Four Lectures on the Differentiable Approach to General Equilibrium Theory

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With few exception the material from the first three lectures is taken from A. Mas-Colell: *The Theory of General Economic Equilibrium: A Differentiable Approach*, Cambridge University Press, 1985. We refer to this text for many extensions and the basic references. The names of the developers of the differentiable approach (at least for the parts covered in these lectures) should, however, be mentioned at the outset: G. Debreu, S. Smale, E. Dierker, and Y. Balasko.

The fourth lecture gives an account of a recent and fascinating development. A major and deep application of the differentiable approach to an area, incomplete market theory, not covered by the above reference.

Lecture I: Single Consumer Theory

I.1 Preference and Utility

The consumers making an appearance in these lectures have preferences defined over nonnegative vectors of R^ℓ, ℓ being the number of commodities. The *consumption set* is thus R^ℓ_+.

A preference relation $\succsim$ is a relation $\succsim \subset R^\ell_+ \times R^\ell_+$ with the properties:

(i) $x \succsim x$ for all $x \in R^\ell_+$ (*reflexivity*).

(ii) "$x \succsim y$ and $y \succsim z$" $\Rightarrow$ "$x \succsim z$" (*transitivity*)

(iii) for every x, y we have that either $x \succsim y$ on $y \succsim x$ (*completeness*).

In addition we always assume that $\succsim$ satisfies a topological property (which does not belong to the essence of the concept of preferences).

(iv) $\succsim$ is a closed set (*continuity*).

By a classic theorem (due to Eilenberg and Debreu) every relation $\succsim$ satisfying (i)–(iv) is representable by a utility function, i.e., there is a $u : R^\ell_+ \to R$ such that "$x \succsim y$" $\Leftrightarrow$ "$u(x) \geq u(y)$". Moreover u can be taken to be continuous. Of course, u is not unique What is intrinsic to $\succsim$ are the family of level curves of u (called *indifference sets*), not the particular indexing (see Figure I.1):

We read $x \succsim y$ as "at least as good", if $x \succsim y$ does hold but $y \succsim x$ does not (resp. does) then we say that x is preferred to y (resp., is indifferent to x), denoted $x \succ y$ (resp., $x \sim y$).

I.2 Properties of Preferences

(i) *Monotonicity:* A $\succsim$ is *monotone* (resp., strictly monotone) if "$x \geq y$" $\Rightarrow$ "$x \succsim y$" (resp., $x \geq y$, $x \neq y \Rightarrow$ "$x \succ y$"). That is, commodities are not noxious (resp., they are desirable). See Figure I.2.

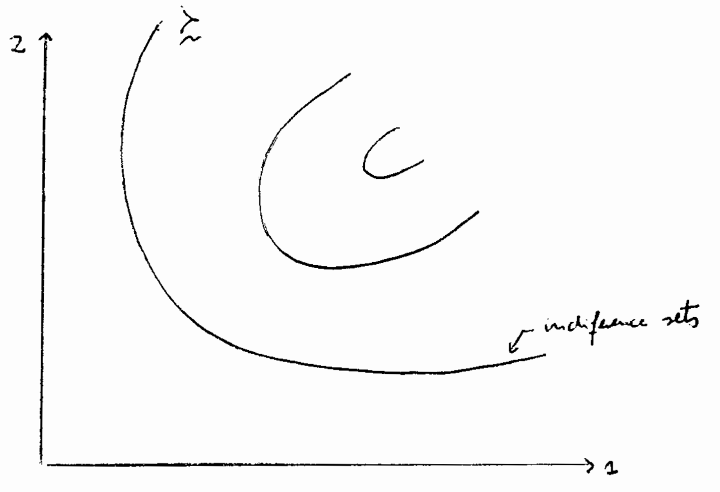

Figure I.1

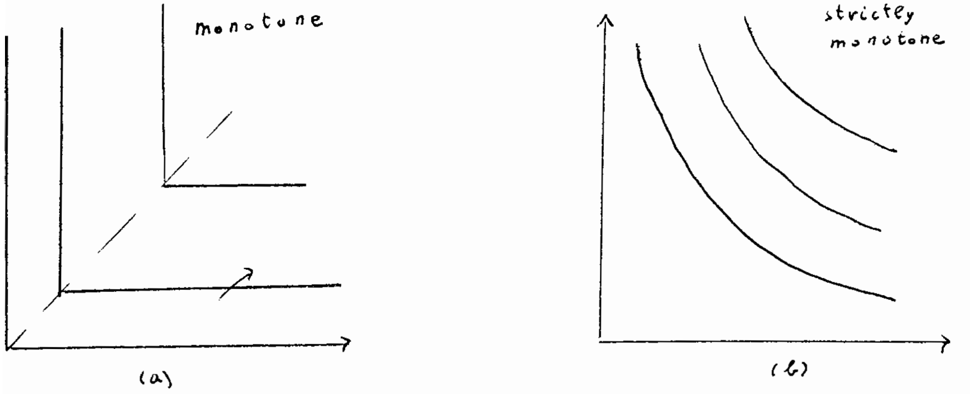

Figure I.2

(ii) *Boundary Condition:* Given $\succsim$, for every $x >> 0$ the at least as good set $\{y: y \succsim x\}$ is closed relative to R^ℓ, i.e., every commodity is indispensable. See Figure I.3.

N.B.: *Unless otherwise stated we assume from now on that preferences satisfy the strict monotonicity and the boundary conditions.*

It is to be emphasized that these restrictions are not essential to the theory. They simply allow for ease of presentation. In particular, the boundary condition allows us to regard $R^\ell_{++} = \{x \in R^\ell: x >> 0\}$ as the consumption set.

(iii) *Convexity:* A $\succsim$ is *convex* (resp., strictly convex) if $\{y: y \succsim x\}$ is a convex set for every y (resp., $\alpha y + (1-\alpha)x \succsim x$ whenever $y \succsim x$ and $0 \leq \alpha < 1$). See Figure I.4.

If $\succsim$ is generated from a concave (resp., strictly concave) utility then $\succsim$ is convex (resp., strictly convex). The converse need not hold (i.e., there are convex preferences not generated by concave utilities).

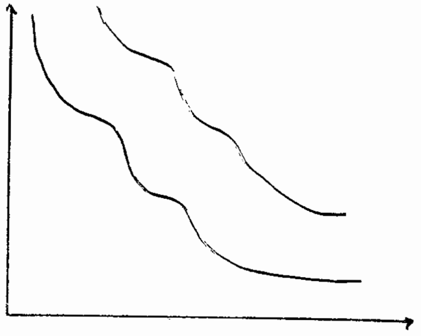

Figure I.3

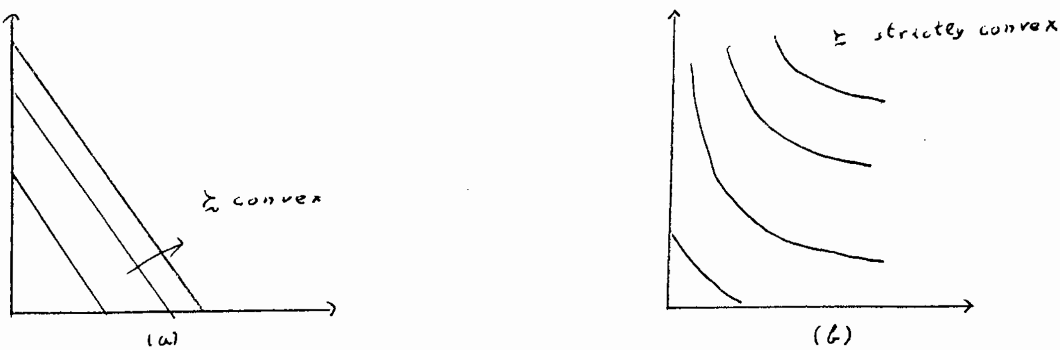

Figure I.4

I.3 Smooth Preferences

Definition. *A $\succsim$ is of class C^r, $r \geq 1$, if the indifference set $I = \{(x,y): x \sim y\} \subset R^\ell \times R^\ell$ (= B dry $\succsim$) is a C^r manifold (i.e., for every $\bar{z} \in I$ there is a C^r function $g: V \to R$ defined on a neighborhood $V \subset R^\ell \times R^\ell$ of $\bar{z}$ s.t. $\partial g(z) \neq 0$ for all $z \in V$ and $g^{-1}(0) = V \cap I$).*

We state without proof.

PROPOSITION. *$\succsim$ is C^r, $r \leq 2$, if and only if $\succsim$ is representable by a C^r utility function $u: R^\ell_{++} \to R$ with no critical point, i.e., $\partial u(x) \neq 0$ for all x.* (The validity of the Proposition does not depend on the maintained monotonicity and boundary conditions.)

Example. $\succsim \subset [-1,1] \times [-1,1]$ is defined by $x \succsim y \Leftrightarrow |x| \leq |y|$. Then $\succsim$ does not have a smooth boundary (Fig. I.5). There is a C^∞ utility for $\succsim$ (i.e., $u(x) = -x^2$) but no C^2 utility with *no vertical point.*

I.4 Curvature

N.B.: *Unless otherwise stated we assume that our preferences are of class C^2 and that utility functions are C^2 have no critical points.*

If a C^2 function $u: R^\ell \to R$ is concave then $\partial^2 u(x)$ is negative semidefinite. If $\partial^2 u(x)$ is in fact negative definite then we say that u is differentiably strictly concave.

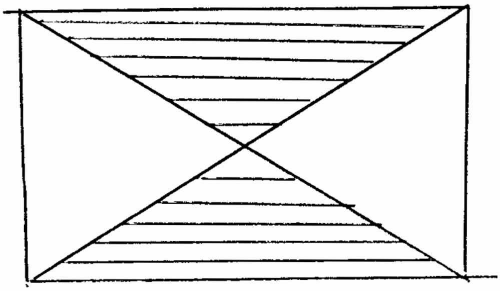

Figure I.5

Given a convex preference relation $\succsim$ we shall now search for a concept of "differentiably strictly convex." Consider $\succsim$ at the point x represented in Fig. I.6. The indifference set at x can be viewed as the boundary of the preferred set. Looking at the curvature of this boundary at x suggests itself.

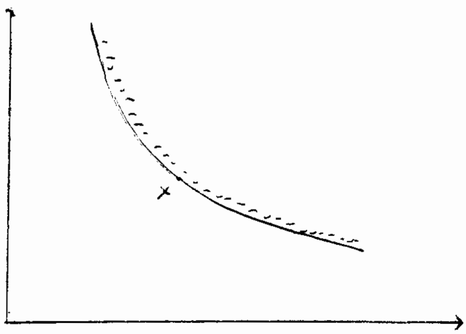

Figure I.6

Definition. *The preference relation* $\succsim$ *is differentiably strictly convex if it is convex (because of the Boundary Condition this is in fact redundant) and for every* $x \in R^{\ell}_{++}$ *the Gaussian curvature of* $I_x = \{y : y \sim x\}$*, viewed as the boundary of* $\{y : y \succsim x\}$*, is nonzero.*

The Gaussian curvature is defined as follows: let $g : R^{\ell}_{++} \to S^{\ell-1}$ be given by $g(y) = (1/\|\partial u(y)\|)\partial u(y)$, where u is an arbitrary utility function for $\succsim$ (the definition is independent of the particular u). See Figure I.7. Let $T_y = \{v \in R^{\ell} : \partial u(y)v = 0\}$. Then at any x, $\partial g(x)$ maps T_x into T_x. The determinant of this linear map is the Gaussian curvature (up to a sign).

There is a simple characterization in terms of utility functions.

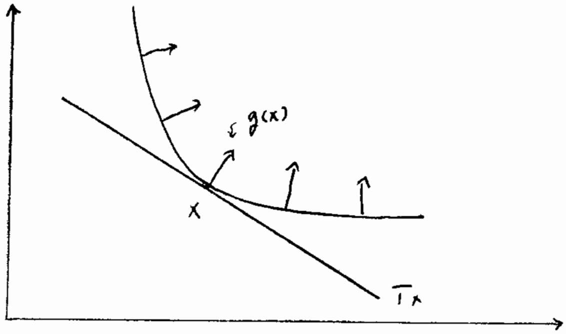

Figure I.7

PROPOSITION: $\succsim$ *is differentiably strictly convex if*

$$\begin{vmatrix} \partial^2 u(x) & \partial u(x) \\ [\partial u(x)]^T & O \end{vmatrix} \neq 0$$

for all x *and some* u *representing* $\succsim$.

The Proposition should be plausible enough if: (i) we remember the definition of Gaussian curvature, and (ii) take into account that for matrices $\underset{n \times n}{A}$ and $\underset{m \times n}{B}$ $(m < n, B$ nonsingular) the determinant of $\begin{bmatrix} A & B \\ -B^T & O \end{bmatrix}$ has the sign of the determinant of the linear map obtained by restricting A to Kernel B and projecting then back into Kernel B, and (iii) we carry out the simple computation allowing one to verify

$$\begin{vmatrix} \partial^2 g(x) & \partial g(x) \\ [\partial g(x)]^T & O \end{vmatrix} \neq 0 \quad \Leftrightarrow \quad \begin{vmatrix} \partial^2 u(x) & \partial u(x) \\ [\partial u(x)]^T & O \end{vmatrix} \neq 0 \ .$$

Yet another interesting characterization is:

PROPOSITION. $\succsim$ *is differentiably strictly convex if, for any smooth utility for* $\succsim$ *and any* x, $\partial^2 u(x)$ *is negative definite on* T_x, *i.e.,* $v \cdot \partial^2 u(x) v < 0$ *for* $v \neq 0, \partial u(x) v = 0$.

Therefore if u has no critical point and it is differentiably strictly concave then $\succsim$ is differentiably strictly convex. There is an (easy) partial converse to this:

PROPOSITION. *If* $\succsim$ *is differentiably strictly convex then for every convex compact* $K \subset R^{\ell}_{++}$ *there is a smooth utility function for* $\succsim$ *which is concave on* K.

Note. We cannot take $K = R^{\ell}_{++}$ in the above. Another necessary condition for concavifiability is that all the at least as good sets $\{y : y \succsim x\}$ have the same asymptotic cone.

I.5 The Demand Function

In this section the strictly convex preference relation $\succsim$ remains fixed.

We shall at last introduce prices. Given a vector of strictly positive prices $p \in R^{\ell}_{++}$ and a level of income (or wealth) w the demand function $\varphi(p, w)$ is defined as the unique maximizer of $\succsim$ on the budget set $\beta(p, w) = \{x \in R^{\ell}_{++} : p \cdot x \leq w\}$. Such a maximum exists by the continuity and boundary hypotheses on $\succsim$. See Figure I.8.

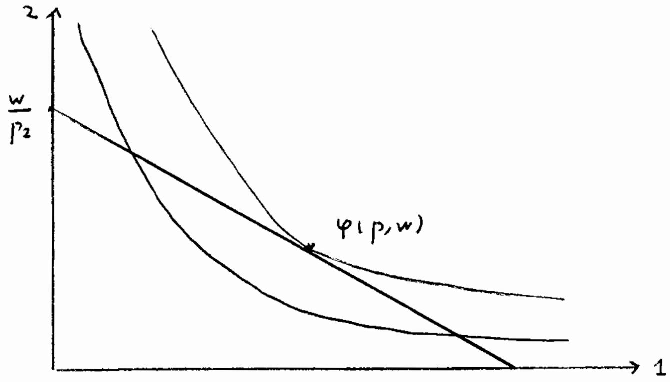

Figure I.8

Some of the obvious, or relatively straightforward properties of the demand function are:

(i) $\varphi(\alpha p, \alpha w) = \varphi(p, w)$ for all p, w and $\alpha > 0$ (*Homogeneity of degree zero*)

(ii) $p \cdot \varphi(p, w) = w$ for all p, w (*Walras law*)

(iii) $\varphi(p, w)$ is singlevalued

(iv) φ is a continuous function on $R^{\ell}_{++} \times R_{+}$.

So far we did not use the smoothness of utility. It is logical to expect that this will be the crucial property in order to get the differentiability of demand.

Suppose that u is a C^2 utility function for $\succsim$ with no critical point. Then, given $(\bar{p}, \bar{w})$, x is the demand vector, i.e., $\bar{x} = \varphi(\bar{p}, \bar{w})$, if and only if (iff) there is $\bar{\lambda}$ such that $(\bar{x}, \bar{\lambda})$ solves the system of equations:

$$(*) \quad \begin{cases} \partial u(x) - \bar{p} = 0 \\ \bar{p} \cdot x = -\bar{w} = 0 \end{cases}$$

Thus, by the Implicit Function Theorem (IFT), $\varphi(p, w)$ will be a differentiable function iff the Jacobian determinant of (*) is nonsingular. But the Jacobian determinant of (*) is

$$\begin{vmatrix} \partial^2 u(x) & \partial u(x) \\ [\partial u(x)^T & O \end{vmatrix}$$

which, as we saw in the previous section, is nonzero iff the nonzero Gaussian curvature condition is satisfied at x. Summarizing: *if there is a smooth utility with no critical point the necessary and sufficient condition for differentiability of demand is that preferences be differentiably strictly convex.*

I.6 The Expenditure Function

Let $\succsim$ be representable by $u(\cdot)$ and fix a $\bar{u} \in u(R^{\ell}_{++})$.

Definition. *The expenditure function* $e_{\bar{u}}: R^{\ell}_{++} \to R$ *is defined as* $e_{\bar{u}}(p) = \min\{p \cdot v : u(v) \geq \bar{u}\}$. *The corresponding (unique) minimizer is denoted* $h_{\bar{u}}(p) \in R^{\ell}_{++}$. *See Figure I.9.*

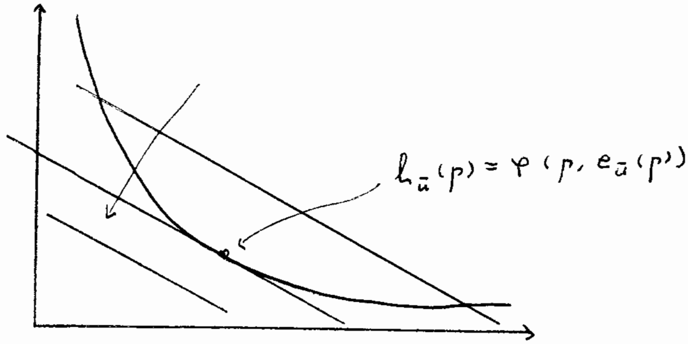

Figure I.9

The function $h_{\bar{u}}(p)$ is called the "compensated demand function." It relates to the demand function of the previous section by $h_{\bar{u}}(p) = \varphi(p, e_{\bar{u}}(p))$.

Mathematically, $e_{\bar{u}}$ is nothing but the support function of a (convex) set. From this some important properties follow:

(i) $e_{\bar{u}}$ is homogeneous of degree one.

(ii) $e_{\bar{u}}$ is concave.

(iii) $e_{\bar{u}}$ is C^1 and $\partial e_{\bar{u}}(p) = h_{\bar{u}}(p)$.

Property (i) is obvious; (ii) is easy to verify directly; for (iii) note that the linear function $p \cdot h_{\bar{u}}(\bar{p})$ majorizes $h_{\bar{u}}(p)$. Hence, if $e_{\bar{u}}$ is differentiable we must have $\partial e_{\bar{u}}(\bar{p}) = h_{\bar{u}}(\bar{p})$.

As we see the compensated demand function $h_{\bar{u}}(\cdot)$ satisfies nice properties but, in contrast with $\varphi(p, w)$, it is not directly observable in the marketplace (the utility function u enters its definition). It turns out however that we can use the properties of $h_{\bar{u}}(\cdot)$ to generate restrictions on the observable market demand function $\varphi(p, w)$. Indeed we have $\partial e_{\bar{u}}(p) = h_{\bar{u}}(p) = \varphi(p, e_{\bar{u}}(p))$ for all p. Hence, letting $u(\varphi(\bar{p}, \bar{w})) = \bar{u}$, $e_{\bar{u}}(\bar{p}) = \bar{w}$, we always have:

$$\partial^2 e_{\bar{u}}(p) = \partial_p \varphi(p, e_{\bar{u}}(p)) + \partial_w \varphi(p, e_{\bar{u}}(p)) \underbrace{(\partial e_{\bar{u}}(p))^T}_{\varphi(p, e_{\bar{u}}(p))} \quad .$$

Evaluating at $(\bar{p}, \bar{w}, \bar{u})$:

$$\partial^2 e_{\bar{u}}(p) = \partial_p \varphi(\bar{p}, \bar{w}) + \partial_w \varphi(\bar{p}, \bar{w})(\varphi(\bar{p}, \bar{w}))^T \quad .$$

The right-hand side only involves the derivatives of φ and it is called the Substitution or Slutzky matrix. The left-hand side is the Hessian matrix of a concave function; therefore, it is negative semidefinite. Note that we always have $p \cdot \partial^2 e_{\bar{u}}(p) = 0$ and $\partial^2 e_{\bar{u}}(p)p = 0$. So, $\partial^2 e_{\bar{u}}(p)$ cannot be negative definite. However, it is always negative definitive on $T_p = \{v{:}p \cdot v = 0\}$. Summarizing:

PROPOSITION. *For all* (p, w), *the substitution matrix*

$$\partial_p \varphi(p, w) + \partial_w \varphi(p, w)(\varphi(p, w))^T$$

is negative definite on $T_p = \{v{:}p \cdot v = 0\}$.

The above is the fundamental economic property of demand.

1.7 The Indirect Utility Function

As before we let $\succsim$ be a C^2, differentially strictly convex preference relation and u a corresponding smooth utility function.

Definition. *The indirect utility function is defined (for* $p >> 0$, $w > 0$*) by* $v(p, w) = u(\varphi(p, w))$.

The theory of the direct (u) and the indirect utility function (v) is rich in duality relations. We shall not get into them now. We merely mention:

PROPOSITION. *The sets* $\{(p, w){:}v(p, w) \leq \bar{v}\}$ *are convex for all* $\bar{v}$. *If* v *is* C^1 *at* $(\bar{p}, \bar{w})$ *then:*

(i) $\partial_w v(\bar{p}, \bar{w}) = -\frac{1}{\bar{w}}\, \bar{p} \cdot \partial_p v(\bar{p}, \bar{w})$

(ii) $\varphi(\bar{p}, \bar{w}) = -\frac{1}{\partial_w v(\bar{p}, \bar{w})}\, \partial_p v(\bar{p}, \bar{w})$.

The fundamental property is (ii) (also called Roy's identity). It is again a consequence of the properties of the support functions of convex sets; in particular, of the fundamental duality fact $\partial e_{\bar{u}}(\bar{p}) = h_{\bar{u}}(\bar{p})$. Indeed, we have the identity $v(p, e_{\bar{v}}(p)) = \bar{v}$. Differentiate and recall that $\partial \varphi_{\bar{u}}(\bar{p}) = \varphi(\bar{p}, \bar{w})$. For (i) differentiate $v(\lambda \bar{p}, \lambda \bar{w}) - v(\bar{p}, \bar{w}) = 0$ with respect to λ and evaluate at $\lambda = 1$.

It is also (ii) which accounts for the usefulness of indirect utility functions. Indeed, (ii) tells us that it is very easy to derive demand from indirect utility (this is in contrast to deriving demand from the direct u). It is "almost" like taking a derivative. The "almost" is for the $1/\partial_w \varphi(\bar{p}, \bar{w})$ factor. In applications it is often possible to go around this factor and get a fully linear dependence of $\varphi(\cdot)$ on $v(\cdot)$. We discuss two illustrative examples.

Example 1. Fix $w = 1$ and denote $v(p) = v(p, w)$. This is just a normalization. We say that the indirect utility function $v(\cdot)$ is logarithmically homogeneous if $v(\alpha p) = v(p) - \ell n\ \alpha$ for $\alpha > 0$ (i.e., it is the ℓn transformation of an homogeneous of degree one function). Then it is easy to verify that $\varphi(p, 1) = -\partial v(p)$ (because $p \cdot \partial v(p) = -1$). So $v \mapsto \varphi$ acts linearly on the convex set of logarithmically homogeneous indirect utility functions.

Example 2. Let v be an indirect utility function. Define quadratic perturbations of $v(p,w)$ by $v_Q(p,w) = v(p,w) + \frac{1}{w^2}\, p \cdot Qp$ where Q is an symmetric matrix. Denote the corresponding demand by φ_Q. With $\mathcal{Q}$ the set of symmetric $\ell \times \ell$ matrices let

$$\mathcal{Q}_p = \{Q \in \mathcal{Q}: p \cdot Q = 0\} \quad .$$

It is then easy to verify that at any fixed $(\bar{p}, \bar{w})$ $\varphi_Q(\bar{p}, \bar{w})$ and $\partial_w \varphi_Q(\bar{p}, \bar{w})$ are invariant to Q and $Q \mapsto \partial_p \varphi_Q(\bar{p}, \bar{w})$ acts linearly.

Lecture II: Pareto Optimality

II.1. Definitions and Preliminaries

We shall now proceed to put n consumers together.

Each consumer $i = 1, \ldots, n$ has a preference relation $\succsim_i$ on R^ℓ_+. We let $\succsim_i$ be represented by a utility function $u_i : R^\ell_+ \to R$. We always assume that $\succsim_i$ satisfies the defining properties of a preference relation plus strict monotonicity. Also, $u_i(0) = 0$, all i.

There is a total endowment of commodities, i.e., a vector $\omega \in R^\ell_+$.

A vector $x = (x_1, \ldots, x_n) \in R^{\ell n}_+$ is an *allocation* if $\sum_i x_i \leq \omega$. In these lectures there shall be no production of commodities by means of commodities. Thus the only economic problem is the allocation of the total vector of goods ω. We shall also not bring to bear during these lectures considerations of fairness. Thus from the welfare point of view we shall not aim at singling out very definite outcomes but just at delimitating the class of nonwasteful ones.

Definition. *The allocation x is a Pareto Optimum* (P.O.) *if there is no other x' such that $x'_i \succ_i x_i$ for every i.*

Because of strict monotonicity this definition is equivalent to the more correct: "there is no other x' such that $x'_i \succsim_i x_i$ for all i with at least one strict preference."

Denote by $P \subset R^{\ell n}_+$ the set of P.O. allocations.

For two consumers and two goods the set of allocations can be represented in the so-called Edgeworth's box, i.e., $\{z : 0 \leq z \leq \omega\}$, where z stands for the allocation $(z, \omega - z)$. The following figures provide some examples of Pareto sets P.

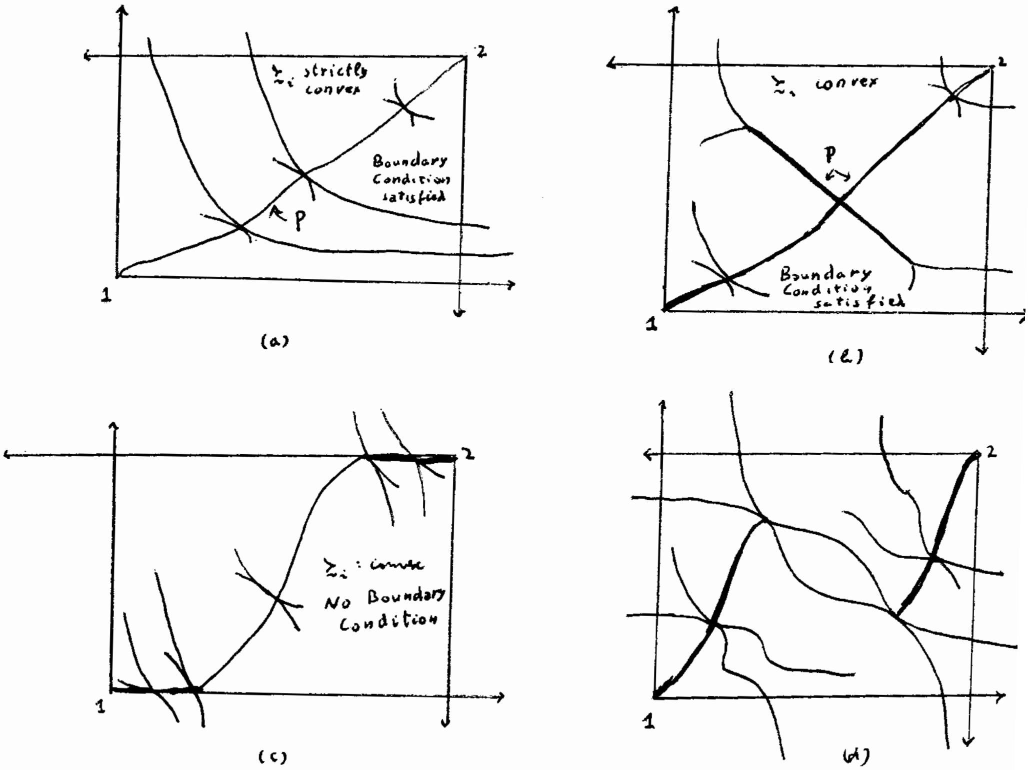

Figure II.1

Another set of interest is:

$$U = \{u = (u_1, ..., u_n) \in R^n : u \leq u(x) = (u_1(x_1), ..., u_n(x_n)),$$
$$\text{for } x = (x_1, ..., x_n) \text{ an allocation}\}$$

Graphically:

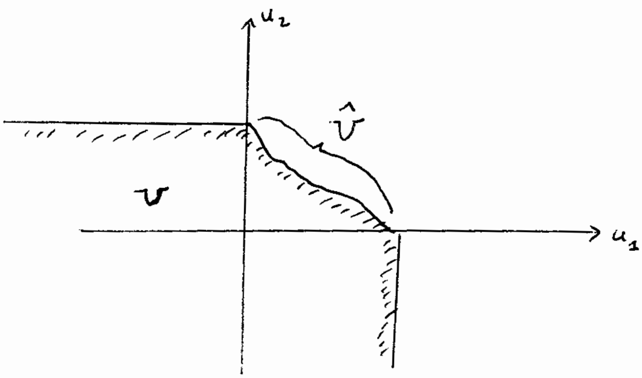

Figure II.2

By monotonicity $U - R^n_+ \subset U$. Hence U is what is called a comprehensive set. The upper frontier of U, $\hat{U} = Bdry U \cap R^n_+$ is, by definition and strict monotonicity (which implies that if $u' \leq u$, $u', u \in \hat{U}$ then $u' = u$) the utility image of the Pareto set P. Note that

$$u\left(\left\{x \in R^{\ell n}_+ : \sum_i x_i \leq \omega\right\}\right) = U \cap R^n_+.$$

It can be shown that $\hat{U}$ is topologically a simplex. We call $\hat{U}$ the utility Pareto set.

II.2. The First Fundamental Theorem

We now proceed to introduce prices.

Definition. *The vector* $x = (x_1, ..., x_n)$ *is a* price equilibrium *if there is* $p \in R^\ell$, $p \neq 0$, *such that* $v \succ_i x_i$ *implies* $p \cdot v > p \cdot x_i$ *for every* i.

PROPOSITION. *If* x *is a price equilibrium then it is a Pareto optimum allocation for* $\omega = \sum_i x_i$.

Proof. Note first that because of monotonicity we must have $p \geq 0$. Let now $x' = (x'_1, ..., x'_n)$ be such that $x'_i \succ_i x_i$, all i. Then $p \cdot x'_i > p \cdot x_i$ for all i. Hence $p \cdot (\sum_i x'_i) > p \cdot \sum_i x_i = \omega$. So $\sum_i x'_i \leq \omega$ cannot occur. ∎

Simple as this result is it is of fundamental economic importance.

The converse to the proposition need not be true (see Fig. II.3 for a one consumer counterexample), although as we shall see it is almost true if convexity hypotheses are added.

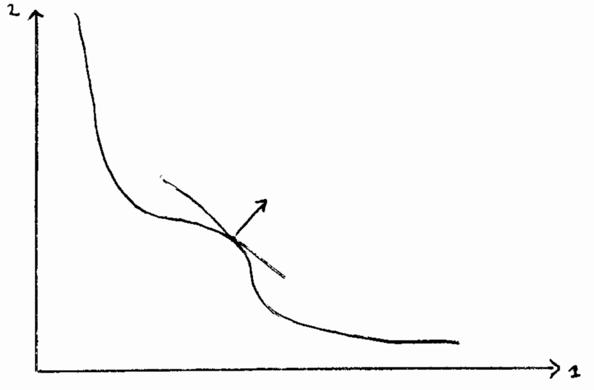

Figure II.3

A "dual" of the Proposition in utility space goes as follows. Let $\lambda \in R^n_+$, $\sum_i \lambda_i = 1$ and consider (for given ω):

$$\text{Max} \sum_i \lambda_i u_i$$
$$\text{s.t. } u \in U.$$

Then if $\bar{u}$ solves this problem it follows that $\bar{u}$ is a P.O. utility allocation. The proof is of course obvious.

II.3. First Order Necessary Conditions

We take for granted the following Kuhn-Tucker like mathematical fact.

Let $f : R^s \to R^t$ be a vector of C^1 objective functions and $h : R^s \to R^m$ a vector of C^1 constraints. The constraint set is $E = \{x \in R^s \ : \ h(x) \geq 0\}$. We say that $x \in E$ is a *local weak optimum* of f subject to h if for some neighborhoods V of x there is no $x' \in E \cap V$ such that $f(x') >> f(x)$.

First Order Necessary Conditions (FONC): If x is a local weak optimum of f subject to h then there are $(\lambda, \mu) \in R^t_+ \times R^m_+$ s.t.:

(i) $(\lambda, \mu) \neq 0$

(ii) If $h_j(x) > 0$ then $\mu_j = 0$

(iii) $\sum_{i=1}^t \lambda_i \partial f_i(x) + \sum_{j=1}^m \mu_j \partial h_j(x) = 0$

Suppose, to get back to the economics, that the utility functions are C^1 and strictly increasing. The next proposition is an easy consequence of the FONC.

PROPOSITION. *Let* x *be a* P.O. *allocation. Then there are* $p \in R^\ell_{++}$ *and* $\lambda \in R^n_{++}$ *such that:* $\lambda_i \partial u_i(x_i) \leq p$ *for all* i *and* $x_i \cdot [p - \lambda_i \partial u_i(x_i)] = 0$ *for all* i.

(Hint for the application of the FONC: forgetting about the nonnegativity constraints we have here: $s = \ell n, \ t = n, \ m = \ell$.)

In particular, if $x >> 0$ then $\lambda_i \partial u_i(x_i) = p$ for all i.

We will see later that under convexity hypotheses the multipliers λ, p are rich in economic interpretations.

The FONC interact nicely with the price equilibrium concept.

PROPOSITION. (i) *If* $\bar{x}$ *is a price equilibrium with respect to* $\bar{p}$ *then for some* $\bar{\lambda}$, $(\bar{p}, \bar{\lambda})$ *solve the* FONC.

(ii) *If* $\bar{u}$ *solves* Max $\sum_i \bar{\lambda}_i u_i$, $u \in U$, $\bar{\lambda} \geq 0$, and $u(\bar{x}) = \bar{u}$, then for some $\bar{p}$, $(\bar{p}, \bar{\lambda})$ solve the FONC at $\bar{x}$.

The proof is easy and will not be given. Figure II.4 illustrates the proposition.

One could ask: when are the FONC sufficient to determine (p, λ) uniquely (up to a positive factor)? An answer is: when x is *linked*, i.e., when it is not possible to split consumers and commodities into two groups in such a way that no consumer of one of the groups consumes any commodity consumed by any consumer of the second group.

II.4 The Second Fundamental Theorem

We assume now that preferences are convex. To obtain the cleanest theory we assume a bit more, that every u_i is concave. Under this hypothesis the set U is convex.

The implications of the last two sections will now also be valid in the reverse direction: if x satisfies the FONC then it is an optimun and if x satisfies the FONC then it can be supported as a price equilibrium or be supported by utility weights. Formally, we begin by this second fact:

PROPOSITION. *Let* x *satisfy the* FONC *with respect to* (p, λ). *Then:*

(i) x *is a price equilibrium with respect to* p.

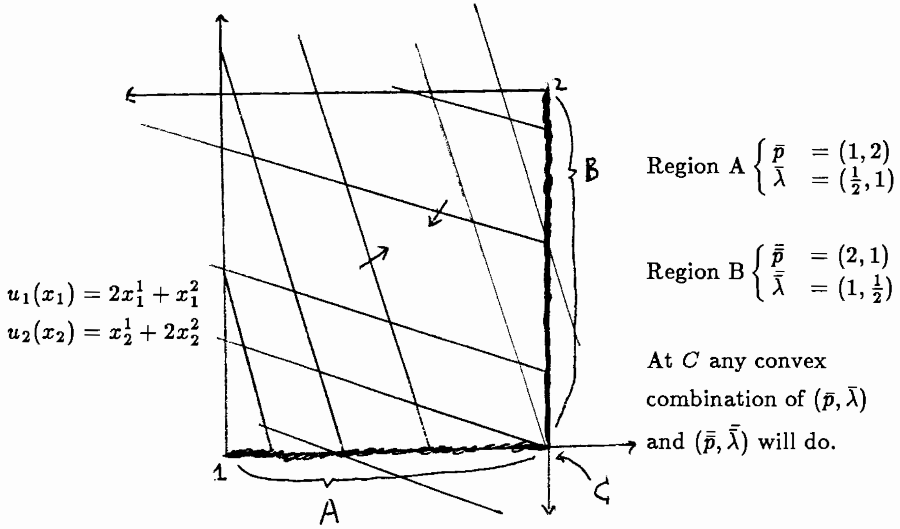

Figure II.4

(ii) $u(x)$ *solves* Max $\sum_i \lambda_i u_i$, $u \in U$.

Proof. (i) To be simple we look at the case $x >> 0$. Then the FONC yield $\partial u_i(x_i) = (1/\lambda_i)p$ which, recalling the definition of the demand function at p, yields $x_i = \varphi_i(p,\ p \cdot x_i)$ for all i. Hence x is a price equilibrium with respect to p.

(ii) For all i, if $\sum_i x_i' \leq \sum_i x_i = \omega$ we have

$$p \cdot (x_i' - x_i) \underset{\text{FONC}}{\geq} \lambda_i \partial u_i(x_i)(x_i' - x_i) \underset{\text{Concavity}}{\geq} \lambda_i(u_i(x_i') - u_i(x_i)).$$

Hence

$$\sum_i \lambda_i u_i(x_i) \geq \sum_i \lambda_i u_i(x_i') - \sum_i p \cdot (x_i' - x_i) \geq \sum_i \lambda_i u_i(x_i') \qquad \blacksquare$$

As a graphical illustration:

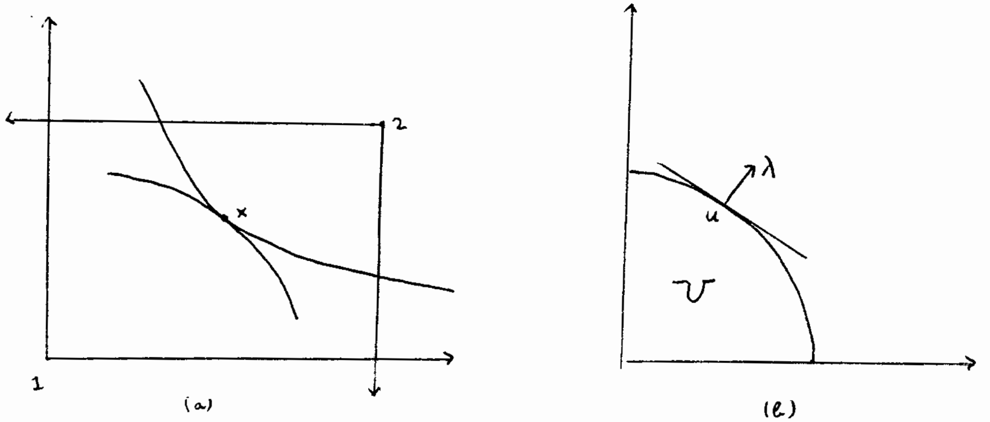

Figure II.5

Of course, an implication of the above proposition is:

PROPOSITION. *If* x *is a* P.O. *allocation then there are* p, λ *such that* (i) *and* (ii) *of the previous proposition are satisfied.*

For yet another, more geometric proof of (i), (ii):

(i) For x a P.O. allocation defines $V = \sum_i \{v_i : v_i \succsim_i x_i\} - \omega$. Then V is convex and $0 \in BdryV$. If we let p support V at 0 then we are done (see Figure II.6.(a)).

(ii) Let $\bar{u}$ be a P.O. utility allocation. Then $\bar{u} \in B(dry)U$. If we let λ support U at $\bar{u}$ we are done (see Fig. II.6.(b)).

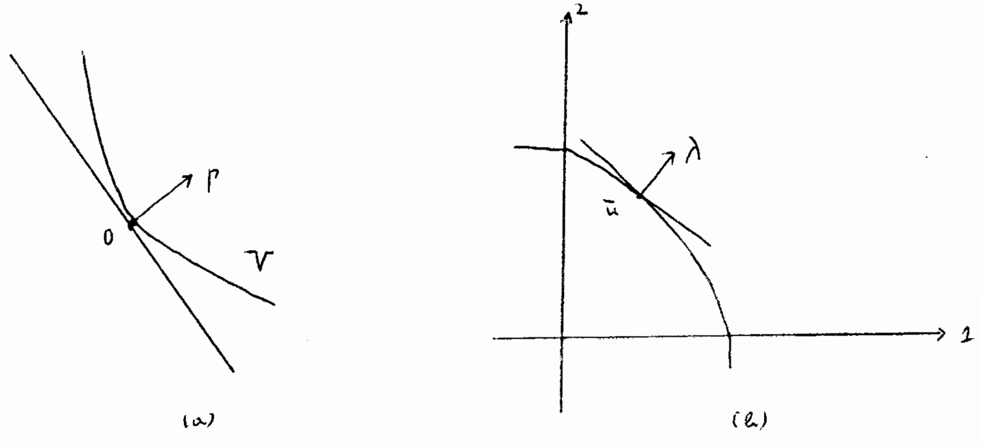

Figure II.6

We can at this point interpret economically the multipliers $(\bar{p}, \bar{\lambda})$ associated to a P.O. allocation $\bar{x}$:

(i) p is the vector shadow prices of the commodities for the social evaluation function $\sum_i \lambda_i u_i(x_i)$. That is, consider:

$$\text{Max} \sum_i \bar{\lambda}_i u_i(x_i)$$

$$\text{such that} \sum_i x_i - \omega \leq 0 \ .$$

Then the $\bar{p}_j$ are the multipliers for this problem and so p_j is the social value of increasing the endowment of the j good by a small infinitesimal amount.

(ii) $\bar{\lambda}_i$ is the reciprocal of the marginal utility of income of consumer i at the price-income combination $(\bar{p}, \bar{p} \cdot \bar{x}_i)$. Indeed, recall that (neglecting inequalities) $\partial u_i(x_i) - (1/\bar{\lambda}_i)\bar{p} = 0$. So $1/\bar{\lambda}_i$ is a multiplier for the problem:

$$\text{Max } u_i(x_i)$$

$$\text{s.t. } \bar{p} \cdot x_i - \bar{p} \cdot \bar{x}_i = 0 \ .$$

Assuming the strict convexity of $\succsim_i$ we can say something of interest about the global topology of P.

PROPOSITION. *P is homeomorphic to the $(n-1)$ simplex.*

Proof. Clearly this is the case for $\hat{U}$. Note then that the natural map $u \mapsto \{x : u(x) = u\}$ from $\hat{U}$ to $R_+^{\ell n}$ is singlevalued (obvious) and continuous (easy). ■

If preferences are C^2, differentially strictly concave and satisfy the boundary condition then $\hat{U} \cap R_{++}^n$ and $P \cap R_{++}^{\ell n}$ are in fact diffeomorphic under this natural map and both are also diffeomorphic to the open unit simplex under, for example, $\lambda \mapsto \bar{u}$: solution to Max $\sum_i \lambda_i u_i$, $u \in U$.

II.5 Second Order Conditions

In this section preferences are C^2 but not necessarily convex.

The analysis of second order conditions can be quite subtle. Here I will be very rough. I will only look at sufficient conditions and stay away from boundaries.

Let $x >> 0$ satisfy the FONC with respect to (p, λ).

PROPOSITION. *A sufficient condition for x to be a local* P.O. [*the definition of local* P.O. *is the obvious one*] *is that the bilinear form*

$$B(v, v') = \sum_i \lambda_i \partial^2 u_i(x_i)(v_i, v_i')$$

(defined on $R^{\ell n} \times R^{\ell n}$) be negative definite on $K = \{v \in \prod_{i=1}^n T_p : \sum_{i=1}^n v_i = 0\} \subset R^{\ell n}$.

Illustrations:

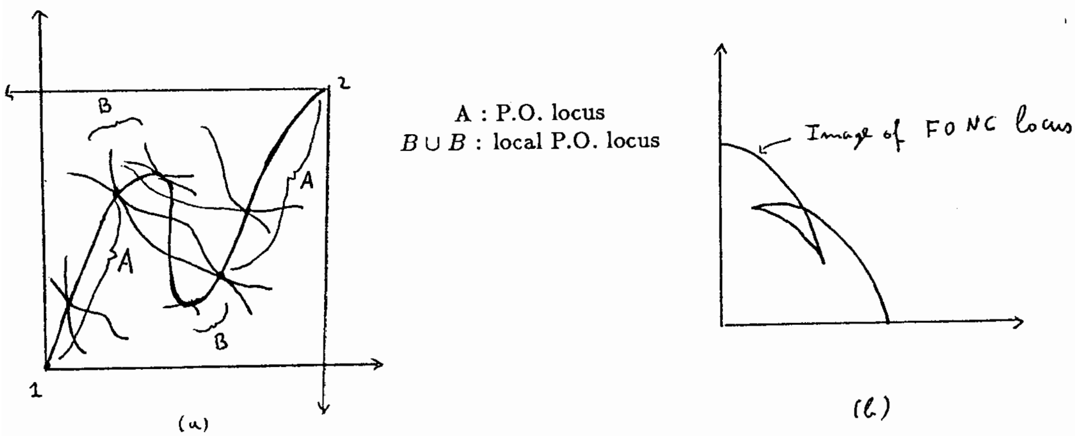

Figure II.7

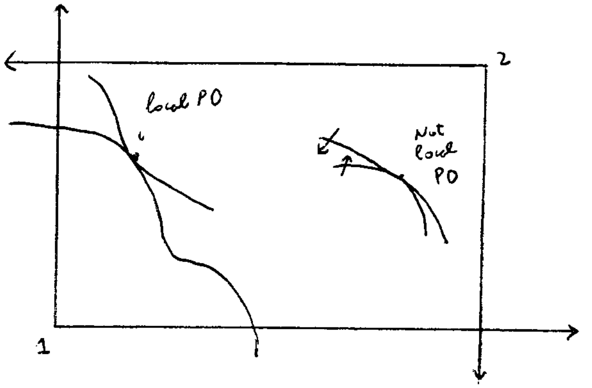

Figure II.8

For an interesting application:

PROPOSITION. *Let $x >> 0$ satisfy the second order sufficient conditions. Then, except for at most $\ell - 1$ agents, $\partial^2 u_i(x_i)$ is negative definite on T_p.*

That is, if n is much larger than ℓ then only exceptionally consumers will not be in local price equilibrium.

Proof. Define $J_i = \{v_i \in T_p : \partial^2 u_i(x_i)(v_i, v_i) \geq 0\}$. Note that $-J_i = J_i$. Suppose that $J_i \neq \{0\}$ for ℓ consumers. Since T_p is $(\ell - 1)$ – dimensional we can find $v_i \in J_i$, not all zero, such that $0 = \sum_i v_i$. But this contradicts the negative definiteness of $\sum_i \lambda_i \partial^2 u_i(x_i)$ on K. ■

Lecture III: Walrasian Equilibrium

III.1 Basic Definitions

To the exchange set-up of the previous lecture we now add a further consideration: individual consumers have entitlements (i.e., own) to a part of the social endowments. In consequence we impose as an equilibrium condition that the value of individual consumptions be the same as the value of individual endownments.

As before we have N consumers endowed with preferences $\succsim_i$ on R^ℓ_+. Every $\succsim_i$ is strictly monotone and representable by a concave utility function u_i. Every consumer i is also endowed with an initial endowment vector $\omega_i >> 0$. We put $\omega = (\omega_1, ..., \omega_N)$.

Definition. *The allocation x is a Walrasian equilibrium if there is a price vector $p \neq 0$ such that x is a price equilibrium with respect to p and $p \cdot x_i = p \cdot \omega_i$ for all i. In other words, for every i, x_i maximizes u_i on the budget set $\{z : p \cdot z \leq p \cdot \omega_i\}$.*

Because of the First Fundamental Theorem a Walrasian equilibrium is a Pareto Optimum. It also follows from strict monotonicity that at an equilibrium we must have $p >> 0$.

The following figure illustrates the concept of equilibrium.

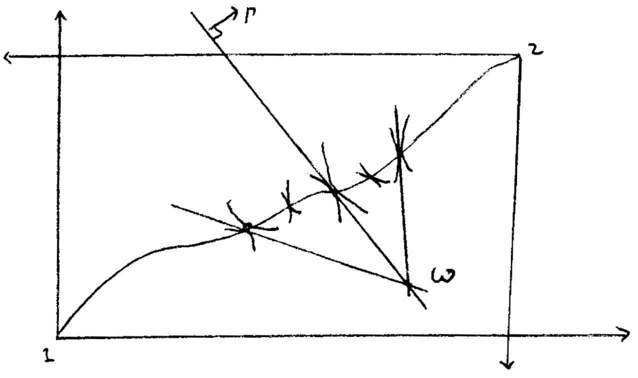

Figure III.1

III.2 Equilibrium Equations

In order to study the properties of equilibria it is convenient to express them as the zeroes of a system of equations. This can be done in several different ways.

Let every u_i be C^1 and assume for simplicity that at equilibrium every consumer consumes strictly positive amounts of every good (i.e., goods are indispensable). Then $x = (x_1, \ldots, x_N)$ is a Walrasian equilibrium if and only if for some $p >> 0$ and $\lambda = (\lambda_1, \ldots, \lambda_N) \in R^N$ the following system of equations is satisfied:

[I] $$\partial u_i(x_i) - \lambda_i p = 0\,, \qquad \text{all } i$$

[II] $$p \cdot (x_i - \omega_i) = 0\,, \qquad \text{all } i$$

[III] $$\sum_i x_i - \sum_i \omega_i = 0\,.$$

This would be called the universal system of equations. In applications it may be convenient to look at more consolidated systems. For example, [I] and [III] can be used to solve p and x_i as a function of λ, i.e., $p(\lambda)$, $x_i(\lambda)$. Replacing this in [II] we end up with an equation system $p(\lambda) \cdot (x_i(\lambda) - \omega_i) = 0$, $1 \le i \le N$, that involves only λ. The search for equilibrium is viewed as the search for the social weights having the property that if a Pareto Optimum is chosen by maximizing the weighted sum of utilities then the value of individual consumptions (evaluated at the imputed shadow prices) is equal to the value of individual endowments. If we are dealing with few consumers and many commodities this is a particularly convenient way to formalize equilibria.

Nonetheless we shall focus on a more traditional reduction of the universal system. It is in a sense dual to the one just described. Assume that every u_i is strictly concave. Using [I] and [II] we can express λ and x_i as a function of p, i.e., $\lambda(p)$, $x_i(p)$. Replacing in [III] we have $\sum_i (x_i(p) - \omega_i) = 0$. The function $f(p) = \sum_i (x_i(p) - \omega_i)$ is called the (aggregate) *excess demand function.*

The function $f(p)$ is related to the demand function $\varphi_i(p, w_i)$ of the first lecture as follows: $x_i(p) = \varphi_i(p, p \cdot \omega_i)$. There we can immediately conclude on some of the basic properties of $f : R^{\ell}_{++} \to R$:

(i) f is homogeneous of degree zero, i.e., $f(\alpha p) = f(p)$, all p and $\alpha > 0$.

(ii) f is continuous (and differentiable if every $\succsim_i$ is differentiably strictly concave).

(iii) f is bounded below. Indeed, $f(p) \geq -\sum_i \omega_i$ for all p.

(iv) f satisfies the so-called Walras' law: $p \cdot f(p) = 0$ for all p.

(v) f is a proper map on any domain $\{p : p \cdot v = 1\}$ (where $v \geq 0$, $v \neq 0$). That is to say, if $p_n \to p$, $p_n \cdot v \geq 1$ and $p^j = 0$ for some j then $\|f(p^j_n)\| \to \infty$.

Property (v) follows from (iii) and the strict monotonicity of every $\succsim_i$.

III.3 Existence of Equilibrium

The existence of a Walrasian equilibrium is not difficult to prove. The traditional tool for establishing the existence of a zero of the excess demand function has been Brouwer's fixed point theorem or any of its variants. For later reference we will avoid an explicit fixed point route.

Denote by $S = \{p \in R^{\ell}_{++} : \|p\| = 1\}$ the strictly positive part of the unit sphere. Because of the homogeneity of degree one of f if p is an equilibrium price vector then so is $(1/\|p\|)p$. Hence without loss of generality we can confine our search of equilibrium to S. Walras' law tells us that $p \cdot f(p) = 0$ for all p, or $f(p) \in T_p = \{v \in R^{\ell} : p \cdot v = 0\}$, i.e., $f(p)$ is nothing but a tangent vector fields on S. Properties (iii) — boundedness below — and (v) — properness — of f imply that f point inwards at the boundary. See Figure III.2. This inward pointing property will be preserved if we replace S by a slightly trimmed closed subset $\bar{S}$ having a smooth boundary. (More precisely, what is preserved is the property of being homotopic to an inward pointing vector field.) As it is well-known: (i) the mod 2 Euler number of an inward pointing vector field on a (connected) manifold with boundary is nonzero if the Euler characteristic of the manifold is nonzero, and (ii) if the mod 2 Euler number of the vector field is different from zero then the vector field has at least one zero. Because $\bar{S}$ is homotopic to the $\ell - 1$ ball it has a nonzero Euler characteristic. Hence f has at least one zero.

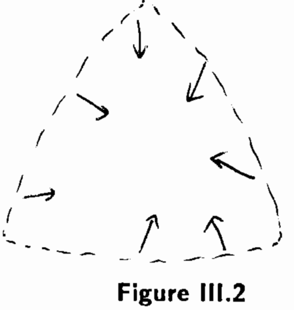

Figure III.2

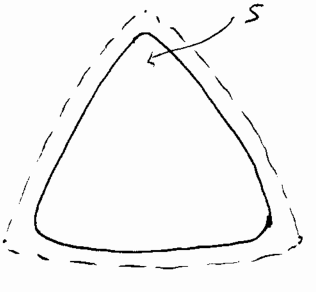

Figure III.3

For later reference it is convenient to rephrase the above argument in a more general manner. Admittedly at this point it will appear as uncalled for generality.

A tangent vector field f or, more precisely, its graph, can be viewed as a section of the tangent bundle $\tau_{\bar{S}}$ of $\bar{S}$. The total space of $\tau_{\bar{S}}$ is $T\bar{S} = \{(p, v) \in \bar{S} \times R^{\ell} : p \cdot v = 0\}$.

Denote by $\sigma_0 : \bar{S} \to T\bar{S}$ the zero section of $\tau_{\bar{S}}$, i.e., $\sigma_0(p) = (p, 0)$, all $p \in \bar{S}$. Then p is an equilibrium if and only if $\sigma_0(p) = (p, 0)$. Therefore the existence of a zero for f is equivalent to σ_0 and Graph f having a nonempty intersection. Because the fibers of $\tau_{\bar{S}}$ are convex, any two sections are homotopic. Furthermore, inward pointing sections will not only be homotopic but will have the same Euler intersection number with the zero section. If this number is different from zero then the intersection must be nonempty. Summarizing: in order to prove the existence of equilibrium it is enough to exhibit a section having a nonzero intersection number with σ_0. But this is easy. Pick an arbitrary $\bar{p}$ and let $g(p)$ equal the perpendicular projection of $p - \bar{p}$ on T_p. Clearly, $g(p) = 0$ only for $p = \bar{p}$. Strictly speaking in order to prove that the intersection number with σ_0 is nonzero we should argue that $\bar{p}$ is not a coincidental zero. This should be obvious enough (it is geometrically trivial — and easy to verify: see the next three sections — that g and σ_0 intersect transversally).

III.4 Local Uniqueness

Is the equilibrium, which existence has already been established, unique? Figure III.1 tells us immediately that not necessarily. We are dealing with highly nonlinear problem (e.g., the excess demand function can never be linear) and there is no general hope of uniqueness (of course, uniqueness is possible in particular and well studied cases).

On the other hand a count of equations and unknowns tells us that there are $\ell - 1$ effective unknowns (the dimension of S) and $\ell - 1$ possibly independent equations (because of Walras' law one component of excess demand is dependent on the others). So one may hope that the equilibrium be locally determinate. As Figure III.4 shows, this need not be the case. In the figure we have a continuum of equilibria. Nonetheless, the situation seems quite pathological (i.e., coincidental) and prompts the following question: is the local uniqueness of equilibrium a generic property of economies?

By using a differentiable approach we shall see in the next two sections that the answer is affirmative. Although we shall not go into it here, it is worth pointing out that this is no longer true in more general contexts, e.g., it may fail in economies with infinitely many commodities and agents. This is one of the active areas of current research.

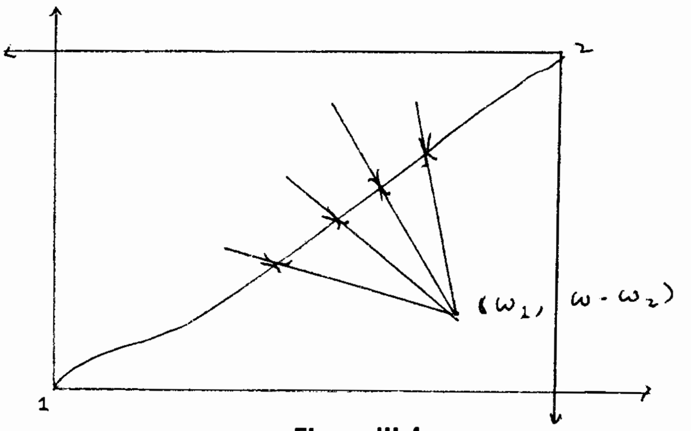

Figure III.4

III.5. Regular Economies

From now on we assume that the excess demand function $f : S \to R^\ell$ is C^1. Go back to Lecture I for the conditions on preferences that imply this. In particular, we are assuming that consumption always takes place in the interior of the positive orthant.

Viewed as a matrix $\partial f(p)$ is $(\ell-1)\times\ell$ and therefore always singular. However, because of Walras' law, $\partial f(p)$ maps T_p into T_p whenever $f(p) = 0$, i.e., at equilibrium (**proof:** differentiate $p \cdot f(p) = 0$ to get $p \cdot \partial f(p) + f(p) = 0$). This motivates the next definition.

Definition. *The equilibrium price vector* p *is regular if* $\partial f(p)$ *maps* T_p *into* T_p.

There are many equivalent forms of the regularity definition. Thus, p is regular if and only if the $(\ell-1)\times(\ell-1)$ matrix obtained by deleting any row and corresponding column from $\partial f(p)$ is nonzero, or if and only if Graph f and σ_0 are transversal at $(p, 0)$ when viewed as submanifolds of TM, etc.

We say that the economy is *regular* if every equilibrium price is regular. The following fact is easy to prove:

PROPOSITION. *A regular economy has a finite number of equilibrium.*

From now on we implicitly let preferences be fixed but want to consider variations on initial endowments $\omega \in R^{\ell N}_{++}$. Thus, we identify the economy with ω and denote the corresponding excess demands by $f_\omega(\cdot)$ or $f(\cdot, \omega)$.

By the implicit function theorem, if $\bar{p}$ is a regular equilibrium for the economy $\bar{\omega}$ then locally p can be solved as a function of ω. This immediately implies that the set of regular economies is an open subset of $R^{\ell N}_{++}$.

We shall see in the next section that the set of nonregular (or critical) economies has Lebesgue measure zero (which constitutes quite a demanding test of negligibility). In particular, the set of regular economies is dense in $R^{\ell N}_{++}$. To prove this will require the use of a comparatively powerful mathematical tool: Sard's theorem. Thus it may be worthwhile to show that a weaker result can be proved by elementary means:

PROPOSITION. *The set of economies for which some equilibrium is regular is dense in* $R^{\ell N}_{++}$.

Proof. Let ω be an arbitrary economy and $f_\omega(p) = 0$. Put $w_i = p \cdot \omega_i$, $x_i = \varphi(p, w_i)$, $u_i = u(x_i)$. For $0 \le t \le 1$ define $\omega(t) = (tx_1 + (1-t)\omega_1, \ldots, tx_N + (1-t)\omega_N)$. Of course, $\sum_i \omega_i(t) = \sum_i \omega_i$ and $p \cdot \omega_i(t) = w_i$. Therefore, $f_{\omega(t)}(p) = 0$ for all t. Let $\alpha(t)$ be the determinant of the linear map $\partial f_{\omega(t)}(p)$ from T_p to T_p.

Denote by $S_i = \partial^2 e_{u_i}(p)$ the substitution matrix for i at (p, w_i); see Section I.6. Simple computations give:

$$\partial f_{\omega(t)}(p) = \sum_i (S_i - (1-t)\partial_w \varphi_i(p_i, w_i)(x_i - \omega_i)^T) \ .$$

Hence $\alpha(t)$ is a polynomial of t. Also $\alpha(1) \neq 0$ because $\sum_i S_i$ is negative definite on T_p. Therefore $\alpha(t)$ is a nondegenerate polynomial which implies $\alpha(t) \neq 0$ for a t arbitrarily close to 0. Since then p is a regular equilibrium for $\omega(t)$ we have what we wanted. □

III.6 Genericity of Regular Economies

Now we shall prove:

PROPOSITION. *The set of regular economies has measure zero in* $R^{\ell N}_{++}$.

Proof. Let $E = \{(p,\omega) : f_\omega(p) = 0\}$ be the equilibrium set. The proof proceeds in three steps: (i) E is a C^1 manifold of dimension ℓN, (ii) ω is a regular economy if and only if it is a regular value of the projection $\pi : E \to R^{\ell N}$, (iii) the set of critical values of π has measure zero. See Figure II.5.

Step (i) follows from the implicit function theorem once one notices that rank $\partial_\omega f(p,\omega) \geq \ell - 1$ for all p, ω (in fact $\partial_\omega f(p,\omega)(v) = -\sum_i v_i$ whenever $p \cdot v_i = 0$ for all i).

Step (ii) is a simple exercise.

Step (iii) is precisely the easy part of Sard's theorem: the set of critical values of a C^1 function between C^1 manifolds of the same (finite) dimension is null in its range. □

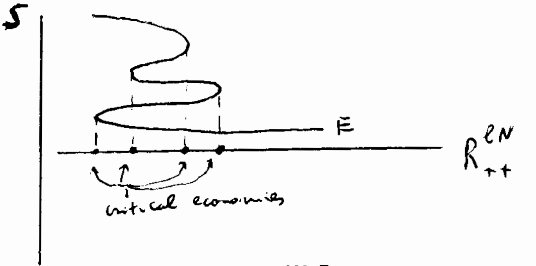

Figure III.5.

Lecture IV: Equilibrium with Incomplete Markets

IV.1 Basic Definitions

We now consider sequential trade under uncertainty. A basic reference is R. Radner, "Existence of Plans, Prices and Price Expectations in a Sequence of Markets," *Econometrica*, 1972. To be simple we consider only two dates: 0 and 1. At date 1, one of S states occurs and spot trade in ℓ commodities takes place according to prices $p_s \in R^{\ell}_{++}$. Given a system of contingent prices $p = (p_1, \ldots, p_S)$, at date 0 there is trade in $K \leq S$ assets. A unit of asset k delivers a (return) vector $a_{ks} \in R^{\ell}_{+}$ of goods if state s occurs. Thus the result of asset trade at date 0 determines the initial endowment at every of the possible states in date 1. At date 0 economic agents have correct anticipation on date 1 prices (conditional, of course, on the state).

Denote by $y_i \in R^K$, $x_i \in R^{\ell S}_{+}$ the trade and consumption plans of agent i. Put $y = (y_1, \ldots, y_N)$, $x = (x_1, \ldots, x_N)$.

Definition. *The plans* $(\bar{y}, \bar{x})$ *and prices* $q \in R^K$, $p \in R^{\ell S}_{++}$ *constitute an equilibrium if:*

(i) *Every* $\bar{y}_i, \bar{x}_i$ *maximizes* $u_i(x_i)$ *subject to* $q \cdot y_i \leq 0$ *and* $p_s \cdot x_{is} \leq p_s \cdot \omega_{is} + \sum_k y_{ik}(a_{ks} \cdot p_s)$ *for every* s.

(ii) $\sum_i y_i = 0, \quad \sum_i (x_i - \omega_i) \leq 0$.

Note: During this lecture we are assuming that the $u_i(\cdot)$ satisfy all the appropriate technical conditions. We also let $\omega_i >> 0$.

Given a vector $p \in R^{\ell S}_{++}$ the return vector (in value terms) for asset k is $g_k(p) = (p_1 \cdot a_{k1}, \dots, p_S \cdot a_{kS}) \in R^S$.

It is not difficult to see that in the definition of equilibrium the trades $\bar{y}$ and prices q of assets can be, so to speak, swept under the rug. In fact, the previous definition is equivalent (in the sense of yielding the same real allocations) to the following (we leave the proof of this as an exercise).

Definition. *The pair* $(\bar{p}, \bar{x}) \in R^{\ell S}_{++} \times R^{\ell S N}_{+}$ *constitutes an equilibrium if:*

(i) *Every* $\bar{x}_i$ *maximizes* $u_i(x_i)$ *subject to* $p \cdot x_i \le p \cdot \omega_i$ *and* $(p_1 \cdot (x_{i1} - \omega_{i1}), \dots, p_S \cdot (x_{iS} - \omega_{iS})) \in L(p)$ *where* $L(p) \subset R^S$ *is the subspace spanned by* $g_1(p), \dots, g_S(p))$.

(ii) $\sum_i (x_i - \omega_i) \le 0$.

A further, and trivial, redefinition will prove helpful.

Denote by $G^{S,K} = \{L \subset R^S : L$ is a K-dimensional subspace$\}$. This is the so-called Grassman manifold of K planes in R^S. Denote $\tilde{G}^{S,K} = \bigcup_{K' \le K} G^{S,K}$.

For every p and $L \in \tilde{G}^{S,K}$ let $f(p, L) \in R^{\ell S}$ be the aggregate excess demand vector for commodities obtained in the usual way except that the consumption set of every consumer is not $R^{\ell S}_{+}$ but $\{x_i \in R^{\ell S}_{+} : (p_1 \cdot (x_{i1} - \omega_{i1}), \dots, p_S \cdot (x_{iS} - \omega_{iS})) \in L\}$.

We can then rephrase the definition of equilibrium as:

Definition. *The pair* $(p, L) \in R^{\ell S}_{++} \times \tilde{G}^{S,K}$ *constitutes an equilibrium if:*

(i) $f(p, L) = 0$, *and*

(ii) $L =$ span$\{g_1(p), \dots, g_K(p)\}$.

In this lecture, we investigate the existence problem for this equilibrium concept. Note that if rank$\{g_1(p), \dots, g_K(p)\} = S$ for all p then putting $L = R^S$ we have a problem identical to the one in Lecture III. This is called the complete market case. Beyond the budget constraint there is no restriction on transfers of purchasing power across states. We will be interested in the case where there may be such restrictions (typically because $K < S$). This is the incomplete market case.

IV.2. Equilibrium May Not Exist

It was shown by O. Hart ("On the Optimality of Equilibrium when Markets are Incomplete," *Journal of Economic Theory*, 1975) that under the standard conditions an equilibrium may not exist in this model. The idea of the example can be briefly explained.

Let $\ell = 2$ and $S = 2$. There are two assets. Each asset is a future contract delivering one of two commodities independently of the state of the world, i.e., $a_{11} = (1,0)$, $a_{12} = (1,0)$, $a_{21} = (0,1)$, $a_{22} = (0,1)$. Note that then $L(p) = R^2$, i.e., markets are complete, if and only if p_1 and p_2 are not collinear (i.e., relative prices are not the same in the two states). If p_1 and p_2 are collinear, then $L(p)$ is one-dimensional and contains some positive nonzero vector. Because of the budget constraint this means that whatever the preferences $f(p, L(p)) = f(p, \{0\})$, i.e., it is as if no transfer were possible across the two states.

Hence, if p is an equilibrium price vector, then either (i) $f(p, L) = 0$ and p_1, p_2 are not collinear, or (ii) $f(p, \{0\}) = 0$ and p_1, p_2 are collinear. It should be reasonably clear that an f can be found so that neither (i) or (ii) holds (i.e., every equilibrium, if markets are complete, should have the same relative prices in the two states while every equilibrium of the incomplete markets situation should have distinct relative prices in the two states). For this f no equilibrium can exist.

There is something that appears coincidental in this example, namely, the equality of relative prices in the two states under the complete market structure. It prompts the following question: if not always, can we at least assert that equilibrium exists generally? As we shall try to suggest in the rest of this lecture the answer is positive. The basic paper is by Duffie and Shafer ("Equilibrium in Incomplete Markets: I. Basic Model of Generic Existence," *Journal of Mathematical Economics* **14**, 1985). We follow M.D. Hirsch–Magill–Mas-Colell ("A Geometric Approach to a Class of Equilibrium Existence Theorem," forthcoming in *Journal of Mathematical Economics*). These papers contain further references.

IV.3. Pseudoequilibrium

We shall focus on a concept more general than equilibrium.

Definition. *The pair* $(p, L) \in R^{\ell S}_{++} \times G^{S,K}$ *constitutes a pseudoequilibrium if:*

(i) $f(p, L) = 0$, *and*

(ii) $g_k(p) \in L$ *for all* $k = 1, \dots, K$.

Note that a pseudoequilibrium (p, L) is an equilibrium if $\{g_1(p), \dots, g_K(p)\}$ are linearly independent. Thus, this allows us to divide the proof of generic existence into two parts: (i) show that generically a pseudoequilibrium is an equilibrium, and (ii) show that a pseudoequilibrium exists under standard conditions.

IV.4. The Genericity Argument

The pseudoequilibrium conditions are:

[I] $f(p, L) = 0$

[II] $g_k(p) \in L$, all $k \leq K$.

How many equations and unknowns do these represent? As for unknowns p gives ℓS minus normalizations. There is one normalization for the budget constraint and $S - K$ for the fact that we always have $f(p, L) \in L$. Thus, p gives $\ell S - (1 + S - K)$. It is immediate that [I] yields the same number of possibly independent constraints. As for L the grassmanian $G^{S,K}$ is a C^∞ manifold of dimension $K(S - K)$. This is precisely the same number of equations needed to represent [II].

Summarizing: the system [I]–[II] has the same number of equations as of unknowns. This means that if we throw in any other constraint, say the linear dependency of $\{g_1(p), \dots, g_K(p)\}$, then the system is overdetermined and an application of Sard's theorem similar to the one of Lecture III should yield the result that for almost every parameter of the model, the solutions of [I]–[II] will have $\{g_1(p), \dots, g_K(p)\}$ linearly independent, i.e., will be true equilibrium

Which parameters should be used as perturbation variables? As in Lecture III, the initial endowment vectors ω_i are a clear possibility. They will certainly provide enough variation directions for [I]. Unfortunately, they are of no help with [II] since ω_i does not enter into this part of the equation system. There seems no alternative (using p itself does not quite do it) to using the return vectors a_{ks}. This is the state of the art. It is a bit unsatisfactory because from many points of view it would be preferable to have a generic result for arbitrarily given return vectors.

IV.5. Existence of Pseudoequilibrium

It is a remarkable fact that in order to prove the existence of a pseudoequilibrium tools stronger than Brouwer's fixed-point theorem or any of its variants (such as the ones used in Lecture III) appear to be required. This is remarkable because it is a novelty in economics.

To focus on essentials let us make the strong simplification that for fixed $L \in G^{S,K}$ there is a single $p(L)$ of unit norm such that $f(p(L), L) = 0$. Defining $g_i(L) = g_i(p(L))$ the existence of pseudoequilibrium reduces then to the question: *Given K functions $g_i : G^{S,K} \to R^S$ is there an L such that $g_i(L) \in L$ for every i?* We assume that each of the functions is continuous (in the obvious topology of $G^{S,K}$).

The answer is positive and a proof can be given following the same section of vector bundles approach of Lecture III. The difference is that we should now work with a vector bundle different to the tangent vector bundle having as base space the positive portion of the sphere. Clearly, our base space will now be $G^{S,K}$. The fiber associated with L will be $\{L\} \times \overbrace{L^\perp \times \ldots \times L^\perp}^{K \text{ times}}$. Note that the dimension of the fibers is the same as the dimension of the base space. For every L let $\pi_{L^\perp}$ denote the projection map of R^S on $L^\perp$. Define a section σ of the vector bundle by $\sigma(L) = (L, \pi_{L^\perp} g_K(L), \ldots, \pi_{L^\perp} g_K(L))$. Then L yields a pseudoequilibrium if and only if $\sigma(L) = (L, 0)$, or, in other words, if and only if $\sigma(L) = \sigma_0(L)$ where σ_0 is the zero section.

The existence of a pseudoequilibrium follows then from the fact that any section of the above vector bundle must intersect the zero section. To see this note: (i) any two sections are homotopic (the base space is compact and boundaryless), hence all sections have the same mod 2 intersection number with σ_0, (ii) the mod 2 intersection number must be nonzero since there is a smooth section σ' which intersects σ_0 transversally at a single point. Indeed, let $a_1, \ldots, a_K \in R^S$ be linearly independent. Put $\sigma'(L) = (L, \pi_{L^\perp} a_1, \ldots, \pi_{L^\perp} a_K)$. Obviously $\sigma'(L) = (L, 0)$ if and only if $L = \text{span}\{a_1, \ldots, a_K\}$. To see that σ' is transversal to σ_0 is straightforward but tedious — we would need among other things to specify the differentiable structure on $G^{S,K}$. Finally, (iii) if the mod 2 intersection number of σ and σ_0 is nonzero then the intersection is actually nonempty.

The above is of course no more than a sketch of proof. There is an interesting point of comparison with the arguments of Lecture III. There phrasing or not our proof as a fixed point, argument was simply a matter of taste. This is not the case here. The base space $G^{S,K}$ may not have the fixed point property (consider $S = 2$, $K = 1$). Thus a rigid adherence to the fixed point proves in this case not to be helpful.

IV.6. An Open Problem

We have seen that proving the existence of pseudoequilibrium necessitates tools stronger than Brouwer's fixed point theorem. (Incidentally, it is a useful exercise to verify that the fixed-point like result on $G^{S,K}$ claimed in the last section implies Brouwer's and also the Borsuk-Ulam theorem, a result more advanced than Brouwer's.) Strictly speaking, however, to establish this one must, roughly speaking, show that the underlying economics places no restrictions on the functions g_i beyond continuity. A careful analysis of this problem leads to the following question.

Open Problem. *Let $J < M$ and consider $G_+^{M,J} = \{L \in G^{M,J}\} : L \cap R_+^M = \{0\}\}$. Suppose that $F : G_+^{M,J} \to R^M$ is a continuous function such that $F(L) \in L$ for all $L \in G_+^{M,J}$.*

Let $E \subset G_+^{M,J}$ *be closed. Is there an economy such that for every* $L \in E$, $F(L)$ *represents the aggregate excess demand vector when every consumer maximizes utility subject to its net trade* $x_i - \omega_i$ *to lie in* $L - R_+^M$*?*

For $J = M - 1$ the function F is nothing but the classical excess demand function of Lecture III. The positive answer to the problem is the well known Sonnenschein-Mantel-Debreu theorem (see A. Mas-Colell, "The Theory of ..." for more on this theorem). Thus we are asking for a generalization of this theorem (the comparatively simpler additional case seems to be $J = 1$).