^A(p) = 0$ for every $p \in S_A$ so that f^A is a tangent vector field on S_A.

For any $p \in S$ call $A(p) = \{a \in A\colon\ p \cdot a = 0\}$ the *base* of p. Of course $p \in L_{A(p)}^\perp$ and so $p \in S_{A(p)}$.

Then $p \in S$ is an equilibrium price system if, besides $p \cdot a \leqslant 0$ for all $a \in \mathscr{A}$, we have $f(p) \in L_{A(p)}^+$. Clearly, this latter condition implies $f^{A(p)}(p) = 0$.

REFERENCES. The (competitive) equilibrium notion described is the classical one of Walras [51]. See Arrow and Hahn [2] Chs. 2, 5 and Debreu [7] Ch. 5.

3. A SPACE OF ECONOMIES

We will now introduce a parameterized set of economies.

We begin with a set $\mathscr{M}$ such that to every $m \in \mathscr{M}$ there is associated an economy $(\mathscr{A}_m, f_m)$. Since so general a setting could scarcely be useful we further postulate:

H.6: $\mathscr{M}$ is a compact metric space.

H.7: f is jointly continuous on p and m.

H.8: H.3 holds uniformly on $\mathscr{M}$.

H.9: H.4 holds uniformly on $\mathscr{M}$, i.e., if $m_n \to m$, $p_n \to p$, $p_n \in S$, $p \notin S$, then $\|f_{m_n}(p_n)\| \to \infty$.

H.10: If $m_n \to m$ then, as sets, $\mathscr{A}_{m_n} \to \mathscr{A}_m$ (in, say, the Hausdorff distance for the nonempty, closed subsets of R^l).

Except for the convenience hypothesis H.9 (the same comment as in Part 2 applies here), the framework H.6–H.10 encompasses most of the specific situations one encounters in the economics literature.

For simplicity, in all the subsequent examples we have a fixed $\mathscr{A}$ and put $\mathscr{A}_m = \mathscr{A}$ for all $m \in \mathscr{M}$.

EXAMPLE 3. With reference to Example 1 and with utility functions fixed we could let $\mathscr{M}$ be a compact subset of R^{ln}_{++} and interpret every $m \in \mathscr{M}$ as a n-tuple of initial endowments (see Debreu [8]).

EXAMPLE 4. Again with reference to Example 1 but now keeping the initial endowments fixed (this is really immaterial) we could let $\mathscr{M}$ be a compact subset of $\mathscr{U}^m$ where $\mathscr{U}$ is a suitable topologized space of utility functions or, perhaps, of individual excess demand functions (see, respectively, Smale [45] and Dierker and Dierker [17]).

EXAMPLE 5. $\mathscr{M}$ is the set of measures on a compact metric space of agents characteristics (endowed with the topology of the weak convergence); see Hildenbrand [27].

Define

$$E = \{(m, p) \in \mathscr{M} \times S: p \text{ is an equilibrium price vector for } (\mathscr{A}_m, f_m)\}.$$

We call E the equilibrium set. It would be more appropriate at this stage to include the equilibrium level of activities in the definition of E but this will become very soon (Part 4.1) unimportant and so, we shall save on notation.

THEOREM 1. *E is a compact set.*

Proof. Let $(m_n, p_n) \in E$, $m_n \to m$. We show first that $\{p_n\}$ has an accumulation point in S. Indeed, suppose $p_n \to p \notin S$. By H.9 $\|f_{m_n}(p_n)\| \to \infty$; by H.8 this means that, denoting $b_n = f_{m_n}(p_n)$, $b_n/\|b_n\|$ has an adherent point b in R^l_+. Since $A_{m_n} \to A_m$ and $b_n/\|b_n\| \in Y(A_{m_n})$ we have $b \in Y(A_m)$ which contradicts H.5. Therefore we can assume $(m_n, p_n) \to (m, p) \in \mathscr{M} \times S$. By H.7 and H.10 $f_{m_n}(p_n) \to f_m(p)$, $f_m(p) \in Y(A_m)$, and $x \in Y(A_m)$ implies $p \cdot x \leqslant 0$. Furthermore, $f_m(p) = \lim_n f_{m_n}(p_n) \in L^+_{A_m(p)}$, since each $f_{m_n}(p_n)$ lies in $L^+_{A_{m_n}}$ and $A_{m_n} \to A_m$. Hence, $(m, p) \in E$. ■

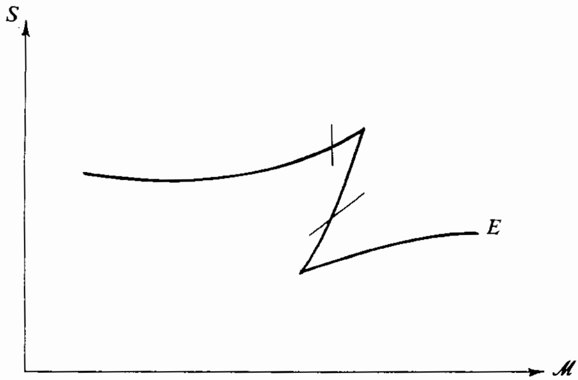

FIGURE 5

The attentive reader of the preceding proof can verify that if instead of H.6 we had just assumed that $\mathcal{M}$ is a separable metric space, then the projection of E on $\mathcal{M}$ would be a proper map. See Figure 5.

REFERENCES. The consideration of whole spaces of economies was initiated by Kannai [30] and Hildenbrand [27]. The need for such a study arose with the continuum approach to the Core Equivalence Theorem (see Aumann [3]) and has led to the heavy utilization of measure theoretical tools in economics.

4. REGULAR ECONOMIES

In this part we shall place the theory of Part 2 and 3 in a smoothness framework.

We consider first a single fixed economy $(\mathscr{A}, f)$ and postulate that, besides H.1–H.5, it satisfies:

H.11: f is a C^1 function.

This is a natural hypothesis and not difficult to justify in the context of Example 1 (see Debreu [9], [10], Mas-Colell [37]).

Smoothness allows us to speak meaningfully about "degenerate" and "nondegenerate" equilibria. Consider, for example, the zeros of

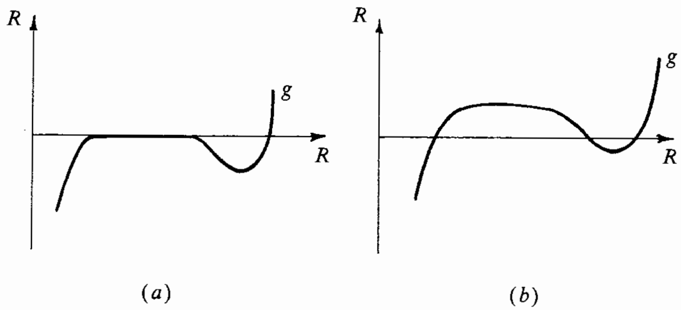

FIGURE 6

a C^1 function $g: R \to R$; it is intuitive that a situation such as Figure 6a is "degenerate" and that, consequently the set of zeros of the function is very "unstable"; on the contrary the situation of Figure 6b is nondegenerate, i.e., regular, and the zero set is "stable." What distinguishes the two cases? Clearly, an important difference is the fact that in the first case the derivative map at a zero of the function vanishes. With this hint we proceed to formulate a notion of regular economy.

4.1. Some definitions and a theorem. We begin by hypothesizing a general position condition on the activities $\mathscr{A}$.

H.12: If $A \subset \mathscr{A}$ and $\#A \leqslant l$ then A is a linearly independent collection of vectors.

Obviously, if $\mathscr{A}$ does not satisfy H.12, a small perturbation will.

If H.12 holds then for any $p \in S$ the basis $A(p) = \{a \in \mathscr{A}: p \cdot a = 0\}$ is a linearly independent collection and so, if $p \in S$ is an equilibrium price vector the equilibrium activity levels $\{\alpha_a: a \in \mathscr{A}\}$ are uniquely determined (remember that at equilibrium $\alpha_a = 0$ if $a \notin A(p)$); they will be denoted $\alpha_a(p)$. Thus an equilibrium is unambiguously determined by the price vector.

Condition H.12 will be part of our definition of regularity. It will be assumed from now on.

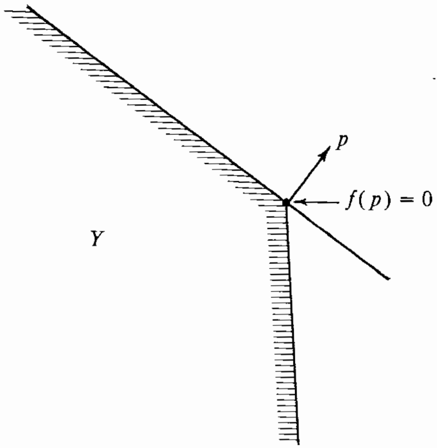

FIGURE 7

Next, we want to eliminate the possibility that at an equilibrium a basis vector not be in operation, i.e., if an activity does not operate it must be because it would make losses otherwise.

H.13: If $p \in S$ in an equilibrium, then $\alpha_a(p) > 0$ for every $a \in A(p)$ (geometrically, $f(p)$ belongs to the interior of the face $Y(\mathscr{A}) \cap \{v \in R: p \cdot v = 0\}$).

Figure 7 describes the situation ruled out.

A consequence of H.13 is that if p is an equilibrium price vector then there is no change in the base $A(p)$ over a sufficiently small neighbour of p *on* $S_{A(p)}(= \{q \in S: q \cdot a = 0 \text{ for all } a \in A(p)\}$.

Suppose that $\bar{p}$ is an equilibrium price vector satisfying H.12 and H.13.

Consider $S_{A(\bar{p})}$, the intersection of S with $L^{\perp}_{A(\bar{p})}$ in R^l. In Section 2.3, we defined

$$f^{A(\bar{p})}: L^{\perp}_{A(\bar{p})} \to L^{\perp}_{A(\bar{p})}$$

as $f^{A(\bar{p})} = \Pi_{A(\bar{p})} \circ f$, where $\Pi_{A(\bar{p})}$ is the projection of R^l onto

$L^{\perp}_{A(\bar{p})}$. As noted near the end of Section 2.3, $f^{A(\bar{p})}$ is a tangent vector field on $S_{A(\bar{p})}$ since, for all $p \in S_{A(\bar{p})}$, we have $f^{A(\bar{p})}(p) \in L^{\perp}_{A(\bar{p})}$ and $p \cdot f^{A(\bar{p})}(p) = 0$, that is to say,

$$\begin{aligned} f^{A(\bar{p})}(p) \in T_p\left(S_{A(\bar{p})}\right) &= T_p(S) \cap L^{\perp}_{A(\bar{p})} \\ &= \left\{ v \in L^{\perp}_{A(\bar{p})} : p \cdot v = 0 \right\}. \end{aligned}$$

Since $\bar{p}$ is an equilibrium price vector, $f^{A(\bar{p})}(\bar{p}) = 0$ and we may conclude that $Df^{A(\bar{p})}(\bar{p})$ maps $T_{\bar{p}}(S_{A(\bar{p})})$ into $T_{\bar{p}}(S_{A(\bar{p})})$—indeed, differentiating $p \cdot f^{A(\bar{p})}(p) = 0$ at $\bar{p}$, we have

$$0 = \bar{p} \cdot Df^{A(\bar{p})}(\bar{p}) + f^{A(\bar{p})}(\bar{p}) = \bar{p} \cdot Df^{A(\bar{p})}(\bar{p}).$$

We shall add to our definition of regularity that this map be onto

H.14: Assume H.12 and H.13. At every equilibrium price vector p, $Df^{A(p)}(p)$ maps $T_p(S_{A(p)})$ onto $T_p(S_{A(p)})$ or, *equivalently*, rank $Df^{A(p)}(p) = l - 1 - \#A(p)$. By definition, $Df^{A(p)}(p) = \Pi_{A(p)} \cdot Df(p)$.

In spite of its apparent technicality, the meaning of H.14 is transparent: It says that infinitesimally there are as many nonredundant equations to determine the equilibrium price as there are unknowns.

Definition. An economy $(\mathscr{A}, f)$ is regular if H.12, H.13, and H.14 are satisfied.

Definition. Let $(\mathscr{A}, f)$ be regular and p be an equilibrium price vector. The index of p denoted $i(p)$ is defined to be $(-1)^{\#A(p)}$ times the sign of the determinant of the linear map $\Pi_{A(p)} \circ Df(p)$: $T_p(S_{A(p)}) \to T_p(S_{A(p)})$. If $\#A(p) = l - 1$ then $T_p(S_{A(p)}) = \{0\}$; by convention the determinant sign is then $+1$ (hence, if $\#A(p) = l - 1$, $i(p) = (-1)^{l-1}$).

Example 6. Let $Y(\mathscr{A}) = -R^l_+$ as in Example 2. Then an economy is regular if whenever $f(p) = 0$, rank $Df(p) = l - 1$.

This is the situation studied by Debreu [8] and Dierker and Dierker [17].

For the purposes of demonstrating our main theorem it will be convenient, but not at all necessary, to strengthen somewhat conditions H.13 and H.14.

H.13′: If $p \in S$ is such that $f(p) = \Sigma_{a \in A(p)} \alpha_a a$, then $\alpha_a \neq 0$ for every $a \in A(p)$.

H.14′: Assume H.12 and H.13. For every $p \in S$, if $f(p) \in L_{A(p)}$ then $Df^{A(p)}(p)$ maps $T_p(S_{A(p)})$ onto $T_p(S_{A(p)})$ or, equivalently, rank $Df^{A(p)}(p) = l - 1 - \#A(p)$. By definition, $Df^{A(p)}(p) = \Pi_{A(p)} \circ Df(p)$.

Note that H.13′ and H.14′ are the same as H.13 and H.14 except we no longer require $p \cdot a \leqslant 0$ for every $a \in \mathscr{A}$.

From now on, we take H.12, H.13′, and H.14′ as our definition of a regular economy.

Let us reintroduce our space of economies $\mathscr{M}$ satisfying H.7–H.10 and impose an appropriate smoothness hypothesis. For the moment we postulate that there is a fixed $\mathscr{A}$ such that $\mathscr{A}_m = \mathscr{A}$ for all $m \in \mathscr{M}$. Hence, only f_m depends on m.

H.15: $Df_m(p)$ is jointly continuous in m and p.

The next theorem summarizes the sharp implications for equilibrium that lie in the concept of regularity. It justifies the name "regular economies." It shows that at regular economies the equilibrium set E_m is finite and "locally stable," i.e., as m in Figure 8.

THEOREM 2. *Let $\bar{m} \in \mathscr{M}$ be regular. Then there is a neighborhood of $\bar{m}$, $U \subset \mathscr{M}$ and continuous functions $\phi_1, \ldots, \phi_h$: $U \to S$ such that for every $m \in U$:*

(i) $$\phi_j(m) \neq \phi_{j'}(m) \quad \textit{if } j \neq j', \quad \textit{and}$$

(ii) $$\{ p \in S : (m, p) \in E \} = \bigcup_{j=1}^{h} \phi_j(m).$$

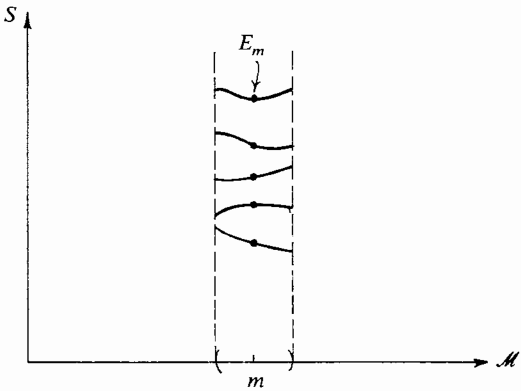

FIGURE 8

Proof. By Theorem 1 it suffices to show that if $\bar{p} \in E_{\bar{m}}$ then there are neighbourhoods $\bar{m} \in U \subset \mathscr{M}$, $\bar{p} \in Q \subset S$ and a continuous function ϕ: $U \to S$ such that $(m, p) \in (U \times Q) \cap E$ if and only if $p = \phi(m)$.

By H.12 and H.13 there are neighborhoods $\bar{m} \in U \subset \mathscr{M}$ and $\bar{p} \in Q \subset S$ such that $(m, p) \in (U \times Q) \cap E$ if and only if $f_m^{A(\bar{p})}(p) = 0$.

The result will now be a consequence of the following version of the Implicit Function Theorem (L. Schwartz [43], pg. 278):

Let $\mathscr{M}$ be a topological space, $S \subset R^n$ an open set and $G: \mathscr{M} \times S \to R^n$ a function such that for all $(m, p) \in \mathscr{M} \times S$ $D_p g(m, p)$ exists and depends jointly continuously on m and p. Suppose that $g(\bar{m}, \bar{p}) = 0$ and rank $D_p g(\bar{m}, \bar{p}) = n$. Then there are open sets $\bar{m} \in U \subset \mathscr{M}$, $\bar{p} \in Q \subset S$ and a continuous function ϕ: $U \to Q$ such that $(m, p) \in U \times Q$, $g(m, p) = 0$ if and only if $p = \phi(m)$.

To get our result from the previous theorem we only have to perform a trivial normalization on $f^{A(\bar{p})}$. Take, without loss of generality, the case $A(\bar{p}) = \varnothing$ so that $f^{A(\bar{p})} = f$. We can identify S with its projection in the first $l - 1$ coordinates and regard therefore the price domain as an open subset of R^{l-1}. Further, letting g: $S \to R^{l-1}$ be the first $l-1$ coordinates of f, we have because of H.2 that $g(p) = 0$ if and only if $f(p) = 0$ and rank $Df(\bar{p}) = l - 1$ if and only if rank $Dg(\bar{p}) = l - 1$. By putting $n = l - 1$ in the previous theorem we are done. ∎

4.2. A treatment of regularity via transversality. This section is in the nature of a detour and can be skipped without further consequence.

The theorem of the previous section hypothesized a space of economies $\mathcal{M}$ with a fixed production set $\mathcal{A}$. It was shown that if $(\mathcal{A}, f_m)$ is regular then the equilibrium set is "stable" (in the sense of Figure 7) if f_m is slightly perturbed. But, what happens if the perturbation is made on $\mathcal{A}$? The answer is that the equilibrium set is stable in this case as well. In fact, loosely speaking, at a regular economy there is stability against any (small, C^1) perturbation. This is not well seen from the phrasing chosen to define regularity. Hence in this section we will state an equivalent definition, based on the notion of transversality of smooth manifolds, which will make the stability property intuitively most plausible. We shall not give proofs nor be very rigorous here; everything stated however can be proved without much difficulty. The books of Guillemin and Pollack [24], and Hirsch [29] can be consulted for the mathematical concepts to be introduced.

Let Q be a C^1 manifold and $M, N \subset Q$ be two C^1 submanifolds. One says that M and N are transversal, denoted $M \pitchfork N$ if whenever $x \in M \cap N$ the sum $T_x(M) + T_x(N)$ equals $T_x(Q)$; in Figure 9a, M and N are transversal, in Figure 9b they are not. In particular, if $\dim M + \dim N = \dim Q$ and $M \pitchfork N$, then $M \cap N$ is a discrete set of points (see Figure 9a) and it is intuitive, and at any rate a consequence of the Implicit Function Theorem, that in bounded regions of Q the intersection set is stable against small, C^1 perturbations of M and N. If $\dim M + \dim N < \dim Q$ and $M \pitchfork N$, then $M \cap N = \varnothing$.

We want to show that the notion of regular economy can be expressed in terms of manifold transversality.

Let $T(S) = \{(p, x) \in S \times R^l\colon\ p \cdot x = 0\}$; $T(S)$ is a C^∞ manifold of dimension $2(l-1)$ called the *tangent bundle* of S.

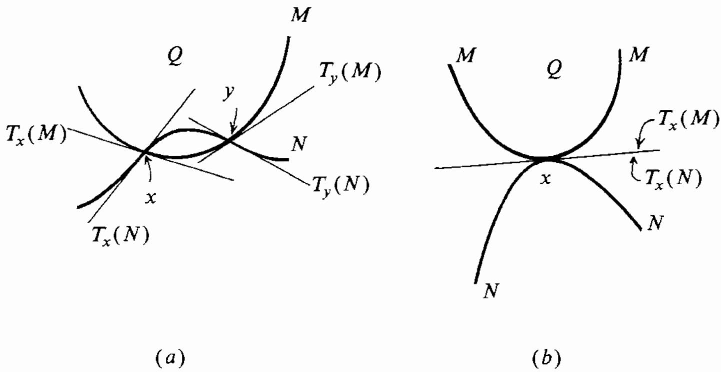

FIGURE 9

Given a C^1 excess demand function f: $S \to R^l$ its graph is Graph(f) = $\{(p, f(p))\colon\ p \in S\} \subset T(S)$. So, Graph($f$) is a $(l-1)$ submanifold of $T(S)$.

Let a set of activities $\mathscr{A}$ be given. For every $A \subset \mathscr{A}$ and $A' \subset A$ put

$$J(A, A') = \left\{(p, x) \in S \times R^l\colon p \in L_A^{\perp}, x \in L_{A'}\right\} \subset T(S).$$

Then $J(A, A')$ is a submanifold of $T(S)$ of dimension $(l-1) - \#(A \setminus A')$.

Observe that if $p \in S$ is an equilibrium price for $(\mathscr{A}, f)$ then $(p, f(p)) \in$ Graph(f) $\cap\, J(A(p), A(p))$.

It is not hard to verify that conditions H.13′ and H.14′ are equivalent to the following condition:

H.16: $(\mathscr{A}, f)$ is regular if H.12 is verified and for every $A \subset \mathscr{A}$, $A' \subset A$, Graph(f) $\pitchfork J(A, A')$.

In words, $(\mathscr{A}, f)$ is regular if the "consumption side" of the economy is transversal to the "production side."

REFERENCES. The seminal paper in the theory of regular economies is Debreu's [8]. The concept of regular exchange economy was introduced there while, for the exchange case, the definition given here first appeared in Dierker and Dierker [17]. Again for the exchange case the index of an equilibrium was first recognized by Dierker [14]. Definitions of regular economies for more general models of production have been proposed in G. Fuchs [23], and Mas-Colell [36]. A theory of regular economies not centered on the concept of aggregate excess demand has been extensively studied (including production) by Smale [45], [46]. Except in a somewhat unsatisfactory manner in Mas-Colell [36] the linear activity model had not previously been studied. K. Hildenbrand [26] was first to call attention to the version of the Implicit Function Theorem used. For further references see Dierker [16] and Debreu [11], [13].

5. GENERICITY OF REGULAR ECONOMIES

In this part we shall argue that, provided the parameterized set of economies $\mathcal{M}$ is rich enough, most economies will be regular. "Most" could mean open-dense if $\mathcal{M}$ is a general metric space and "of full Lebesgue" measure if $\mathcal{M}$ is, say, an open subset of an Euclidean space. Thus the situation is analogous to the one depicted in Figures 6a and b where 6a appears clearly pathological. In the mathematical literature, properties of this kind are referred to as *generic*.

Since the theory of general economic equilibrium is a static one, the genericity of regular economies provides justification for restricting our analysis to equilibria of regular economies. The "static" qualification must be emphasized. For the purposes of a *dynamic theory* the in-depth analysis of *critical* (i.e., nonregular) *economies* is most important since, in the first place, it is there that any qualitative change takes place and, second, they may unavoidably appear in the trajectory of a dynamic path.

We will give a very rough treatment of the genericity question. It is assumed that there is a fixed set of activities $\mathcal{A}$ and that $\mathcal{M}$ is an open subset of R^s for some $s > 0$. We write $f_m(p)$ in the form $f(p, m)$ and require:

H.17: $f\colon S \times \mathcal{M} \to R^l$ is a C^1 function.

H.18: For any $p \in S$, $D_m f(p, m)$ has rank $l - 1$.

It is hypothesis H.18 that captures the "rich enough" requirement on $\mathcal{M}$; given any economy m it is possible to distort it in any direction.

EXAMPLE 7. Example 3 satisfies both H.17 and H.18.

There is an unpalatable aspect to the above treatment: It postulates the inexistence of purely intermediate commodities, i.e., produced commodities which only become valuable in the process of production but are neither final consumption goods nor non-produced original inputs supplied by consumers. While a satisfactory genericity treatment has to, and no doubt can, allow for intermediate goods there appears to be a gap in the literature which we shall not fill for the occasion. Given the illustrative purposes (of types of reasonings, tools and techniques) which dominate this presentation H.18 will be good enough.

An economy $m \in \mathcal{M}$ which is nonregular shall be called *critical.*

THEOREM 3. *The set of critical, i.e., nonregular, economies form a closed set of (Lebesgue) measure zero.*

The proof constitutes an (immediate) application of Sard's theorem, a recent (1942) but most important theorem of analysis. For the present version see, for example, Guillemin and Pollack [24] p. 68, where the phrasing is in terms of transversality but the translation is immediate; the usual version is on page 39. See also Spivak [50].

SARD'S THEOREM. *Let $U \subset R^s$, $V \subset R^t$ be open sets and g: $U \times V \to R^n$, $s \leqslant n$, a C^1 function. Suppose that g satisfies the condition: for every $(u, v) \in U \times V$ $Dg(u, v)$ has rank n, i.e., maps onto, then for almost every v $f(u, v) = 0$ implies rank $D_u f(u, v) = n$.*

Call a function g regular if it satisfies the conditions of Sard's theorem. If $s = n$ then by the usual version of the Implicit Function Theorem the set $\{u \in U: f(u, v) = 0\}$ must be discrete for every v satisfying the conclusion of the Theorem. If $s < n$ the only possibility is that $\{u \in V: f(u, v) = 0\}$ be empty.

Let us now apply Sard's theorem to our problem. Hypotheses H.13′ and H.14′ can be strengthened and written jointly as follows:

H.19: For any $A' \subset A \subset \mathscr{A}$, $\#A \leqslant l - 1$, if $f(p) \in L_{A'}$, $A = A(p)$, then $Df^A(p)$ maps $T_p(S_A)$ onto $T_p(S_{A'})$.

Clearly H.19 implies H.14′. To see that it also yields H.13′ note that if $A' \subset A$ and $A' \neq A$ then $\dim T_p(S_A) < \dim T_p(S_{A'})$. So, the conclusion can only be satisfied if $f(p) \in L_{A'}$, $A = A(p)$, is impossible. But this is precisely H.13′.

To deduce H.19 from H.17, H.18 and Sard's theorem, we need to "normalize" f, i.e., to work in a space of dimension $l - 1$ rather than l. The simplest and most standard (although not the most elegant) way to do this is to project all the spaces and functions on the $l - 1$ space spanned by the first $l - 1$ commodities; in other words, we drop from consideration the price and demand-supply of the lth commodity. With this understanding the measure zero property of critical economies follows from Sard's theorem if for every $A' \subset A \subset \mathscr{A}$, $\#A \leqslant l - 1$, we interpret $V = \mathscr{M}$, $U = S_A$, $g = f^{A'}$, and $R^n = T_p(S_{A'})$; note also there is only a finite number of $A' \subset A \subset \mathscr{A}$ combinations.

The closedness properties of critical economies is an immediate consequence of the definition, Theorem 1, and the continuity of determinant functions.

REFERENCES. The prototypical version of Theorem 3 was given first by Debreu [8] in the context of Example 7. Versions encompassing more general and complete situations than Debreu's have been offered by Dierker and Dierker [17], Smale [45], [46], Fuchs [23], Mas-Colell [36], Balasko [4], Dierker [18], Cheng [5], Hildenbrand [26], and others. See Debreu [13], for an extensive reference list.

6. THE MAIN THEOREM

For every economy $(\mathscr{A}, f)$ let $E(\mathscr{A}, f)$ be the set of its equilibrium prices. It is assumed that economies satisfy the hypotheses H.1–H.5 and H.11. Clearly, if an economy is regular then $\#E(\mathscr{A}, f) < \infty$.

MAIN THEOREM. *If the economy $(\mathscr{A}, f)$ is regular then*

$$\sum_{p \in E(\mathscr{A}, f)} i(p) = (-1)^{l-1}.$$

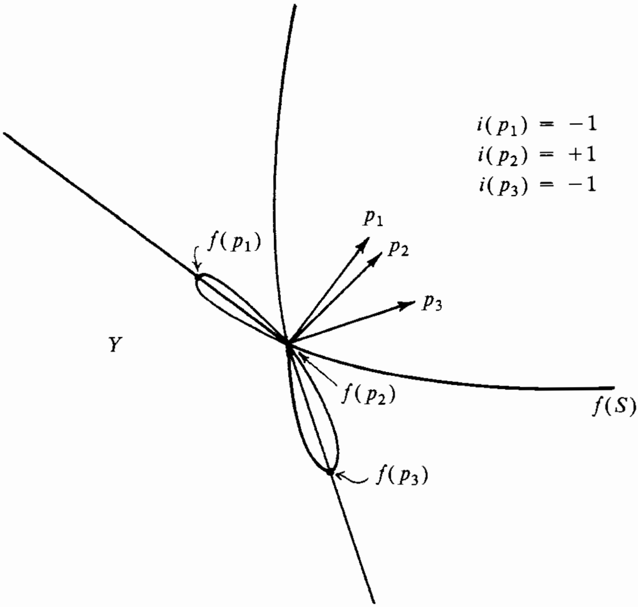

FIGURE 10

Some consequences of the theorem are:

(i) Since $i(p) = \pm 1$, $\#E(\mathscr{A}, f)$ is an odd number. In particular $\#E(\mathscr{A}, f) \neq 0$, i.e., a price equilibrium *exists*.

(ii) Suppose that a regular economy $(\mathscr{A}, f)$ is such that for every $p \in E(\mathscr{A}, f)$ there is a full set of activities in operation, i.e., $\#A(p) = l - 1$ for every $p \in E(\mathscr{A}, f)$. Since then $i(p) = (-1)^{(l-1)}$ for every $p \in E(\mathscr{A}, f)$ we conclude that the equilibrium is unique.

An illustration of the theorem is provided in Figure 10.

REFERENCES. For the case of Examples 2, 6, and 7, the so-called exchange economies, the index theorem was given by Dierker [14]. For the present linear activity case the theorem as well as consequence (ii) seem to be new. Both have been the object of simultaneous and independent finding by Timothy Kehoe [31] at Yale. The production equilibrium problem can be formally subsumed in the so-called "non-linear complementary problem." An index theory for the latter is implicitly contained in Saigal and Simon [40]. See also Eaves and Scarf [20].

7. PROOF OF THE MAIN THEOREM BY PATH-FOLLOWING

For all of this part a fixed *regular* economy $(\mathscr{A}, f)$ is given.

7.1. An auxiliary construction: one parameter family of economies. Let $e = (1, \dots, 1) \in R^l$ and $\Delta = \{q \in R^l_{++}\colon q \cdot e = 1\}$. To every $q \in \Delta$, we associate an excess demand function

$$f_q(p) = (p \cdot e)\left(\frac{q^1}{p^1}, \dots, \frac{q^l}{p^l}\right) - e.$$

Denote $\hat{q} = (1/\|q\|)q$. It is easily verified that $f_q(\hat{q}) = 0$ and $q \cdot f_q(p) > 0$ for $p \neq \hat{q}$. Furthermore, for any $q \in \Delta$ and $p \in S$, $D_q f_q(p)$ maps $T_q\Delta = \{w\colon w \cdot e = 0\}$ onto $T_p(S)$. This is clear since for any $w \in R^l$, $D_q f_q(p)w = (p \cdot e)(w^1/p^1, \dots, w^l/p^l)$. Note also that $T_q(\Delta) = T_e(S)$ for any $q \in \Delta$.

We will choose $q \in \Delta$ satisfying:

H.20: $q \cdot y < 0$ for any nonzero $y \in Y(\mathscr{A})$.

See Figure 11. If there is a q in Δ satisfying H.20, then there is a whole open set of such q since the property will be preserved under small perturbations. The existence of one such q follows from the separating hyperplane theorem (see, for example, Debreu [7]) and H.5.

An economy $(\mathscr{A}, f_q)$ which satisfies H.20 has the unique equilibrium $\hat{q}$. Furthermore,

$$D_p f_q(p)w = (w \cdot e)\left(\frac{q^1}{p^1}, \dots, \frac{q^l}{p^l}\right) - (p \cdot e)\left(\frac{q^1 w^1}{(p^1)^2}, \dots, \frac{q^l w^l}{(p^l)^2}\right).$$

One checks easily that $D_p f_q(p)w = 0$ if and only if w is a multiple of p. Therefore, rank $D_p f_q(p)|T_p(S) = l - 1$ for all $p \in S$ and $(\mathscr{A}, f_q)$ is a regular economy.

To compute the index of the equilibrium q, we need to compute the sign of $\det D_p f_q(\hat{q})$ on $T_{\hat{q}}(S)$, which will clearly

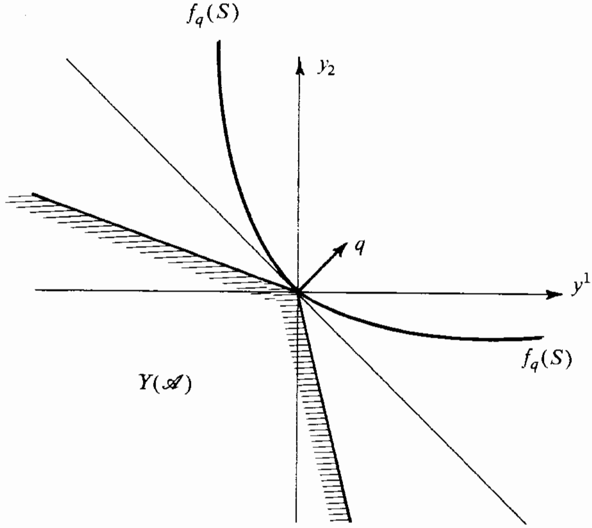

FIGURE 11

equal the sign of det $D_p\hat{f}_q(q)$ on R^{l-1} where $\hat{f}_q(p) = (f_q^1(p), \ldots, f_q^{l-1}(p))$. But, $D_p f_q(q)$ has a negative diagonal matrix; hence, the sign of its determinant is $(-1)^{l-1}$. Thus, the index of $\hat{q}$ is $(-1)^{l-1}$.

For every $q \in \Delta$ we define a *one-parameter family of excess demand functions* as follows: for every $0 \leq t \leq 1$, let $f_{q,t} = tf + (1-t)f_q$. Of course, $f_{q,0} = f_q$ and $f_{q,1} = f$. We will also write $f_{q,t}(p)$ as $f_q(p, t)$. The *one-parameter family of economies* is then $(\mathscr{A}, f_{q,t})$, $0 \leq t \leq 1$. Note that the family satisfies H.3 and H.4 uniformly.

Relative to a fixed q, let

$$E = \{(p, t) \in S \times [0,1]: p \text{ is an equilibrium for } (\mathscr{A}, f_{q,t})\},$$

$E_t = \{p \in S: (p, t) \in E\}$, and, for every $A \subset \mathscr{A}$ with $\#A \leq l-1$, put $I_A = \{(p, t) \in S_A \times [0, 1]: f_q^A(p, t) \equiv \Pi_A \circ f_q(p, t) = 0\}$. Disregarding the nonnegativity requirement, I_A is a kind of equilibrium set relative to the activities in A. Of course,

$$E \subset \bigcup_{\substack{A \subset \mathscr{A} \\ \#A \leq l-1}} I_A.$$

The one-parameter family $(\mathscr{A}, f_{q,t})t \in [0,1]$ will be called *regular* if it satisfies:

H.21: For every $A \subset A' \subset \mathscr{A}$ with $\#A' \leq l-1$ we have that if $p \in S_{A'}$ and $f_q^A(p,t) = 0$ i.e., $f_q(p,t) \in L_A$, then $D_{p,t} f_q^A(p,t)$ maps $T_p(S_{A'}) \times R$ onto $T_p(S_A)$.

The implications of the regularity property will be spelled out later on. The next section will be devoted to showing that by perturbing q slightly we can guarantee that the one-parameter family be regular. If the reader is willing to believe this, the section can be skipped.

7.2. Genericity of regular one-parameter families. We shall now use Sard's theorem in a manner entirely analogous to its use in Part 5.

Let $Q = \{q \in \Delta: q$ satisfies H.20$\}$; Q is nonempty and open.

Consider two fixed $A \subset A' \subset \mathscr{A}$, $\#A' \leq l-1$.

By construction at any $\bar{p} \in S_{A'}$, $\bar{t} \in [0,1)$ and $\bar{q} \in Q$, $D_q f_{\bar{q}}^A(\bar{p}, \bar{t})$ maps $T_e(S)$ onto $T_{\bar{p}}(S_A)$. If $\bar{t} = 1$ and $f_{\bar{q}}^A(\bar{p}, 1) = 0$ then, by the regularity assumption on $f_{\bar{q},1}$, $D_p f_{\bar{q}}^A(\bar{p}, 1)$ maps $T_{\bar{p}}(S_{A'})$ onto $T_{\bar{p}}(S_A)$—if $A' = A$ this is true by H.14′; if $A' \neq A$ this is true vacuously because by H.13′ $p \in S_{A'}$ and $f_{\bar{q}}^A(\bar{p}, 1) = 0$ cannot simultaneously hold. Hence, at any $\bar{p} \in S_{A'}$, $\bar{q} \in Q$ and $\bar{t} \in [0,1]$ if $f_{\bar{q}}^A(\bar{p}, \bar{t}) = 0$ then $D_{q,p,t} f_{\bar{q}}^A(\bar{p}, \bar{t})$ maps $T_{\bar{p}}(S_{A'}) \times R \times T_e(S)$ onto $T_{\bar{p}}(S_A)$. Therefore, by Sard's theorem—see Part 5—almost every $q \in Q$ will be such that f_q satisfies H.21 with respect to A' and A. Since there is only a finite number of pairs $\{A, A'\}$, the proof of the existence of one $q \in Q$ fulfilling H.21 is concluded. Strictly speaking, to apply Sard's theorem we must normalize f_q in the usual manner; this is done, without difficulty, as in Part 5 (project $T_p(S_{A'})$, into the first $l-1$ coordinates of R^l and drop from f_q the last coordinate function).

N.B.: From now on, a fixed q making $(\mathscr{A}, f_{q,t})$, $t \in [0,1]$, regular is retained. We drop reference to it.

7.3. The structure of I_A. We now consider a fixed $A \subset \mathscr{A}$, $\#A \leq l-1$, with $S_A \neq \varnothing$ and study $I_A \subset S_A \times [0,1]$. Without loss of generality and merely for notational convenience we take $A = \varnothing$. Then, we replace S_A by S and f_t^A: $S_A \times R \to L_A^{\perp}$ by f_t: $S \times R \to R^l$. We also put $I_A = I$ and let z stand for (p, t).

By H.21, whenever $f(z) = 0$, $Df(z)$ maps $T_p(S) \times R$ onto $T_p(S)$. Hence, we can apply the following form of the Implicit Function Theorem (see Milnor [38] pg. 13 and Appendix):

Let M be an m dimensional smooth manifold with boundary. Let f: $M \to R^{m-1}$ be a C^1 function such that f and $f|\partial M$ have 0 as regular value (i.e., if $f(z) = 0$—resp. $(f|\partial M)(z) = 0$—then $Df(z)$—resp. $D(f|\partial M)(z)$—maps onto; ∂M denotes the boundary of M) then $f^{-1}(0)$ is a 1-dimensional manifold and $\partial f^{-1}(0) = f^{-1}(0) \cap \partial M$. Further, if $f^{-1}(0)$ is compact then it consists of a finite number of components each diffeomorphic to the unit circle or the unit interval

(see Figure 12 or 13).

In our case, relying on the usual normalization of f (i.e., project S onto R^{l-1} and drop the last coordinate of f) we see that the conditions of the theorem are met with $M = S \times [0,1]$. Also, by

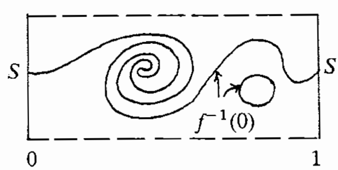

FIGURE 12

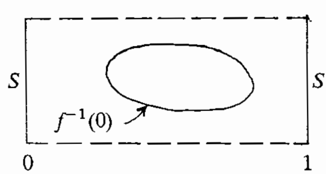

FIGURE 13

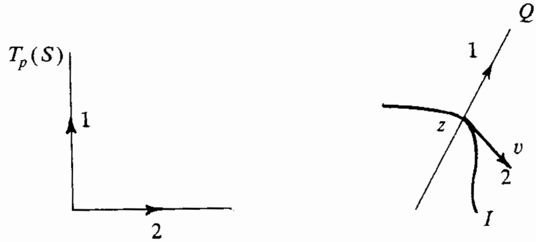

FIGURE 14

H.4, $f^{-1}(0)$ is compact (remember the proof of Theorem 1 and note that $S_A \neq \varnothing$ implies $L_A \cap R^l_+ = \{0\}$). So, up to diffeomorphism, $f^{-1}(0)$ is composed of a finite number of pairwise disjoint segments and circles.

For each $p \in S$ we can choose a set of $(l-1)$ vectors $w_1(p), \ldots, w_{l-1}(p)$ tangent to S at p which form a basis of $T_p(S)$ and vary smoothly with p. Adding to the basis the positive unit vector of R, we have a smooth basis for $T_p(S) \times R$ for each p, which we will call the main basis of $T_p(S) \times R$.

Consider a point $z = (p, t) \in I$. Take a vector $v \neq 0$ tangent to the one-dimensional set I at z. Since I is defined as $f^{-1}(0)$, $Df(z)v = 0$. Supposing that $l > 1$, consider any $(l-1)$ dimensional subspace $Q \subset T_p(S) \times R$ which does not include v. Give Q an ordered basis in such a manner that by adding v to this basis as the lth vector, this new basis has the same orientation as our main basis of $T_p(S) \times R$ (i.e., the linear map which carries our new jth basis vector to the jth vector in the main basis has positive determinant). See Figure 14. Since rank $Df(z) = l - 1$ and since $Df(z)v = 0$, $Df(z)\colon T_p(S) \times R \to T_p(S)$ maps Q onto $T_p(S)$ in a one-to-one manner and so $Df(z)|Q$ has a nonzero determinant. It is a simple argument (use the continuity of the determinant function) to verify the sign of this determinant is independent of the choice of Q and of the choice of ordered basis of Q and $T_p(S)$. We will call the sign of the determinant "sign v."

If $\#A = l - 1$, then S_A is a single point $p_A \in S$ and $I_A = \{p_A\} \times [0, 1]$. In this case if v is tangent to I_A and is not zero, sign v has the natural meaning.

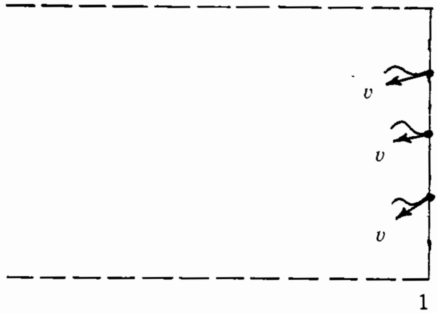

FIGURE 15

Define the index of z relative to A and v, written as $i(z, A, v)$, as $(-1)^{\#A} \cdot \text{sign}\, v$.

Of course, we always have $i(z, A, v) = -i(z, A, -v) \neq 0$.

Let $p \in S$ be an equilibrium price vector of our given economy $(\mathscr{A}, f_1)$. Then, $(p, 1) \in I_{A(p)}$. Denote $(p, 1)$ by z. For a tangent vector $v \in T_p(S_{A(p)}) \times R$ to $I_{A(p)}$ at z, let's study $i(z, A(p), v)$. Since $(\mathscr{A}, f)$ is regular, the last coordinate of v—denoted v_t—must be different from zero (see Figure 15). Suppose $v_t > 0$. Letting $Q = T_p(S)$, we can then take as basis for Q the main basis $w_1(p), \ldots, w_{l-1}(p)$, of $T_p(S)$. Hence, if $v_t > 0$,

$$i(z, A(p), v) = i(p) = (-1)^{\#A(p)} \times \text{sign}\det Df_1^{A(p)}(p)$$

(with the convention that $\text{sign}\det Df_1^{A(p)}(p) = 1$ if $\#A(p) = l - 1$). Of course, if $v_t < 0$, $i(z, A(p), v) = -i(p)$.

7.4. Fitting together different I_A. Let $\mathscr{V} = \{z \in S \times [0,1] | z \in I_A \cap I_{A'},\ A \neq A'\}$. For reasons to become clear in the next section, points of $\mathscr{V}$ shall be called *switch points*.

Let $z = (p, t) \in I_A$ for some $A \subset \mathscr{A}$ with $\#A \leq l - 1$. So, $f(p,t) \in L_A$. Then, $A \subset A(p) = \{a \in \mathscr{A}\colon\ p \cdot a = 0\}$ and $z \in I_{A(p)}$. Hence, we can write $f^{A(p)}(p, t) = \sum_{a \in A(p)} \alpha_a a$ in a unique manner, by our assumption H.12 on $\mathscr{A}$. Put $A'(p) = \{a \in A(p)\colon \alpha_a \neq 0\}$. Then, $A'(p) \subset A$ and $z \in A'(p)$ also. Therefore, if

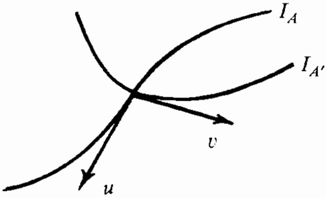

FIGURE 16

$z \in V$, we must have $A'(p) \neq A(p)$. Let $A' = A'(p)$ and $A = A(p)$. Since $A' \subset A$, $S_A \subset S_{A'}$ and $\#A' + 1 \leq \#A$. By the regularity hypothesis H.21, $Df^{A'}(z)$ maps $T_p(S_A) \times R$ onto $T_p(S_{A'})$. Therefore, $\dim[T_p(S_A) \times R] \geq \dim T_p(S_{A'})$, or $(l-1) - \#A + 1 \geq (l-1) - \#A'$, or $\#A' + 1 \geq \#A$. We conclude that $\#A' + 1 = \#A$ and that $A(p) = A'(p) \cup \{a\}$ for some $a \in \mathscr{A} \setminus A'(p)$. In words, the only way that (p, t) may belong to two distinct I_A, $I_{A'}$, is for one of $\{A, A'\}$ to be $A(p)$ and the other to be $A(p)$ with exactly one activity vector a removed.

Let $z = (p, t) \cup V$. Set $A' = A'(p)$ and $A = A(p)$ with $A = A' \cup \{a\}$. Take $v \neq 0$ (resp. $u \neq 0$) to be a vector tangent to $I_{A'}$ (resp. I_A) at z, i.e., $v \in T_p(S_{A'}) \times R$ and $Df^{A'}(z)v = 0$ (resp., $u \in T_p(S_A) \times R$ and $Df^A(z)u = 0$). See Figure 16.

We claim that (i) $(a, 0) \cdot v \neq 0$, and (ii) $a \cdot Df^{A'}(z)u \neq 0$. Indeed,

(i) if $(a, 0) \cdot v = 0$, then $v \in T_p(S_A) \times R$ because $v \in T_p(S_{A'}) \times R$. However, $v \neq 0$ and $Df^{A'}(z)v = 0$. This contradicts the fact that $Df^{A'}(z)$ maps $T_p(S_A) \times R$ one-to-one and onto $T_p(S_{A'})$, (by H.21 and $\dim(T_p(S_A) \times R) = \dim T_p(S_{A'})$).

(ii) if $a \cdot Df^{A'}(z)u = 0$, then $Df^{A'}(z)u$ is perpendicular to a and lies in $T_p(S_A)$. Hence, $Df^A(z)u = Df^{A'}(z)u$. So, $Df^{A'}(z)u = 0$, $u \in T_p(S_A) \times R$, and $u \neq 0$, which is impossible for the same reason as in (i).

Let us interpret (i) and (ii). At z the activity a breaks even and it is operated at zero level; (i) says that as we move along $I_{A'}$ in the v direction we pass from the region of positive (or negative) potential profits of the activity to the region of negative (or positive) poten-

FIGURE 17

tial profits without even infinitesimally dwelling in the region of zero profits. The interpretation of (ii) is entirely analogous with respect to the level of operation of the activity.

An implication of (i) and (ii) is that any two I_A, $I_{A'}$ cross at most a finite number of times. Hence, V is a finite set.

7.5. Defining some piecewise smooth paths. Pick any $z \in E \cap (\partial(S \times [0,1]))$ i.e., $z = (p, t) \in E$ and $t \in \{0,1\}$. Then $z \in I_{A(p)}$ and $z \notin V$. Since z belongs to the boundary of $S \times [0,1]$, see Figure 17, the orientation of a tangent vector $v \in T_p(S_{A(p)}) \times R$ to $I_{A(p)}$ at z, which is required to point inwards, is well determined.

We begin now a path at z by following $I_{A(p)}$ in the v direction. We claim that if the rule of never leaving the E region is enforced there is a uniquely determined path beginning at z (i.e., there is never the possibility of bifurcation) and this path necessarily terminates at a point of the boundary of $S \times [0,1]$ *distinct* from z. Indeed, z is an endpoint of a component of $I_{A(p)}$ which is a segment, we now show that the other endpoint belongs also to the boundary of $S \times [0,1]$ (see Section 3). The path starting at z follows this segment "at a steady pace." Subject to the constraint of staying in E, only two things are possible: (i) a point of V is reached, (ii) the other endpoint of the segment is reached. In case (ii) we are done. Consider case (i).

Let $z' = (p', t') \in V$ be the point reached and take $v_{z'}^-$ to be the tangent vector to $I_{A(p)}$ in the direction of arrival (i.e., in the direction compatible with v); see Figure 18. At z', $A(p') = A(p)$ and $A'(p') = A(p') \setminus \{a\}$ for an activity $a \in \mathscr{A}$. Denoting $A =$

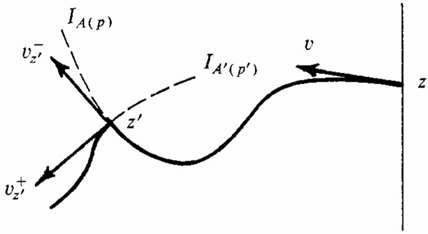

FIGURE 18

$A(p')$, $A' = A'(p')$, we saw in Section 4 that $a \cdot Df^{A'}(z')v_{z'}^- \neq 0$ which because we arrive from region E implies $a \cdot Df^{A'}(z')v_{z'}^- < 0$. So, if we keep to I_A we are led to a negative level of activity for a, i.e., we abandon region E. There is no choice but to switch from I_A to $I_{A'}$. In which direction? In Section 4, we saw that if u is tangent to $I_{A'}$ at z' then $(a,0)u \neq 0$. So, there is only one direction leading to negative potential profits for a, i.e., we must choose $v_{z'}^+$, the exit direction from z', so that $(a,0)v_{z'}^+ < 0$. We have now a new starting point and a new starting direction to follow. The ruling principle is always the same: if a switch point is reached, either a new activity enters and we join it to the basis and follow in the direction where the activity comes into operation, or an activity reaches zero level of operation and we drop it from the basis and follow in the direction where it would give potential losses. There are only a finite number of points in V and, subject to the constraint of staying in E, the path constructed never has bifurcations or crossroads, i.e., every point of V can be visited at most once. Therefore, the path must terminate and since it can only do so at the boundary of $S \times [0,1]$ we obtain our conclusion. See Figure 19.

We shall prove in the next section:

INVARIANCE OF INDEX PROPERTY. *Let $\bar{z} = (\bar{p}, \bar{t})$ be the end-point of the path started at $z = (p,t)$ in the direction v. Let $\bar{v}$ be the arrival direction at $\bar{z}$ (see Figure 19). Then*

$$i(z, A(p), v) = i(\bar{z}, A(\bar{p}), \bar{v}).$$

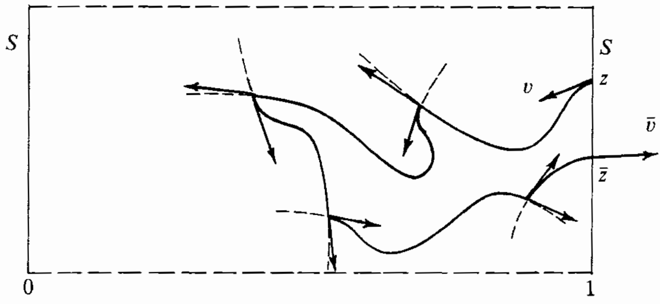

FIGURE 19

The invariance of index property yields the main theorem. Indeed, take first $z = (q, 0)$ as starting point of our path. Then, the corresponding ending point $\bar{z}$ must necessarily be of the form $(\bar{p}, 1)$—remember $E \cap \partial(S \times [0,1]) = (q, 0)$—i.e., $\bar{p}$ is an equilibrium of $(\mathscr{A}, f_1)$. Since v points inward, $\bar{v}$ must point outward, i.e., $v_t > 0$, $\bar{v}_t > 0$, see Figure 20. Therefore, $i(\bar{p}) = i(\bar{z}, A(\bar{p}), \bar{v}) = i(z, \varnothing, v) = (-1)^{l-1}$ because, given $v_t > 0$, $i(z, \varnothing, v)$ equals the index of q in the economy $(\mathscr{A}, f_0)$. Any path starting at $z \in S \times \{1\}$, $z \neq \bar{z}$, can only terminate at $S \times \{1\}$ (the endpoint $(q, 0)$ has already been preempted by $(\bar{p}, 1)$ and only one path begins, hence ends, at $(q, 0)$). So, letting aside $\bar{p}$, the equilibria of $(\mathscr{A}, f_1)$ are associated in pairs. Consider one of those pairs, say p' and p''; there is a path connecting $z' = (p', 1)$ and $z'' = (p'', 1)$. The exit

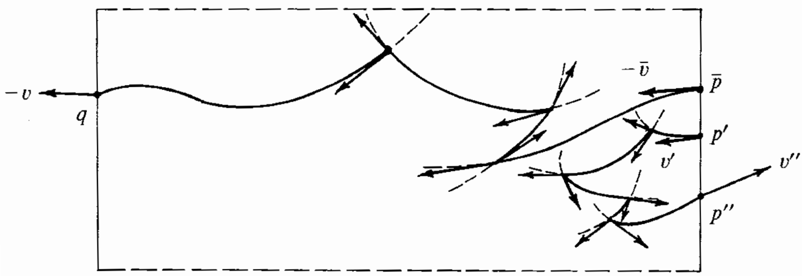

FIGURE 20

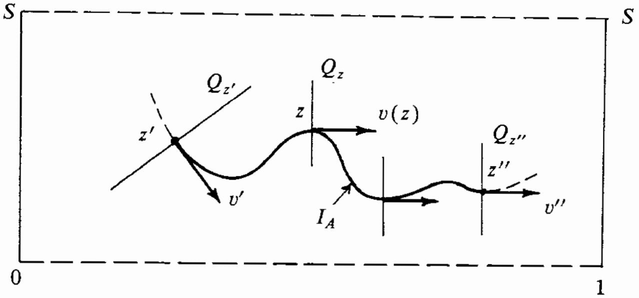

FIGURE 21

direction v' at z' points inwards, i.e., $v'_t < 0$, while the arrival direction at z'' points outwards, i.e., $v''_t > 0$. See Figure 20. So, $i(p') = -i(z', A(p'), v') = -i(z'', A(p''), v'') = -i(p'')$. Therefore,

$$\sum_{p \in E(\mathscr{A}, f)} i(p) = (-1)^{l-1}.$$

7.6. Proof of the invariance of index property. We first note that for any $A \subset \mathscr{A}$, $\#A \leq l-1$, if two $z', z'' \in I_A$ belong to the same component of I_A and v', v'' are tangent vectors to I_A at z', z'', respectively, which are oriented in a compatible manner, then $i(z', A, v') = i(z'', A, v'')$. Indeed, there is a nonzero vector field $v(z)$ along the component of I_A having $v(z') = v'$, $v(z'') = v''$, see Figure 21. Hence, if for any $z = (p, t)$ we let Q_z be the orthogonal complement of $v(z)$ in $T_p(S_A) \times R$, the determinant of the map $Df(z)$: $Q_z \to T_p(S_A)$ moves continuously with z and cannot vanish. So, it cannot change signs, i.e., $i(z', A, v') = i(z'', A, v'')$.

Therefore, it is the behavior of the index at switch points which must occupy us.

Let $z = (p, t) \in V$ be reached along the path. Denote by A' (resp. A) the base associated with the incoming (resp. outgoing) paths and v^- (resp. v^+) the arrival (resp. exit) direction. We need to show that $i(z, A', v^-) = i(z, A, v^+)$.

Without loss of generality, we can take $A = A' \cup \{a\}$. Since $a \in T_p(S_A)$ we can assume that the main base of $T_p(S_A)$ is orthogonal and has, as the first coordinate, the vector a. For notational ease we put $T = T_p(S_{A'})$, $T_a = T_p(S_A) = \{u \in T\colon a \cdot u = 0\}$.

Set $h = \dim T = (l - 1) - \#A' \geq 1$. Let B be the $h \times (h + 1)$ matrix of the linear map $Df^{A'}(z)\colon T \times R \to T$. We can write B as

$$B = \begin{bmatrix} \beta & \eta & \xi \\ \gamma & \hat{B}_a & \theta \end{bmatrix},$$

where β is a number, η is an $(h - 1)$ row vector, ξ is a number, $\hat{B}_a$ is a $(h - 1) \times (h - 1)$ matrix corresponding to T, and γ and θ are $(h - 1)$-column vectors.

Similarly, let B_a be the $(h - 1) \times h$ matrix of the linear map $Df^A(z)\colon T_a(S^A) \times R \to T_a$. We can write B_a as

$$B_a = \begin{bmatrix} \hat{B}_a & \theta \end{bmatrix}.$$

If $h = 1$ (i.e., T is one-dimensional), then $\hat{B}_a$, η, γ, and θ are empty symbols. Let $\hat{B} = \begin{bmatrix} \beta & \eta \\ \gamma & \hat{B}_a \end{bmatrix}$. The indices $i(z, A', v^-)$ and $i(z, A, v^+)$ are properties of the matrices B and B_a and of the vectors v^- and v^+. Since both indices are different from zero, they are invariant under a small perturbation of the matrices B and B_a.

Carrying out, if necessary, such a small perturbation, we can assume that $\hat{B}$ and, if $h > 1$, $\hat{B}_a$ are nonsingular. Since $B_a v^+ = 0$ and $\hat{B}_a$ is nonsingular, $v_t^+ \neq 0$. Since $Bv^- = 0$ and $\hat{B}$ is nonsingular, $v_t^- \neq 0$. Then, in order to compute the index, we can take as complementary space to v^- (resp. v^+) the space T (resp. T_a) itself. If $v_t^- > 0$ (resp. $v_t^+ > 0$), we can take as oriented basis in T (resp. T_a) the main basis itself. So, if we take into account that $i(z, A', v^-) = -i(z, A', -v^-)$ and $i(z, A, v^+) = -i(z, A, -v^+)$, we can conclude

$$i(z, A', v^-) = (-1)^{\#A'} \operatorname{sign}(v_t^-) |\hat{B}|$$

$$i(z, A, v^+) = (-1)^{\#A} \operatorname{sign}(v_t^+) |\hat{B}_a|,$$

where $|\ |$ denotes determinant. If $\hat{B}_a$ is an empty symbol (i.e., $h = 1$), then $|\hat{B}_a| = 1$.

Since $(-1)^{\#A} \cdot (-1)^{\#A'} = -1$, $i(z, A', v^-) \cdot i(z, A, v^+) = -\operatorname{sign}[(v_t^-) \cdot (v_t^+) \cdot |\hat{B}| \cdot |\hat{B}_a|]$. To demonstrate $\operatorname{sign} i(z, A', v^-) = \operatorname{sign} i(z, A, v^+)$, we need only show that

$$\operatorname{sign}(v_t^-)(v_t^+) = -\operatorname{sign} |\hat{B}| \cdot |\hat{B}_a|.$$

Hence, the following Lemma concludes the proof.

LEMMA. $\operatorname{sign}(v_t^-)(v_t^+) = -\operatorname{sign} |\hat{B}| \cdot |\hat{B}_a|$.

Proof. Recall that $(a, 0)$ is the first basis vector of $T \times R$. Since v^- is the direction coming from the region E of economic equilibria, on $I_{A'}$, the new activity a enters the picture at z (as argued in Section 7.5) and we must have $v_1^- > 0$. Similarly, v^+ is the direction going to E and we must have $(Bv^+)_1 > 0$, where $(Bv^+)_1$ is the first coordinate of Bv^+.

Put $\hat{v}^- = (\operatorname{sign} v_t^-)v^-$ and $\hat{v}^+ = (\operatorname{sign} v_t^+)v^+$. Then, $\hat{v}_t^- > 0$, $\hat{v}_t^+ > 0$, and $\operatorname{sign}(\hat{v}_1^-)(B\hat{v}^+)_1 = \operatorname{sign}(v_t^-)(v_t^+)$. We will show that $\operatorname{sign}(\hat{v}_1^-)(B\hat{v}^+)_1 = -\operatorname{sign}|\hat{B}||\hat{B}_a|$.

We can take $\hat{v}_t^- = \hat{v}_t^+ = 1$. Consider the equation

$$B\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \beta & \eta & \xi \\ \gamma & \hat{B}_a & \theta \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} w \\ 0 \end{bmatrix},$$

where u and w are scalars and y is an h-column vector. If $y_t = 1$, we have that

$$\hat{v}_1^- = u \quad \text{when} \quad w = 0,$$
$$(B\hat{v}^+)_1 = w \quad \text{when} \quad u = 0.$$

Let $\bar{w}$ be the solution of the equation with $u = 0$. Then $\bar{w} = -\eta\hat{B}_a^{-1}\theta + \xi$ (where if $h = 1$ an "empty" term of the sum such as $-\eta\hat{B}_a^{-1}\theta$ is taken to be zero. This comment applies as well to similar situations in the next paragraph).

Let $\bar{u}$ be the solution of the equation with $w = 0$. If we denote by (δ, ω)—δ being a number—the first row of $\hat{B}^{-1}$ then $\bar{u} =$

$-(\delta\xi + \sum_{i=2}^{h} \omega_i \theta_i)$. More explicitly:

$$\delta = \frac{|\hat{B}_a|}{|\hat{B}|}$$

and

$$\omega_i = \frac{(-1)^{i+1}}{|\hat{B}|}\left(\sum_{k=2}^{h} (-1)^k \eta_k |\hat{B}^T_{a(k,i)}|\right)$$

where $\hat{B}^T_a$ is the transpose of $\hat{B}_a$ and $\hat{B}^T_{a(k,i)}$ is the matrix obtained from $\hat{B}^T_a$ by deleting the kth row and ith column. Remember, also, that if $h = 1$ then $|\hat{B}_a| = 1$.

Therefore,

$$\bar{u} = -\frac{1}{|\hat{B}|}\left(|\hat{B}_a|\xi + \sum_{i=2}^{h}\sum_{k=2}^{h} (-1)^{i+k+1} \eta_k \theta_i |\hat{B}^T_{a(k,i)}|\right)$$

or, simply,

$$\bar{u} = -\frac{|\hat{B}_a|}{|\hat{B}|}\left(\xi - \eta \hat{B}_a^{-1}\theta\right).$$

We conclude that

$$\overline{uw} = -\frac{|\hat{B}_a|}{|\hat{B}|}\left(\xi - \eta \hat{B}_a^{-1}\theta\right)^2$$

and so,

$$\operatorname{sign}(\hat{v}_1^-)(B\hat{v}^+)_1 = -\operatorname{sign}|\hat{B}_a||\hat{B}|. \qquad Q.E.D.$$

7.7. Final comment. The reader will have noticed that the previous proof has a constructive and computational flavor. In fact, the path constructed from $(q, 0)$ and arriving at an equilibrium of the given economy could be (approximately) followed either by complementarity pivot methods (Scarf [42], Eaves [19], Eaves-Scarf

[20]) or by numerical solution methods for differential equations (see Kellogg, Li and Yorke [32] and Smale [47]).

It should come as no surprise that we get an existence result without appealing to any fixed point theorems. It was shown by Hirsch [28] (see also Milnor [38] pg. 14) that Brouwer's fixed point theorem can be obtained via Sard's theorem.

REFERENCES. The path following procedure of the proof is well known. Mathematically, it is a quite standard homotopy argument. The same applies to the orientation arguments used. The interested reader may consult the book by Milnor [38] which has had a substantial influence in mathematical economics.

In the context of theoretic simplicial methods of computation, Eaves [19] pointed out the usefulness of adding a parameter and appealing to homotopy arguments. The technique has been further developed and applied to the computation of solutions of piecewise linear equations by Eaves and Scarf [20]; their paper also contains an analysis of the index.

Kellogg, Li and Yorke [32] and Smale [47] have proposed algorithms for differentiable functions based on differential equations but not relying on the addition of a parameter. The one-parameter trick seems to have the advantage that if we actually want to compute a solution then, a priori information on its approximate location can be exploited by appropriately choosing the function at $t = 0$.

Our problem is a mixture of differentiable and piecewise linear and this is reflected in the piecewise differentiable nature of our solution paths.

A simplicial type algorithm to compute equilibria of economies with linear activities was given by Scarf in his book [42], Ch. 5. Different algorithms for piecewise linear economies specified by production activities and utility functions (i.e., as Example 1) rather than excess demand functions have been recently proposed by Dantzig, Eaves and Gale [6], Wilson [56] and Elken [21]. The latter contains an extensive survey of path-following methods.

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