NOTES ON PRICE AND QUANTITY TÂTONNEMENT DYNAMICS*

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1. Introduction

1.1 In these notes we present an exposition of some modern versions of L. Walras' tâtonnement theory (see (1926)) with special emphasis on simultaneous price and quantity dynamics.

1.2 Walras proposed his theory as an approach to the determination of equilibrium prices and productions which rested on dynamic laws intended to mimic the actual functioning of competitive markets. His main laws were two: (I) prices move according to the difference of demand and supply ("loi de l'offre et la demande"), and (II) production moves according to the difference of price and cost ("loi de prix de revient").

1.3 By modern versions we mean the formalization via differential equations first suggested by Samuelson (1947) and pursued by Arrow-Hurwicz (1958). Although a convenient and powerful approach, there is no intrinsic reason why Walras' laws should be embodied into differential equations. Difference equations have a long tradition in economics (e.g. the cobweb). A stochastic treatment would also be quite natural.

1.4 The literature on the tâtonnement is very extensive. See the above reference and also the recent survey by Hahn (1982). Much of the research, however, has concentrated on the limit pure price dynamics (with production, if at all there, automatically adjusted to equilibrium) or, to a lesser extent, on the limit pure quantity dynamics (see the contribution of Novshek and Sonnenschein to this volume). The general case treated here, where prices and quantities stand on the

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same footing, is much less familiar and this in spite of the fact that, as we shall see, it displays interesting and, relative to the two limit cases, novel and illuminating dynamic features. So, we feel this exposition is justified. Of course, there is some work on price-quantity dynamics, e.g. Arrow and Hurwicz (1960), which is relevant. More directly related to these notes are Morishima (1959), Beckmann-Ryder (1969) and Mas-Colell (1974).

1.5 There is a key aspect to tâtonnement dynamics which needs to be emphasized. Actual production and trade only takes place at equilibrium. Outside of it all production and price "decisions" are to be thought of as tentative plans. Thus, it is better not to visualize the dynamics as engaging the economy in real time. For a criticisim of this and alternative approaches with production and trade out of equilibrium see the recent book by Fisher (1983).

2. The Basic Economic Model

2.1 There are ℓ perfectly divisible commodities and we take the economy as composed of a production and a consumption side.

2.2 The production side is composed of n sectors, each one of which can be thought of as an aggregate of several firms. Every sector j is characterized by a convex and closed set of technically feasible input-output vectors $Y_j \subset R^{\ell}$. According to customary convention the negative (resp. positive) entries of a vector $y_j \in Y_j$ stand for inputs (resp. outputs). Denote by $y = (y_1,\ldots,y_n) \in \prod_j Y_j$ the economy wide vector of productions.

2.3 Except for section 5 we shall be interested in the more particular production model where every sector produces only one output (this is called the no joint production model). Without loss of generality we assume that every sector produces a different good. Choosing indices appropriately we can view the j-sector as producing the j good and we can express its technology by a concave production function $g_j\colon R_+^{\ell} \to R_+$. We assume that $m < \ell$, i.e., there are non producible factors of production.

2.4 An important particular case is when the technology exhibits constant returns to scale, i.e. if $y_j \in Y_j$ then $\lambda y_j \in Y_j$ for any $\lambda > 0$. In the no joint production model this translates into the homogeneity of degree one of the production functions g_j.

2.5 Every commodity has a price. We fix the price of the ℓ-th commodity (to be thought of as, say, labor services) to be equal to one. The price vector for the first $\ell - 1$ commodities is denoted p. Given p we let $\hat{p} = (p,1) \in R^{\ell}$.

2.6 Given y_j and p the profit (or loss) of the j-sector is simply $\pi_j(y_j,p) = \hat{p} \cdot y_j$. Given y and p the overall vector of profits is $\pi(p,y) = (\hat{p}\cdot y_1,\ldots,\hat{p}\cdot y_m) \in R^m$.

2.7 The demand side of the economy is given to us in a highly reduced form by means of an excess demand function $f(p,\pi) \in R^{\ell-1}$ which assigns net demands (positive entries) or supplies (negative entries) for the first $\ell - 1$ commodities to every $\ell - 1 + m$ vector of prices and profits of the different production sectors $\pi \in R^m$. The excess demand of the ℓ-th commodity is implicitly taken to be $\sum_{j=1}^{m} \pi_j - p\cdot f(p,\pi)$. The complete excess demand system, inclusive of the ℓ-th commodity, is denoted $\hat{f}(p,\pi)$. Of course, it satisfies $\hat{p}\cdot\hat{f}(p,\pi) = \sum_{j=1}^{m} \pi_j$.

2.8 An excess demand function satisfies the <u>Representative Consumer Hypothesis</u> (RC) if, for every p, π, $f(p,\pi)$ could have been generated by a consumer maximizing a utility function $u(z)$ subject to a budget constraint $\{z: p\cdot z < \sum_{j=1}^{m} \pi_j\}$.

2.9 An excess demand function f satisfies the <u>Weak Axiom of Revealed Preference</u> (WA) if <u>whenever</u> $f(p,\pi) \neq f(p',\pi')$ <u>and</u> $\hat{p}\cdot\hat{f}(p',\pi') < \sum_j \pi_j$ <u>we have</u> $\hat{p}'\cdot\hat{f}(p,\pi) > \sum \pi_j$, i.e. if $f(p,\pi)$ is "revealed preferred" to $f(p',\pi')$ then $f(p',\pi')$ cannot be revealed preferred to $f(p,\pi)$. Obviously, the RC Hypothesis implies the WA but the converse need not be the case (except for $\ell = 2$). In fact, as an hypothesis predicated on observed demand behaviour RC is pretty implausible as its satisfaction depends on the fulfillment of delicate integrability conditions. The same cannot be said of the WA. The WA has a clear economic interpretation as a sort of Generalized Law of Demand (more on this later) and while its fulfillment in the aggregate is known to be restrictive it is, at least, a robust property (for the appropriate notions of perturbations). It is just possible that, as an empirical matter, it may hold. A good background reference for all of this is Shafer-Sonnenschein (1982).

2.10 It is time to define the concept of Walrasian equilibrium. Neglecting the possibility of zero prices a pair (p,y) will constitute an equilibrium if demand is equal to supply and every firm maximizes profits. Formally:

Definition: The pair (p,y) constitutes an equilibrium if:

(I) $\hat{f}(p,\pi(p,y)) = \sum y_j$, and

(II) $\hat{p}\cdot y_j > \hat{p}\cdot z$ for all $z \in Y_j$ and j.

2.11 Let $F(p,y) = f(p,\pi(p,y))$. Assuming the F is C^1 denote by $S(p,y)$ the $(\ell - 1) \times (\ell - 1)$ matrix $\partial_p F(p,y)$. Consider the property:

(NQD). The matrix $S(p,y)$ is negative quasidefinite, i.e. $v\cdot S(p,y)v < 0$ for $v \neq 0$.

Under NQD the diagonal entries of $S(p,y)$ are negative. Hence, the Law of Demand holds at (p,y) for the first $(\ell - 1)$ commodities in the sense that demand increases if price decreases (y remains fixed but profit income is adjusted when p changes). Suppose that the Law of Demand were to hold at (p,y) for any possible choice of coordinates (i.e. for any way to define composite commodities among the first $(\ell - 1)$). It can be checked that this Generalized Law of Demand is precisely the NQD condition.

2.12 Suppose that (p,y) is an equilibrium. Then the WA implies the slightly weaker version of NQD where the strict inequality is replaced by a weak one. Conversely, NQD implies the WA in a neighborhood of (p,y). Thus, roughly speaking, in a vicinity of equilibrium the WA and NQD amount to the same condition. See Khilstrom, Mas-Colell and Sonnenschein (1976) for all this.

3. Dynamics

3.1 We will now formalize the dynamic laws proposed by Walras for the determination of an equilibrium. The situation studied by Walras was one of no joint production and constant returns. The no joint production aspect does indeed make matters conceptually simpler. Hence, we shall follow his example in this respect. Section 5 contains comments on the general case.

3.2 We first introduce some notation and reformulate the notion of equilibrium for the no joint production case. Let $x \in R^m$ be the vector of gross productions for the m outputs. Given p and x_j denote by $C_j(p,x_j)$ the minimum cost of producing x_j. Assuming that $C_j(\cdot)$ is differentiable we put $c_j(p,x_j) = \partial_{x_j} C_j(p,x_j)$ and $c(p,x) = (c_1(p,x_1),\ldots,c_m(p,x_m))$. Neglecting the possibility of zero production the maximization of profits by sector j is equivalent to the condition $p_j = c_j(p,x_j)$. Denote $p_M = (p_1,\ldots,p_m)$. Given p and x_j let $y_j(p,x_j) \in R^{\ell-1}$ (resp. $\hat{y}_j(p,x_j) \in R^{\ell}$) be the cost minimizing plan for the first $\ell - 1$ commodities (resp. for all of them). Put $y(p,x) = \sum_j y_j(p,x_j)$ and accordingly for $\hat{y}(p,x)$. The equilibrium conditions (i) and (ii) then take the form:

$$\text{(I)} \quad F(p,\hat{y}(p,x)) - y(p,x) = 0 \qquad \text{(Demand = Supply)}$$

$$\text{(II)} \quad p_M - c(p,x) = 0 \qquad \text{(Maximum Profits)}$$

3.3 In part II, section 3, of the Eléments Walras proposes as a counterpoint of (I) and (II) two dynamic principles for an economy which at a tentative (p,x) is not in equilibrium:

(I') If the planned demand of a commodity is larger (resp. smaller) than the planned supply then the tentative price increases (resp. decreases). This is the "loi de l'offre et la demande".

(II') If the tentative price of a commodity is larger (resp. smaller) than the (marginal) cost then the tentative planned production increases (resp. decreases). This is the "loi du prix de revient".

Neither production nor trade does actually take place until equilibrium has been reached. Thus the term "tâtonnement" dynamics. For the purposes of local analysis and given the equilibrium conditions (I) and (II) it is now fairly clear how to formalize the dynamic principles (I') and (II') by means of differential equations (the use of differential equations for the study of Walrasian dynamics goes back to Samuelson (1947)). The state space is constituted by the price and quantity variables (p,x). The dynamic equations are:

$$(I') \quad \dot{p} = K(F(p,\hat{y}(p,x)) - y(p,x))$$
$$(II') \quad \dot{x} = Q(p_M - c(p,x))$$

where K and Q are positive diagonal matrices of speeds of adjustment. Economic considerations do not give many clues about the values of K and Q. It is therefore important that positive results do not depend too precisely on them.

3.4 Suppose that around an equilibrium we can use equation (II) to solve for productions as function of prices, i.e., x(p) (a salient situation where this cannot be done is the constant returns case). Substituting x(p) for x in (I') we then obtain a <u>pure price dynamics</u> of the form $\dot{p} = G(p)$. This is the type of tâtonnement dynamics which has been most extensively studied (see Hahn (1982) for a survey). It can legitimately be considered as the limit of the general dynamics (I')-(II') obtained by letting quantitites adjust much faster than prices.

3.5 Suppose that around an equilibrium we can use equation (I) to solve for prices as a function of productions, i.e. p(x). Substituting p(x) for p in (II') we then obtain a <u>~~price~~ pure quantity (or Marshallian) dynamics</u> of the form $\dot{x} = G(x)$. The symmetry with the pure price dynamics is not quite complete. There is at least one important difference. The ~~price~~ pure quantity dynamics cannot always be legitimately considered as the limit of the general dynamics (I')-(II') obtained by letting prices adjust much faster than quantities. For it to be so a further and substantial requirement is needed, namely that for x fixed the dynamic systems (I') must have p(x) as a locally asymptotically stable equilibrium (the corresponding condition for the pure price dynamics can be proved to be automatically satisfied).

4. <u>Local Stability</u>

4.1 To analyze the local asymptotic stability of (I')-(II') we shall assume that the system is C^1 in a neighborhood of a fixed reference equilibrium $(\bar{p},\bar{x})$. We proceed by analyzing the first order approximation to (I')-(II') at $(\bar{p},\bar{x})$.

4.2 Some facts from the duality theory of production (see Diewert (1982) for a survey) will be very helpful. Let $\beta_j(p,x_j)$ be the maximum profits of sector j if production must equal x_j, i.e. $\beta_j(p,x_j) = p_j x_j - C_j(p,x_j)$. Put $\beta(p,x) = \sum_j \beta_j(p,x_j)$. The function β is convex in p, concave in x and, assuming enough differentiability in the relevant region, $\partial_p \beta(p,x) = y(p,x)$. Hence, $\partial_p y(p,x)$ is positive semidefinite (p.s.d.). Clearly, $\partial_x \beta(p,x) = p_M - c(p,x)$ and therefore

$$\partial_p(p_M - c(p,x)) = \partial_{x,p}\beta(p,x) = (\partial_{p,x}\beta(p,x))^T = (\partial_x y(p,x))^T$$

where T denotes matrix transposition. Finally, note that $-\partial_x c(p,x) = \partial_x(p_m - c(p,x)) = \partial_{x,x}\beta(p,x)$ is negative semidefinite (n.s.d.).

4.3 At $(\overline{p},\overline{x})$ profits are maximal given $\overline{p}$. Hence, $\partial_{\hat{y}}\pi(\overline{p},\hat{y}(\overline{p},\overline{x})) = 0$. Using this and the equalities of the last paragraph the Jacobian matrix of the system (I')-(II') at $(\overline{p},\overline{x})$, denoted J, is easily seen to be the $(\ell - 1 + m) \times (\ell - 1 + m)$ matrix

$$J = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & -B \\ B & C \end{bmatrix}$$

where:
$$A = S(\overline{p},\hat{y}(\overline{p},\overline{x})) - \partial_p y(\overline{p},\overline{x})$$
$$B = (\partial_x y(\overline{p},\overline{x}))^T$$
$$C = -\partial_x c(\overline{p},\overline{x})$$

The matrix C is n.s.d. In the constant returns case $C = 0$. From now on we shall assume that B has maximal rank m. This is an extremely weak and economically sensible hypothesis. It is automatically satisfied if, for example, outputs are not required (or not "too much") for the production of other outputs. Under the NOD hypothesis on the demand function (see section 2) the matrix A will be negative quasidefinite (n.q.d.).

4.4 Suppose that the NOD hypothesis is satisfied and B has maximal rank. Then J has the form $J = LE$ where L is a positive diagonal matrix and E is non-singular and negative quasisemidefinite (i.e. $E + E^T$ is n.s.d.). These two properties of E are straightforwardly verified. Clearly, any eigenvalue of J

will then have negative real part. Hence, we have shown:

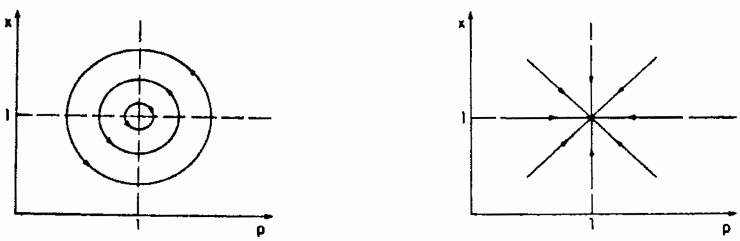
Proposition 1: If at the equilibrium $(\bar{p},\bar{x})$ the NQD condition on demand is satisfied and $\partial_x y(\bar{p},\bar{x})$ has full rank then $\bar{p},\bar{x}$ is locally asymptotically stable for any speeds of adjustment K, Q.

Trivial examples for $\ell = 2$ exhibiting violations of the Law and Demand show that the NQD condition cannot be dispensed with (see section 8). Remember from section 2 the interpretation of the NQD condition as a generalized, coordinate-free, version of the Law of Demand.

4.5 Observe that the matrix E has a sort of skew-symmetric structure. Hence even if A is n.q.d. and, therefore, J a stable matrix, some of the eigenvalues of J will typically exhibit non-zero imaginary parts, i.e. trajectories will converge to equilibrium but they will do so in a spiraling manner with prices and quantities systematically overshooting their equilibrium values. This can best be illustrated in a limit degenerate case where NQD does not hold with strict inequality and the system is stable but not asymptotically stable. Let $\ell = 2$ and suppose that every unit of output can be produced with a unit of input (hence there are constant returns). For a region around p = 1 the demand of output is constantly equal to one. Hence an equilibrium is $(\bar{p},\bar{x}) = (1,1)$ and with unitary speed of adjustmenet the dynamic system (I')-(II') reduces to

$$\begin{cases} \dot{p} = 1 - x \\ \dot{x} = p - 1 \end{cases}$$

which has the phase diagram of figure 1(a).



(a) (b)

4.6 Finally, a comment on a peculiarity of the constant returns case that we view as somewhat paradoxical. It appears most clearly for $m = \ell - 1$. Suppose first that C is nonsingular but very small. Equivalently, in a neighborhood of equilibrium we can solve (II) for the competitive "supply function" $x(p)$ and the price effects matrix $\partial x(p)$ is positive definite and very large. In this situation the pure price dynamics will be stable whether or not the NOD condition is satisfied (the price effects from the production side will dominate the possible perverse ones from the consumption side). This is well reflected in the dynamical system (I')-(II'): <u>If Q is sufficiently large relative to K then it is immediately verified that the system is locally asymptotically stable</u>. Now consider the constant returns case. From the economic point of view this is the limit situation where the production price effects are infinitely large relative to the consumption effects (in fact, $x(p)$ is not even defined. The production surface has become flat). Hence it could be thought that, with even more reason, the previous underlined statement will remain valid. But this is not so. Suppose, for example, that A is positive definite. Then the matrix in J is completely unstable (i.e. all eigenvalues have positive real parts) for any speeds of adjustment. Mathematically the source of the qualitative discontinuity is clear: the matrix C becomes singular at constant returns. Nevertheless, from the economic viewpoint the discontinuity is surprising. All this will again be illustrated in Section 8.

5. <u>Local stability with general production</u>

5.1 A key difficulty for the treatment of the general joint production case is how to formulate the profit maximization equilibrium condition (II) in a manner that lends itself to the specification of Walras' second dynamic principle. When, as in sections 3 and 4, every sector produces only one output, and input use is instantaneously adjusted to be cost minimizing, there is for every sector a single profit improving direction. Hence, except for speeds of adjustments, the dynamics of the system are completely determined by the requirement that every output moves in the profit maximizing direction. This ceases to be so if there is joint production (or, for that matter, if input used is not instantaneously adjusted).

5.2 The most general version of the second principle could be formulated as follows. Suppose that $p(t)$ and $y(t) \in Y_1, \ldots, Y_m$ are differentiable trajectories of prices and production plans. Then we shall call $p(t), y(t)$ admissible if, for all t, :

(I') $\dot{p}(t) = K(F(p,y(t)) - y(t))$

(II") for all j, $p(t) \cdot \dot{y}_j(t) \geq 0$, with equality if and only if $y_j(t)$ maximizes profits on Y_j for $p(t)$.

In rigour condition (II") should be strengthened to require that every trajectory $y_j(t)$ not be sluggish, i.e. if at $p(t)$ the potential profit improvement is significant then $p(t) \cdot \dot{y}_j(t)$ should be significantly positive.

5.3 It is a disappointing fact that even under the NQD condition an equilibrium need not be locally asymptotic stable under the dynamic restrictions (I'), (II"). In other words, there are admissible trajectories starting arbitrarily close to equilibrium and diverging from it. In this paragraph we describe an example in the setting of the limiting pure quantity dynamics (this is not restrictive). There are two outputs $x = (x_1, x_2)$ and one input. Every unit of output 1 or 2 can be produced with a unit of input. Although the technologies for the two outputs can be separated they have to be thought of as being under the same management. Without loss of generality the outputs are measured as deviations from equilibria (so that $x = 0$ is an equilibrium) and output prices $p = (p_1, p_2)$ as deviations from unit cost (so that at equilibrium $p = 0$). For productions x let $p(x)$ be the market clearing prices (with the price of the input fixed at one). The NQD condition translates into $\partial p(x)$ being n.q.d. in a neighbourhood of the origin. In the example production always takes place at the efficiency frontier. There is no difficulty in guaranteeing that $p(\ )$ can be extended to the interior of the production set in a manner consistent with NQD.

Let $p(x) = \begin{bmatrix} -3 & 5 \\ -5 & -3 \end{bmatrix} x \equiv Ax$. The matrix A is n.q.d.

Consider now $\dot{x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x \equiv Bx$ (or, equivalently, $\dot{x} = BA^{-1}p$).

Then $A^T B = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$ is p.q.d. Hence $p(x) \cdot \dot{x} = x \cdot A^T Bx > 0$ at all $x \neq 0$. So, any solution trajectory $x(t)$ is profit improving. But the eigenvalues of B are $1 \pm i$ and so, all the solutions to the linear system $\dot{x} = Bx$ are unstable. A glance at figure 2 may help to understand what is happening.

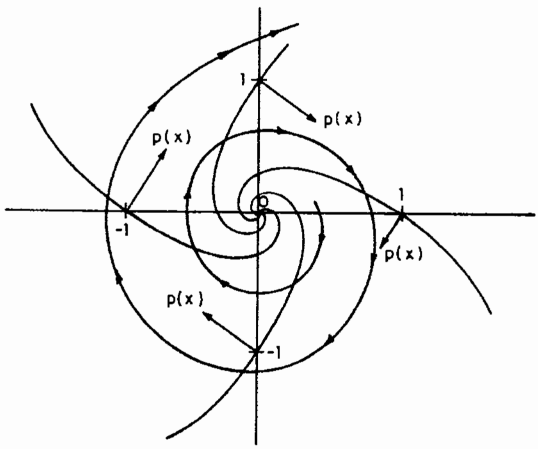


Figure 2

5.4 In the context of the previous examples (except for the particular numerical values) is there a more restrictive dynamics on $x(t)$ than (II") guaranteeing local stability? Clearly the answer is yes. Just take $\dot{x} = Qp(x)$ where Q is positive definite (e.g. a diagonal positive matrix). Then

$$V(x) = p(x) \cdot Qp(x)$$

provides a Lyapunov function. This, of course, agrees with the (II') dynamics.

5.5 It will be mentioned in Section 7 that if demand satisfies the Representative Consumer hypothesis then the pure quantity dynamics of the system (I')-(II') is stable. The contrast with the situation of the two previous paragraphs is clear and it illustrates well that there is a price to pay for relaxing the RC hypothesis to the WA. In order to obtain local stability one must, at the very least, tighten the specification of Walras' second dynamic principle from (II") to something like (II'). The precise form of this more restricted dynamics in the general join production case we do not know. A final observation: it is instructive to see in figure 2 why the RC hypothesis is violated. Under RC the matrix A would have to be symmetric and the field of perpendiculars to $p(x)$ would draw a family of closed curves from the inside of which $x(t)$ would not escape (in other words, the utility function of the RC serves as a Lyapunov function).

5.6 The observations of the previous paragraph on the (I')-(II") dynamics and the RC hypothesis referred only to the pure quantity dynamics, i.e. to K large. For the general case we do not know if this dynamics is locally stable. More formally (but not absolutely precisely):

Problem 1: Determine if under the RC hypothesis any non-sluggish and admissible for (I'), (II") trajectory $(p(t),y(t))$ must coverge to equilibrium provided it starts near enough to it.

6. Different Dynamics

6.1 In this section we take a brief glimpse at price-quantity dynamics which cannot be considered as Walrasian in nature because they do not attempt to model plausible behaviour of competitive markets. The purpose of the section is merely to make the point that price-quantity dynamics does not end with Walras' dynamic principles.

6.2 Suppose there was a planner that had to devise a dynamic system to reach equilibrium. Would he come up with Walras' principles (I')-(II')? It is unlikely. If one forgets about the underlying competitive market economics the system (I')-(II') has to appear as a peculiar one because of its indirectness. Indeed, prices react to quantities and quantities to prices. This indirectness is, incidentally, the source of the skew symmetric structure of the matrix J and therefore of the spiraling trajectories. Our planner would, more likely, first think of a <u>direct</u> method where quantities produced react to demand and prices move to match marginal costs. The contrast between the indirect and the direct approach is most striking as applied to the example in 4.5. For the same data a direct dynamics would be

$$\begin{cases} \dot{p} = 1 - p \\ \dot{x} = 1 - x \end{cases}$$

which has the phase diagram of figure 1b).

6.3 Is there anything general in the stability of the previous example for the direct method? First of all let us formalize the latter. Suppose that we are in the no joint production world of section 3 and 4. To take the simplest case assume also that $m = \ell - 1$, i.e. there is a single nonproducible factor of production. Then we can define the dynamics of the direct method by

$$(I^*) \qquad \dot{p} = K(c(p,x) - p)$$

$$(II^*) \qquad \dot{x} = Q(F(p,\hat{y}(p,x)) - y(p,x))$$

The Jacobian matrix of the system (I^*)-(II^*) at the equilibrium $(\bar{p},\bar{x})$ is

$$J^* = \begin{bmatrix} K & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} -B & -C \\ A & -B \end{bmatrix}$$

where the matrices A, B and C are as in 4.3. We shall make a further hypothesis on the matrix $B = (\partial_x y(\bar{p},\bar{x}))^T$ which is economically natural and quite standard in the input-output analysis of production (e.g. Nikaido (1968)): <u>B^T has positive diagonal and negative off-diagonal entries. Moreover, B is a productive, i.e. there is a strictly positive vector v such that $B^T v$ is a strictly positive vector.</u> In words, the production structure is such that, assuming cost minimization, an increase in the production of an output leads to a net surplus of the output and to an increase in the use of every input (remember that input use is measured in negative units). Also, there is some way to increase the production of every output that leads to a positive net surplus for each of them. It is well known (e.g. Nikaido (1968)) that any matrix with the properties of B (as, for example, KB) is positive quasidefinite. This immediately yields the following result:

Proposition 2: Suppose that:

(i) there is no joint production and a single nonproducible factor of production;

(ii) the matrix $(\partial_x y(\bar{p},\bar{x}))$ has a positive diagonal, negative off-diagonal entries and is productive; and

(iii) there are constant returns to scale, i.e., C = 0.

Then the Jacobian matrix J^* of the system (I*)-(II*) is stable. This is true independently of the particular matrix A.

The conditions of this proposition are precisely the hypothesis of the famous Non-Substitution Theorem (see, for example, Nikaido (1968)) asserting that equilibrium prices can be determined independently of the demand side of the economy. It is interesting that the stability of the Non-Substitution equilibrium with independence of the demand side follows for (I^*)-(II^*) but not, as we saw in 4.6, for (I')-(II'). If we depart from the hypothesis of the proposition by, for example, allowing $C \neq 0$, it is entirely possible, however, for (I^*)-(II^*) to be unstable even if the matrix A is symmetric negative definite (hence not even the RC hypothesis guarantees the stability of (I^*)-(II^*) in the non-constant returns case). As an example let $\ell - 1 = 2$, $K = \Omega = I$ and put

$$J^* = \begin{bmatrix} -20 & 16 & 20 & -4 \\ 1 & -2 & -4 & 1 \\ -20 & -5 & -20 & 16 \\ -5 & -2 & 1 & -2 \end{bmatrix}$$

Note that, for $v = (0,-1,1,1)^T$, $J^*v = v$. Hence, J^* has a real, positive eigenvalue.

6.4 The general setting for the comparative study of the properties and domains of stability of systems such as (I')-(II') or (I^*)-(II^*) is an abstract theory of decentralized adjustment mechanisms (indeed, a system such as (I^*)-(II^*), although not market-like, seems intuitively as decentralized as (I')-(II')). The formulation of such a theory has been rigorously initiated by Jordan (1982) (see, also, Saari-Simon (1978)).

6.5 It is important not to lose sight that Walras' dynamics (I')-(II') deal with planned production and tentative prices. It does not take place in real time. There is, for example, something strange in thinking of an actual expansion of production in a situation that may be of excess supply. Real time dynamics opens up a host of new mathematical and economic problems. This is not the place to go into them. There is an extensive literature. See Hahn (1982), Fisher (1983) and their references.

7. Brief Comments on Global Stability

7.1 Once the local stability of the system has been studied it is almost unavoidable to ask about the global stability of the price quantity dynamics. It is important to keep in mind, however, that the global analysis is more difficult to interpret than the local. In the first place, the farther we are from equilibrium the less confidence we can have in precisely formulated dynamic laws. In the second place, the tatonnement dynamics is not a real system taking place in real time. Remember that there is not trade out of equilibrium. Perhaps we can attach some meaning to a globally convergent dynamics but a limit cycle, say, lacks any real significance. The point is that either the tatonnement works by taking us to equilibrium and then we have a theory or it does not and then we don't. Because of all this our observations on global stability will be cursory and brief.

7.2 Suppose that the Weak Axiom is satisfied, the equilibrium is unique and there is no joint production. Assume further (this is only a minor regularity strengthening) that the pure price and the pure quantity dynamics are well defined for the system (I')-(II'), i.e. for the system with constant speeds of adjustment.

Then both dynamics are globally stable. In fact, the Euclidean distance to equilibrium serves in both cases as a Lyapunov function. This is standard for the pure price dynamics (see Hahn (1982)) and it is entirely analagous for the pure quantity case.

7.3 Maintaining the hypothesis of the previous paragraph suppose that, in addition, the RC hypothesis holds. Can the results be improved? Yes, to the extent of allowing for variable speeds of adjustment and, more generally, for weak versions of the dynamic principles such as (II") for the quantity case. With a RC there is an obvious candidate for a Lyapunov function: the utility function itself. The contrast between the WA and the RC hypothesis in the analysis of global stability seems to be completely parallel to the one drawn in Section 5 for local stability.

7.4 In contrast to the two limit cases of pure price and pure quantity dynamic it is our impression that even in the most favourable situation, i.e. no joint production, constant speed of adjustment dynamics and RC Hypothesis (and, we could add, $\ell = 2$) the global stability of system (I')-(II') is not guaranteed. But this could be wrong because we have no formal example. So, we leave this as:

<u>Problem 2: Determine if under the RC Hypothesis the system (I')-(II') is globally stable, or find a counterexample.</u>

8. Brief Discussion of the One Input-One Output Case

8.1 In this section we will quickly discuss the particular case where: (i) there is only one output, i.e. $\ell = 2$, and (ii) demand is independent of profit income. Both hypotheses facilitate phase diagram analysis. A thorough examination has been carried out by Beckman and Ryder (1969). The (I')-(II') dynamics now takes the form:

$$(I') \quad \dot{p} = F(p) - x$$

$$(II') \quad \dot{x} = q(p - c(x))$$

where x, p are real numbers and, without loss of generality, the speed of adjustment of the price equation is fixed to equal one. In accord with (ii), demand $F(p)$ depends only on p. It is worth mentioning that this is compatible with the Representative Consumer Hypothesis (or the Weak Axiom, there is no distinction for $\ell = 2$) if and only if $F(p)$ is a decreasing function. For simplicity we always assume that $F(p) = 0$ for any sufficiently large p and that (I')-(II') have a single rest point, see figure 3.

8.2 As it should be, the local analysis of (I')-(II') yields the familiar conclusions. The Jacobian matrix at equilibrium is:

$$J = \begin{bmatrix} a & -1 \\ q & -qc \end{bmatrix}$$

where a, c are, respectively, the slopes of demand and cost. We have $c > 0$ and, because equilibrium is unique, $ac < 1$, see figure 3. From trace $J_q = a - qc$ and $\det J_q = q(1 - ac)$ we can reach the following conclusions about the local dynamics of (I')-(II'):

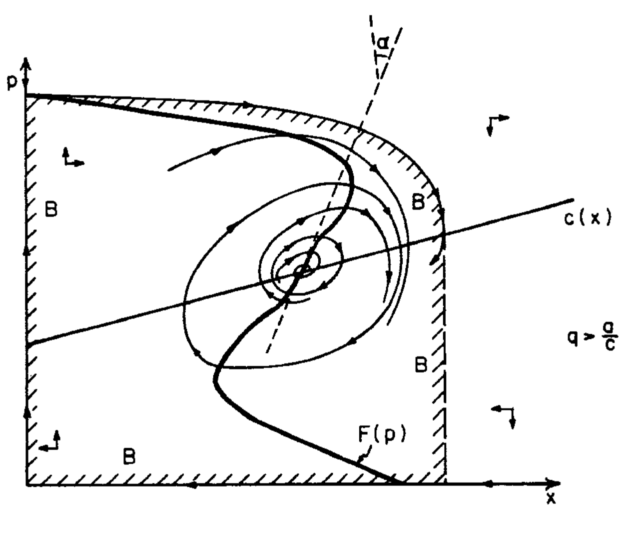


FIGURE 3

(i) If $a < 0$, e.g. there is a Representative Consumer, then the equilibrium is locally stable. For fixed a it goes from a stable node to a stable spiral point as c decreases.

(ii) If $a > 0$ then the local stability depends on q. At low q the system if locally unstable. At $q^* = \frac{a}{c}$ it undergoes a Hopf bifurcation towards stability. Indeed, det $J_q > 0$ for any $q > 0$ but trace $J_q \lessgtr 0$ according to if $q \gtrless q^*$ (see Hassard-Kazarinoff-Wan (1981) for the Hopf bifurcation). Note that if $c = 0$, e.g. there are constant returns, then $q^* = \infty$ and the system is always unstable.

8.3 Some brief words on the global dynamics of (I')-(II') which, incidentally, we take to be defined on the boundary of R^2_+ in the natural way. The key observation, due to Beckmann and Ryder (1969), is that the system has an absorbing, bounded, simply connected region. Region B in figure 3 will do. In the figure the upper boundary of the region is the part above $c(\cdot)$ of any trajectory starting at a $(0,p)$ with $F(p) = 0$. If the system is locally unstable, e.g. $a > 0$, $q < q^*$, then this and Poincaré-Bendixon theory (see Hartman (1964)) yield the existence of a stable limit cycle. This cycle which is obtained from a mixture of local and global considerations and which is always stable should not be confused with the cycle obtained from Hopf bifurcation theory. The existence and stability of the latter is entirely controlled by local information on the demand and cost functions at equilibrium. Thus, it is perfectly possible, and as likely as the contrary, that the higher derivatives of demand and cost at equilibrium force the existence of an unstable (resp. stable) Hopf cycle for some values of $q > q^*$ (resp. $q < q^*$). The existence of an absorbing region does then still imply that there must be at least another stable (Poincare-Bendixon) cycle. The possibility is illustrated in figure 3.

Summing up: except in two cases there is nothing simple about the global dynamics of even the simplest demand and supply model. The two cases are: (i) the demand function is decreasing on its entire domain, or (ii) the cost function is strictly increasing (i.e. $\frac{dc(x)}{dx} > \varepsilon > 0$ for all x) and the speed of adjustment q is high enough. The system is globally stable in both these cases (thus, in figure 3 if q is large the intermediate region of instability disappears). The stability proofs are not difficult to work out.

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