Chapter 17

VALUATION EQUILIBRIUM AND PARETO OPTIMUM REVISITED*

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1. Introduction

Abstract linear spaces constitute an ideal mathematical framework for the study of equilibrium problems with infinitely many commodities. More than thirty years ago Debreu (1954) presented a study on the supportability of Pareto optimal allocations by price (or, as he called them, valuation) functionals in this general setting.

Convexity hypothesis aside, Debreu's work rests on the assumption that the (aggregate) production set of the economy has a non-empty interior. Subject to this key assumption, his analysis is definitive. The problem is cleanly reduced to the supportability of a certain convex set at the origin by a hyperplane. The set has a non-empty interior because of the key assumption, and the existence of a supporting hyperplane is guaranteed then by the Hahn–Banach theorem. It is an argument that in the finite-dimensional case, where there is no need for a non-empty interior hypothesis, originates in Arrow (1952) and Debreu (1951) and that has since become classical.

It was also pointed out by Debreu that if the commodity space has, as it should, a linear order structure, then the production set will have a non-empty interior if, first, the positive orthant of the space has a non-empty interior and, second, the technology allows for free disposal of commodities. The latter is a

*My thinking on the problem of this paper has benefited from conversation with many people: C. Aliprantis, A. Araujo, D. Duffie, C. Huang, L. Jones, J. Ostroy, W. Zame. For the exchange case an announcement of the results in this paper was circulated in February 1984 under the title "Notes on Pareto Optima in Linear Spaces". The extension to production has been stimulated by recent work of W. Zame. Financial support from MSRI, the University of California at Berkeley, NSF and the Guggenheim Foundation is gratefully acknowledged.

strong but natural hypothesis and shall not be questioned in this paper. Debreu's result is thus excellently suited for the space of bounded functions on a measure space, L_∞, endowed with the supremum norm.

Bewley (1972) started the systematic use of L_∞ in equilibrium theory. It is quite often the case that the allocation of commodities through time or over states of the world is well modeled by L_∞. Nonetheless, the last decade has produced a variety of economically important allocation problems which do not lend themselves to be treated by means of L_∞ or, for that matter, of any ordered linear space whose positive orthant has a non-empty interior. Examples are (a) finance models where assets' returns are stochastic processes modeled on the Hilbert space L_2 of square integrable functions on a measure space [e.g., Chamberlain and Rothschild (1983), Duffie and Huang (1985)] or (b) models of product differentiation where the commodity space is taken to be $ca(K)$, the set of countably additive measures on a compact metric space K endowed with the weak-star topology [e.g., Mas-Colell (1975), Jones (1984a)]. In these spaces the economically reasonable hypothesis of free disposal does not automatically entail the non-emptiness of the interior of the production set. The latter may still hold but a moment reflection will reveal how strong a condition it is (how incompatible it is, for example, with a priori restriction on which commodities can be outputs).

It is the purpose of this paper to reconsider the supportability problem in abstract linear spaces without requiring the non-emptiness of the interior of the production set. Our aim is to obtain a general understanding and some significant theorems covering, at least, the two examples above. We may remark, incidentally, that getting conditions guaranteeing the supportability of weak optima is also the key step unblocking a positive solution to the problem of the existence of a Walrasian equilibrium [see Mas-Colell (1985b)].

Suppose that our commodity space is an ordered linear topological space L whose positive orthant L_+ has an empty interior. To be specific consider for the time being the exchange case with consumers having L_+ as consumption sets. It is easy to produce a one-consumer example with continuous, monotone and convex preferences where the optimum cannot be (weakly) supported by any non-trivial functional. This is done, for example, in Mas-Colell (1985b) for the commodity space $ca(K)$. Thus, if we want supportability we will have to require that individual preferences be "well-behaved". The meaning of well-behaved will be left undefined for the moment. It is clear, however, what the desiderata are: (a) it should be sufficient, and not far from necessary, for the supportability by prices of individual preferences at any consumption, and (b) it should be automatically satisfied if preferences are monotone and L_+ has in fact a non-empty interior.

It turns out, perhaps unexpectedly, that the well-behavedness of individual preferences does not automatically entail the supportability of optima. In certain spaces the way in which the different individual consumptions fit together at the weak optimum may prevent the existence of socially supporting price functionals. Jones (1984b) has offered the following example, which is actually quite standard in theoretical urban economics.

Example (Jones). There are two consumers and no production. The commodity space is $L_\infty([0,1])$, the bounded, measurable functions on $[0,1]$, endowed with the topology induced by the duality with $C^1([0,1])$, the continuously differentiable functions on $[0,1]$. That is to say, if $x_n, x \in L_\infty$, then $x_n \to x$ if and only if $\int [x_n(t) - x(t)] f(t)\,\mathrm{d}t$ for every $f \in C^1$. The total endowments of society is the constant functions $\omega(t) = 1$. The two consumers have the linear, continuous utility functions $u_1(x) = \int t x(t)\,\mathrm{d}t$ and $u_2(x) = \int (1-t)x(t)\,\mathrm{d}t$. Clearly, these preferences are as nice and well-behaved as one could wish. The allocation $x_1(t) = 0$ for $t \leq \frac{1}{2}$, $x_2(t) = 0$ for $t > \frac{1}{2}$, is an optimum. Obviously, the only price functional that supports it is given by $\int q(t)x(t)\,\mathrm{d}t$, where $q(t) = (1-t)$ for $t \leq \frac{1}{2}$, $q(t) = t$ for $t > \frac{1}{2}$. This price functional is non-zero, but since $q \notin C^1$ it fails to be continuous.

The previous example feels rather pathological. The topology of the space is far from natural. No better examples are available. For us, this is good news because we aim at ruling out this sort of example. This we shall accomplish by combining an appropriate concept of well-behaved preferences and technologies with some structural properties of the space.

We shall obtain two results. In the first (Section 3) the structural property of the space is relatively strong. In the second (Section 4) the property is weaker and therefore many more spaces are included [in particular $ca(K)$ with the weak-star topology]. However, beyond well-behavedness, we impose an additional smoothness-like restriction on preferences and technologies. Together, the two results seem to cover most cases of interest.

The approaches underlying the two results are variations on a common, and unexpected, theme: that a successful attack on our problem depends on the exploitation of the lattice structure of the commodity space. It is fair to say that one would not have suspected this from the finite-dimensional theory. For the exchange case the above observation was made in Mas-Colell (1985b) and extended to production by Zame (1985). In a different context, vector lattices as models of commodity spaces were first used in economics by Aliprantis and Brown (1983).

The basic formal setting of the paper is in the next section. Sections 3 and 4 do present, respectively, the two more specialized results.

2. The formal setting

The commodity space L will be modelled as a *vector lattice* (also called a Riesz space).

A vector lattice is an ordered linear space where the order is such that any two vectors $x, y \in L$ have a *supremum*, denoted $x \vee y$ (in other words, if $z \geq x$, $z \geq y$, then $z \geq x \vee y$). It follows that any two vectors x, y also have an *infimum*, denoted $x \wedge y$. For any vector x one can then define the *positive part* $x^+ = x \vee 0$, the *negative part* $x^- = (-x) \vee 0$ and the *absolute value* $|x| = x^+ + x^-$. Any vector x can also be written as $x = x^+ - x^-$. An elementary but basic fact of vector lattices is the *decomposition property*: If $x, y \geq 0$ and $0 \leq z \leq x + y$, then we can find x', y' such that $0 \leq x' \leq x$, $0 \leq y' \leq y$ and $x' + y' = z$. Obviously, this extends to any number of vectors.

Mathematically, a vector lattice is a very special ordered linear space. For example, in finite dimensions an ordered linear space is a vector lattice if and only if the positive cone is spanned by a number of independent vectors equal to the dimension of the space. Nevertheless, economically, a vector lattice is a very sensible sort of commodity space. Because the positive and the negative part of any vector are well defined any "aggregate production vector" can be unambiguously written as the difference of a non-negative output vector and a non-negative input vector. The decomposition property also has an economic interpretation. If $x, y \geq 0$ are the consumptions of two consumers, $0 \leq z \leq x + y$, and we view $x + y - z$ as a "tax" to be paid by the two of them, then the decomposition property guarantees that the tax can be paid so as to leave the two consumers with non-negative consumption.

For the purposes of this paper a vector lattice structure of the commodity space is sufficiently general and covers all the cases we are interested in. Only rarely will an economic situation lead to an ordered linear space which is not a vector lattice. One example is $C^r([0,1])$ for $r > 0$. In view of the observations of the previous paragraph it can be questioned if $C^r([0,1])$, as opposed to $C^0([0,1])$, is an appropriate commodity space. At any rate, it is worth mentioning that, if $C^r([0,1])$ comes equipped with a supremum norm, then the hypothesis of free disposal will make the original approach of Debreu (1954) applicable. This remark shows, incidentally, that what we do in this paper is not, strictly speaking, a generalization of Debreu's 1954 contribution. In order to handle production sets with empty interior, we need to impose on the commodity space an additional structure (the lattice order) not required by Debreu.

Good introduction to vector lattice theory are Aliprantis and Burkinshaw (1978), Peressini (1967) and Schaeffer (1970). For a more extensive treatment, see Schaeffer (1974).

In this paper, we shall also assume that L is equipped with a Hausdorff, locally convex, linear topology. The topology and the order fit together well enough for at least the positive orthant L_+ to be closed. A much stronger fit will be considered in Section 3.

An *economy* is composed by N consumers, M producers and an aggregate endowment vector $\omega \geq 0$.

Every consumer i is characterized by a *consumption set* X_i and a *preference relation* $\gtrsim_i$ on X_i. I shall assume that $X_i = L_+$. This is done for definiteness. The essential properties of latter development is that every X_i be closed, convex, satisfy $X_i + L_+ \subset X_i$ and constitute a sublattice of L, that is, if $x, z \in X_i$, then $x \vee z$, $x \wedge z \in X_i$. The preference relation is assumed to satisfy familiar properties, namely *continuity*, *convexity* (i.e., the set $\{z : z \gtrsim_i x_i\}$ is convex for every x_i) and *monotonicity* (i.e., $z \geq x_i$ implies $z \gtrsim_i x_i$). Moreover, there is a *desirable direction* – a $v_i \geq 0$, such that $x_i + \alpha v_i \succ_i x_i$ for all x_i and $\alpha > 0$.

The description of a producer j is less conventional. It proceeds in two steps. In the first one specifies a *pretechnology set* $Z_j \subset L$. The set Z_j is closed, convex, contains the origin, satisfies free disposal (i.e., $Z_j - L_+ \subset Z_j$) and constitutes a *sublattice* of L (i.e., if $z, y \in Z_j$, then $z \vee y$, $z \wedge y \in Z_j$). The set Z_j is not a production set. As a matter of interpretation it specifies the set of commodities which are admissible outputs (because of free disposal every commodity is an admissible input). For example, in R^2 we could have $Z_j = -R^2_+$ or $Z_j = R \times [-\infty, 0]$, which means that no commodity or only the first commodity, respectively, is a conceivable output. In the second step we then specify the *production set* proper $Y_j \subset Z_j$ which is assumed to be closed, convex and satisfy free disposal (i.e., $Y_j - L_+ \subset L_+$). The rationale for the introduction of the intermediate concept of the pretechnology set will become clear as we go along. A producer is therefore specified by the pair (Z_j, Y_j) which we could call the *technology of the j producer*.

We conclude this section by defining the central concept of this paper.

An *allocation* is a list $(x, y) \in L^N \times Y_1 \times \cdots \times Y_M$ such that $\sum_i x_i = \sum_j y_j + \omega$.

An allocation (x, y) is a *weak optimum* if there is no other allocation (x', y') such that $x'_i \succ_i x_i$ for all i.

The weak optimum $(\bar{x}, \bar{y})$ is *supported* by the non-zero linear functional $p : L \to R$, to be called a price or valuation functional, if (a) $py_j \leq p\bar{y}_j$ for every $y_j \in Y_j$ and j (profit maximization) and (b) $px_i \geq p\bar{x}_i$ for every $x_i \gtrsim_i \bar{x}$ and i (cost minimization by consumers).

In the rest of this paper we shall formulate two sets of conditions guaranteeing the supportability of weak optima by non-zero price functionals. We are mainly interested in the supportability by *continuous* functionals, that is by members of topological dual of L, denoted L^*.

3. First approach: Topological vector lattices

3.1. Well-behaved preferences and production sets

As discussed in the introductory section, a first step towards a supportability result is a notion of well-behaved preferences and production sets. We begin by preferences.

Let $\gtrsim$ be a preference relation on L_+ satisfying the maintained hypotheses of Section 2. Consider a fixed vector $z \in L_+$. By assumption there is a desirable vector $v \geq 0$, that is $z + \alpha v \succ z$ for $\alpha > 0$. We shall say that $\gtrsim$ is well-behaved at z if the vector v is so desirable that the marginal rate of substitution of any other commodity for v is bounded away from zero. Formally, the preference relation $\gtrsim$ is *proper* at x if there is $v \geq 0$ and a non-empty open neighborhood V of the origin such that $z \in L$ and $x - \varepsilon v + z \gtrsim x$ implies $z \notin \varepsilon V$ for all ε, i.e., if z is too small, then it cannot compensate for the loss of εv. The preference relation is *uniformly proper* if v, V and ε can be taken independently of x.

Geometrically, properness at $x \in L$ simply says that there is a non-empty, open, convex cone Γ such that $(\{x\} - \Gamma) \cap (w : w \gtrsim x\} = \varnothing$. Therefore, by the Hahn–Banach theorem properness at a point is equivalent to the existence of a non-trivial, continuous, linear functional supporting $(z : z \gtrsim x\}$ at x.

To motivate an analogous condition for production sets note that the uniform properness property for preferences can be reformulated as follows: there are $v \geq 0$ and a non-empty, open neighborhood of the origin V such that for any $x \in L_+$ if $x' \in L_+$ but $x' \notin \{w : w \gtrsim x\}$, then $x' - \varepsilon v + z \gtrsim x$ implies $z \notin \varepsilon V$ for all ε. Given a pretechnology set Z and a production $Y \subset Z$ (see Section 2 for definition) we shall let Z and Y play, respectively, the roles of L_+ and $\{w : w \gtrsim x\}$ in the preference context. Hence, we call Y, or, more rigorously, the technology pair (Z, Y), *uniformly proper* if there are $v \geq 0$ and a non-empty, open neighborhood V of zero such that, whenever $y \in Z$ and $y \notin Y$, we have, for all ε, that $y + \varepsilon v + z \in Y$ only if $z \notin \varepsilon V$. In words: if y is not producible (but it is, nonetheless, a conceivable input–output vector) and we still subtract from it some significant input, then it cannot be made producible by adding a vector which is too small, i.e., marginal rates of substitution with respect to v are bounded away from zero.

We conclude this subsection with some remarks:

(a) The properness and uniform properness conditions are stronger, the weaker is the topology of L. But as long as the space admits a single non-trivial, positive, continuous linear functional, the condition is not vacuous.
(b) Because of free disposal and monotonicity the uniform properness condition is automatically satisfied if $L = L_\infty$ with the norm topology.
(c) If the pretechnology set equals L, then the uniform properness is so strong that it implies the non-emptiness of the interior of Y (this is, incidentally, a good exercise to test comprehension of the properness concept). We would be back to square one. It is precisely this need to limit the strength of the well-behavedness condition which has prompted the introduction of the pretechnology set Z as a natural analog for producers of the consumption set L_+.
(d) The uniform properness condition in production is different from a condition introduced by Zame (1985) with a similar aim. This suggests that a substantial weakening may be possible by combining the two of them (see also Zame's comments in the final section of his paper).

3.2. Topological vector lattices

What distinguishes the approach in this section from the next is not the concept of well-behaved preferences and production sets (it will essentially be the same in the next section) but a structural property of the space. Namely, we will require here that L be a *topological vector lattice*.

A vector lattice is a topological vector lattice if the topology and the lattice structures of the space fit well enough for the lattice operations (it is enough to consider $x \to |x|$) to be uniformly continuous. Equivalently, if the topology is *locally solid*, that is, it admits a basis for zero consisting of solid sets (a set $A \subset L$ is solid if $v \in A$ and $|w| \leq |v|$ implies $w \in A$); see Aliprantis and Burkinshaw (1978, ch. 2) for all this. The most important topological vector lattices are the *Banach lattices*. Those are ordered Banach spaces where the order is a lattice order which satisfies the condition "$|v| \leq |w|$ implies $\|v\| \leq \|w\|$, for all v, w".

Although there are some linear spaces with very weak topologies which are topological vector spaces (e.g., R^∞ with the product topology), the continuity requirement on the lattice operations has to be seen, by and large, as placing strong restrictions on the topology of L. For example, if L is normed, then the lattice operations are uniformly continuous in the weak topology if and only if L is finite-dimensional [Aliprantis and Burkinshaw (1978, p. 42)]. This means in particular that one of the examples we are most interested in, $ca([0,1])$ with the weak-star topology, is not a topological vector lattice (consider the sequence $x_n = \delta_{(1/n)} - \delta_0$; it converges to zero but $|x_n| = \delta_{(1/n)} + \delta_0 \to 2\delta_0 \neq 0$). The same is true for the example by Jones discussed in the Introduction. An interesting case is L_∞ with the Mackey topology which *is* a topological vector lattice.

3.3. A result

Theorem 1. Suppose that L is a topological vector lattice and that the preference relations and production technologies are uniformly proper (and satisfy the maintained hypotheses of Section 2). Then any weakly optimal allocation (x, y) can be supported by a non-zero, continuous linear functional $p \in L^*_+$.

We can add to the conclusion of Theorem 1 that $p \cdot v = 1$, where $v = \sum_i v_i + \sum_j v_j$ and v_i, v_j are as in the definition of properness for, respectively, preferences and production technologies. If we can take $v_i = v_j = \omega$ for all i and j, then we can guarantee $p \cdot \omega > 0$. By viewing this as an essential ingredient of supportability it is possible to illustrate the role of the properness conditions even in the finite-dimensional case [see Mas-Colell (1985a)].

The order dual $L^\sim$ of a vector lattice L is the set of linear functionals which are order bounded, that is, if $f \in L^\sim$ and $a \leq b$, then $f(\{z : a \leq z \leq b\})$ is bounded. The order dual is a vector lattice in its own right (see Section 4.2). Now suppose that our L is a Banach space with the weak topology. In particular, the properness condition holds relative to the weak topology. It is an important fact that there is a locally solid topology, called the absolute weak topology, which is stronger than the weak and has its topological dual contained in $L^\sim$. See Aliprantis and Burkinshaw (1978, sec. 2.6) for all this. Therefore, Theorem 1 can be used in considerable generality to obtain a supporting functional in the order dual.

Proof of Theorem 1. Let v_i, v_j be as in the definition of properness. Without loss of generality, we can take a non-empty, open, convex, solid neighborhood of zero, V, which satisfies $V = -V$, and the definition of properness for every i and j. Put $v = \sum_i v_i + \sum_j v_j$ and let Γ be the cone spanned by $v + V$.

For the given weak optimum $(\bar{x}, \bar{y})$ define $E = \sum_i \{z_i : z_i \succsim_i \bar{x}_i\} - \{\omega\} - \sum_j Y_j$. Note that $L_+ \subset E$.

Suppose that $E \cap (-\Gamma) = \varnothing$. Then by the Hahn–Banach theorem there is a continuous, linear $p \geq 0$, $p \neq 0$, separating the sets E and $-\Gamma$. Standard arguments [see e.g. Debreu's *Theory of Value*, ch. 6] show then that p is a supporting price functional at (x, y).

The heart of the proof is therefore to establish that $E \cap (-\Gamma) = \varnothing$.

Suppose, by way of contradiction, that there are $x_i \gtrsim_i \bar{x}_i$, $y_j \in Y$ such that letting $\hat{x} = \sum_i x_i$, $\hat{y} = \sum_j y_j$, we have $\hat{x} - \omega - \hat{y} \in -\Gamma$ or $\hat{x} - \omega - \hat{y} - (-\varepsilon v) \in \varepsilon V$ for some $\varepsilon > 0$.

We will obtain a contradiction by constructing from (x, y) an allocation (x', y') which Pareto-dominates $(\bar{x}, \bar{y})$.

As $\hat{y} = \sum_j y_j^+ - \sum_j y_j^-$, we have $\hat{x} - \omega - \hat{y} = \hat{x} - \omega - \sum_j y_j^+ + \sum_j y_j^-$. Now, $0 \le (\hat{x} - \omega - \sum_j y_j^+ + \sum_j y_j^- + \varepsilon v)^+ \le \hat{x} + \sum_j y_j^- + \varepsilon v$. Therefore, by the lattice decomposition property we can write

$$\sum_i s_i + \sum_j s_j = (\hat{x} - \omega - \hat{y} + \varepsilon v)^+, \qquad \begin{array}{l} 0 \le s_i \le x_i + \varepsilon v_i, \\ \\ 0 \le s_j \le y_j^- + \varepsilon v_j. \end{array}$$

Also, we have $s_h \in \varepsilon V$ for all $h = i, j$ because $|s_h| \le |(\hat{x} - \omega - \hat{y} + \varepsilon v)^+| \le |\hat{x} - \omega - \hat{y} + \alpha v| \in \varepsilon V$ and V is solid.

Define $x_i' = x_i + \varepsilon v_i - s_i$, $y_j' = y_j - \varepsilon v_j + s_j$. This will be the desired allocation. Note first that $x_i' \ge 0$. Therefore, by uniform properness, $x_i \gtrsim_i x_i'$ is impossible, which implies $x_i' \succ_i x_i$. In the production side observe that $y_j' = y_j - \varepsilon v_j + s_j \le y_j + y_j^- = y_j^+ = y_j \vee 0$. As $0 \in Z_j$, $y_j \in Z_j$, Z_j is a sublattice and $Z_j - L_+ \subset Z_j$, it follows that $y_j' \in Z_j$. Therefore, by the properness property we must have $y_j' \in Y_j$ (if not, $y_j' \notin Y_j$, $y_j' \in Z_j$, $y_j' + \varepsilon v_j - s_j \in Y_j$ and $s_j \in \varepsilon V$, which is a direct contradiction to the uniform properness of Y_j).

It only remains to show that (x', y') is feasible. Let $\hat{x}' = \sum_i x_i'$, $\hat{y}' = \sum_j y_j'$. Then

$$\begin{aligned} \hat{x}' - \hat{y}' - \omega &= \hat{x} + \varepsilon v - \hat{y} - \omega - \sum_i s_i - \sum_j s_j \\ &= \hat{x} + \varepsilon v - \hat{y} - \omega - (\hat{x} - \omega + \varepsilon v - \hat{y})^+ \\ &= -(\hat{x} + \varepsilon v - \hat{y} - \omega)^- \le 0. \end{aligned}$$

This concludes the proof. □

4. *Second approach: The dual is a lattice*

The approach of Section 3 covers some cases of interest (e.g., any model where L is a reflexive L_p space, in particular L_2) but leaves out a number of others. We have seen, for example, that for $L = ca([0,1])$ it cannot be used to obtain supporting prices representable as continuous functions on [0, 1]. This prompts our search for an alternative, conceivably more general, theory. As we will see, the key to getting one is a more intensive exploitation of lattice theoretic properties. But, first, a preliminary.

4.1. *Well-behaved preferences and production sets*

In this section, we shall assume that preferences on L_+ are derived from utility functions that can, in principle, be defined on the entire space. Formally: for every i there is a concave, continuous function $u_i : L \to R$ such that $z \gtrsim_i v$ if and only if $u_i(z) \geq u_i(v)$. Thus, in this section well-behavedness means extendability. It has been shown by Richard (1985a, 1985b) that this property is intimately related to the properness conditions of the previous section. Conceptually, it has to be thought as equivalent.

We do similarly for the production technologies. For every j there is a concave, continuous functional $q_j : L \to R$ such that $Y = \{z \in Z_j : q_j(z) \geq 0\}$. Note that, if $Z = L$ and $q_j(z) > 0$ for some z, then Y would have a non-empty interior. Thus, again, the concept of pretechnology set is decisive to avoid this implications.

4.2. *A lattice structure for the dual*

In the previous section we defined the order dual $L^{\sim}$ of L as the set of order bounded linear functionals. Clearly, every positive linear functional belongs to $L^{\sim}$. The vector space $L^{\sim}$ is ordered by letting $p \geq q$ if $p - q$ is positive. This order constitutes a lattice order and it is not hard to verify that

$$(p \vee q)(v) = \sup\{\, p(v_1) + q(v_2) : v_1, v_2 \in L_+,\ v_1 + v_2 = v \,\} \qquad \text{for} \quad v \geq 0;$$

see Aliprantis and Burkinshaw (1978, p. 21).

If, as in Section 3, L is a locally solid space, then the topological dual L^* is a sublattice of $L^{\sim}$ [Aliprantis and Burkinshaw (1978, p. 36)], that is, $L^* \subset L^{\sim}$ and $p \vee q,\ p \wedge q \in L^*$ wherever $p, q \in L^*$. We shall now isolate and focus on this property of the dual. We assume from now on that L^* *is a sublattice of* $L^{\sim}$. In particular, the operations $p \vee q$, $p \wedge q$ are well-defined in the natural order of the dual. Note that this condition is satisfied for $L = ca([0,1])$ with the weak-star topology [because the dual is $C^0([0,1])$, which is a lattice] but not for the example described in the Introduction [where the dual was $C^1([0,1])$]. It seems to us that this condition is the natural hypothesis for our problem. If the dual fails to be closed under the lattice operations it is not reasonable to expect the fulfillment of the supportability property.

4.3. Linearization

We now introduce a smoothness-like condition. About its dispensability we shall comment at the end of the section.

The utility and production functions $u_i, q_j : L \to R$ have the *unique supporting hyperplane property* if the functions u_i, q_j, which are defined on the entire L, have a unique, non-zero, continuous subgradient at any $z \in L$. Note that the non-negativity constraints do not appear in this definition.

The decisive advantage of this condition is that it allows us to reduce the problem to the linear case.

Lemma. *Suppose that the unique supporting hyperplane property holds and that (x, y) is a weak optimum. Then (x, y) is a weak optimum for the economy obtained by replacing the original u_i, q_j by its linearization at x_i, y_i. Further, any supporting price functional for (x, y) in the linearized economy supports also in the original one.*

Proof. The second claim is clear enough because the linearization only expands the production and preferred sets.

For the first claim, let $\hat{u}_i, \hat{q}_j$ be the linearization at x_i, y_j. Suppose, by way of contradiction, that there is $(\bar{x}, \bar{y})$ such that $\bar{x}_i \geq 0$, $\hat{u}_i(\bar{x}_i) > \hat{u}_i(x_i)$ for all i, $\bar{y}_j \in Z_j$, $\hat{q}_j(\bar{y}_j) \geq 0$ for all j, and $\sum_i \bar{x}_i \leq \sum_j \bar{y}_j + \omega$. Define $x_i(\alpha) = \alpha\bar{x}_i + (1 - \alpha)x_i \in L_+$, $y_j(\alpha) = \alpha\bar{y}_j + (1 - \alpha)y_j \in Z_j$ for $\alpha \in [0,1]$. Then $\sum_i x_i(\alpha) \leq \sum_j y_j(\alpha) + \omega$ for all α. By the concavity of the u_i, q_j functions we have that $\{\alpha : u_i(x_i(\alpha)) > u_i(x_i)\} = (0, \alpha_i)$ and $\{\alpha : q_j(y_j(\alpha)) > 0\} = [0, \alpha_j]$. If $\alpha_i, \alpha_j > 0$ for all i, j, then our original (x, y) is not a weak optimum. Therefore, either $\alpha_i = 0$ for some i or $\alpha_j = 0$ for some j. Suppose that $\alpha_i = 0$. Then $J = \{(\alpha\bar{x}_i + (1 - \alpha)x_i, u_i(x_i)) : \alpha \in R\} \subset L \times R$ is a straight line which does not intersect the interior of $E = \{(z_i, t) : t \leq u_i(z_i)\}$; see Figure 1, where J' is the projection of J on L.

By the Hahn–Banach theorem [see Schaeffer (1971)] there is a closed hyperplane in $L \times R$ which contains J and leaves E at one side. But this yields a linearization of u_i at x_i different from $\hat{u}_i$ (indeed $\hat{u}_i$ is not constant on J'). Contradiction.

The contradiction for $\alpha_j = 0$ is obtained in an entirely analogous manner. □

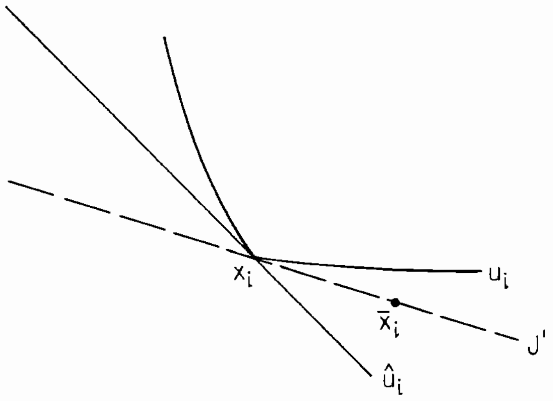

Figure 1

Remarks. (a) Because of monotonicity (respectively free disposal) the linearization of u_i (respectively q_j) is non-negative (respectively non-positive). (b) In practice the linearization may not be difficult to compute. For example, in $L = ca([0,1])$ let $u_i(x_i)$ be given by $u_i(x_i) = g(\int f_1 \, \mathrm{d}x_i, \ldots, \int f_H \, \mathrm{d}x_i)$, where $g : R^H \to R$ is C^1 and $f_h : [0,1] \to R$ is continuous. Then u_i is approximated at x_i by $\sum_h (\partial g / \partial \alpha_h) f_h$, where $\partial g / \partial \alpha_h$ is evaluated at $(\int f_1 \, \mathrm{d}x_i, \ldots, \int f_H \, \mathrm{d}x_i)$.

4.4. A result

The following theorem, with the previous lemma, can be viewed as a Kuhn–Tucker result in an abstract, infinite-dimensional setting where the non-negativity constraints are expressed as the restriction to the positive orthant of a vector lattice (and the remaining constraints are "nice" and finite in number).

Theorem 2. Suppose that L^* is a sublattice of $L^\sim$ and that the utility (respectively production) functions are of the (affine) form $u_i \cdot x_i + b_i$ (respectively $q_j \cdot z_j - c_j$), where the non-zero linear functionals $u_i \geq 0$, $q_j \leq 0$ are continuous. Let (x, y) be a weak optimum. Then there are real numbers $\lambda_i \geq 0$, $\mu_j \leq 0$, not all zero, such that the non-zero, continuous linear functional

$$p = \lambda_1 u_1 \vee \cdots \vee \lambda_N u_N \vee \mu_1 q_1 \vee \cdots \vee \mu_M q_M$$

supports (x, y).

Proof of Theorem 2. To obtain the λ_i, μ_j coefficients we simply mimick the familiar proofs of the Kuhn–Tucker theorem. Define the convex set $V \subset R^{N+M}$ by

$$V = \left\{ (u_1 \cdot x_1, \ldots, u_N \cdot x_N, q_1 \cdot y_1, \ldots, q_M \cdot y_M) : \right.$$
$$\left. x_i \geq 0,\, y_j \in Z_j,\, \sum_i x_i - \sum_j y_j \leq \omega \right\}.$$

Let $(\bar{x}, \bar{y})$ be the weak optimum under consideration and denote $\bar{v} = (u_1 \cdot \bar{x}_1, \ldots, u_N \cdot \bar{x}_N, q_1 \cdot \bar{y}_1, \ldots, q_M \cdot \bar{y}_M)$. We have $\bar{v} \in V$ and $V \cap (\{\bar{v}\} + R^{N+M}_{++}) = \varnothing$. Therefore, by the separating hyperplane theorem (in finitely many dimensions !) there are $\lambda \in R^N_+$, $\mu \in -R^M_+$ such that $(\lambda, \mu) \neq 0$ and $(\lambda, -\mu)$ supports V and $\bar{v}$. Without loss of generality it can be assumed that $\mu_j = 0$ whenever $q_j \cdot \bar{y}_j > c_j$. We claim that these real numbers λ, μ do the job. Call $p = \lambda_1 \mu_1 \vee \cdots \vee \lambda_N u_N \vee \mu_1 q_1 \vee \cdots \vee \mu_N q_M$. We must verify the profit maximization and the preference cost minimization conditions.

(a) Profit maximization

Let $z \in Y_1$, i.e., $z \in Z_1$, $q_1 \cdot z \geq c_1$.

Put $s_1 = \bar{y}_1 \vee z \in Z_1$ and $s_j = \bar{y}_j$ for $j \neq 1$. Define the sets $A, A' \subset R^{N+M}$ by

$$A = \left\{ (x_1, \ldots, x_N, w_1, \ldots, w_M) : \right.$$
$$\left. x_i \geq 0,\, s_j - w_j \in Z_j,\, \sum_i x_i + \sum_j w_j = \sum_j s_j + \omega \right\},$$

$$A' = \left\{ (x_1, \ldots, x_N, w_1, \ldots, w_M) : \right.$$
$$\left. x_i \geq 0,\, w_j \geq 0,\, \sum_i x_i + \sum_j w_j = \sum_j s_j + \omega \right\}.$$

Since $Z_j - L_+ \subset Z_j$ we have $A' \subset A$. Denote $\bar{w}_j = s_j - \bar{y}_j \geq 0$. Because $(\lambda, -\mu)$ supports V at $\bar{v}$ we have $(\bar{x}_1, \ldots, \bar{x}_N, \bar{w}_1, \ldots, \bar{w}_M) \in A'$ maximizes $\sum_i \lambda_i u_i \cdot x_i + \sum_j \mu_j q_j \cdot w_j$ over the set A and therefore over the set A'. By definition of p as a lattice supremum this means that

$$p \cdot \left(\sum_j s_j + \omega \right) = \sum_i \lambda_i u_i \cdot \bar{x}_i + \sum_j \mu_j q_j \cdot \bar{w}_j$$
$$= \sum_i \lambda_i u_i \cdot \bar{x}_i + \mu_1 q_1 \cdot (s_1 - \bar{y}_1).$$

Because $-\mu_1 q_1 \cdot z \geq -\mu_1 q_1 \cdot \bar{y}_1$ we have $\mu_1 q_1 \cdot (s_1 - z) \geq \mu_1 q_1 \cdot (s_1 - \bar{y}_1)$. Therefore,

$$p \cdot \left(\sum_j s_j + \omega\right) \leq \sum_i \lambda_i u_i \cdot \bar{x}_i + \mu_1 q_1 \cdot (s_1 - z) \leq p \cdot \left(\sum_i \bar{x}_i + s_1 - z\right).$$

The last inequality follows from the definition of p and $s_1 - z \geq 0$. We conclude

$$p \cdot \left(\sum_{j \neq 1} \bar{y}_j + \omega\right) \leq p \cdot \left(\sum_i \bar{x}_i - z\right),$$

or

$$p \cdot z \leq p \cdot \left(\sum_i \bar{x}_i - \sum_{j \neq 1} \bar{y}_j - \omega\right) \leq p \cdot \bar{y}_1,$$

which is what we wanted to prove.

(b) Cost minimization

The proof is entirely analogous to (a). Because $(\lambda, -\mu)$ supports V at $\bar{v}$, the vector $(\bar{x}_1, \ldots, \bar{x}_N) \geq 0$ maximizes $\sum_i \lambda_i u_i \cdot x_i$ subject to $x_i \geq 0$ and $\sum_i x_i = \omega + \sum_j \bar{y}_j \geq 0$. Therefore, from the definition of p as a lattice supremum it follows that

$$p \cdot \left(\omega + \sum_j \bar{y}_j\right) = \sum_i \lambda_i u_i \cdot \bar{x}_i.$$

Hence, $\sum_i p \cdot \bar{x}_i \leq \sum_i \lambda_i u_i \cdot \bar{x}_i$. But $p \cdot \bar{x}_i \geq \lambda_i u_i \cdot \bar{x}_i$ for all i because $\bar{x}_i \geq 0$ and $p \geq \lambda_i u_i$. So, we may conclude $p \cdot \bar{x}_i = \lambda_i u_i \cdot \bar{x}_i$ for all i.

Let now $u_i \cdot z \geq u_i \cdot \bar{x}_i$, $z \geq 0$. Then $p \geq \lambda_i u_i$ yields $p \cdot z \geq \lambda_i u_i \cdot z \geq \lambda_i u_i \cdot \bar{x}_i = p \cdot \bar{x}_i$, as we wanted to prove. □

Finally, a comment on the unique supporting hyperplane property. It is on its account that Theorem 2 is not strictly weaker than Theorem 1. We do not know to what extent the conclusion can be weakened and so, this is left as an open problem. It is sure, however, that it can be weakened. Suppose, for example, that everything is as in Theorem 2 except that there is a single firm (this is to lighten notation) with production set $Y = \{z \in Z : q^1 \cdot z \leq c^1$ and

$q^2 \cdot z \leq c^2\}$, where q^1, q^2 are linear, continuous functionals. Then essentially the same proof shows that Theorem 2 remains valid with the supporting linear functional taking the form $\lambda_1 u_1 \vee \cdots \vee \lambda_N u_N \vee (\mu^1 q^1 + \mu^2 q^2)$.

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