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On the Second Welfare Theorem for Anonymous Net Trades in Exchange Economies with Many Agents

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1. Introduction

This chapter is set in the economic environment of an exchange economy with a continuum of traders. It offers sufficient conditions for the efficient net trades attainable by the use of anonymous mechanisms to be Walrasian. In other words, given efficiency, anonymity rules out net transfers of wealth. This is a problem of the Second Fundamental Theorem of Welfare Economics variety. The converse First Fundamental Theorem question, that is, conditions for anonymous mechanisms to yield efficient net trades in continuum exchange economies, was investigated by Dubey, Mas-Colell, and Shubik (1980).

There shall be no need to consider mechanisms explicitly. The equilibrium net trades of any sensible anonymous exchange mechanism do themselves satisfy some anonymity properties. Hence, we simply study arbitrary net trades that satisfy those properties and that are efficient. More specifically, we consider two anonymity concepts for net trades. One we call, simply, anonymity, the other, strict anonymity. Roughly speaking, the underlying difference is that the second allows agents to enter the market (i.e., to use the mechanism) any finite number of times (keep in mind that we are studying continuum economies). We should quickly add that the resulting notions are not at all new. Efficient and anonymous net trades are the extensively investigated fair net trades (Foley 1967; Schmeidler and Vind 1972; Vind 1971; Varian 1976; Kleinberg 1980; Hammond 1979; Champsaur-Laroque 1981; McLennan 1982), while the efficient and strictly anonymous net trades are the strongly fair net trades of Schmeidler and Vind (1972). We apologize

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for introducing new terms, but we find the term *fair* excessively normative in the present context.

It is well known that the hypothesis of a continuum of agents alone will not make an efficient and anonymous, or strictly anonymous, net trade Walrasian. If, for example, all the characteristics and individual net trades fall into two types, there is not, from the standpoint of the present problem, any qualitative difference from a two-agent economy. What turns out to matter decisively is how varied the agents' characteristics are, that is, how rich is the support of the measure that describes the distribution of agents' characteristics.

Concerning efficient and anonymous net trades, an important result has been established by Kleinberg (1980), Hammond (1979), Champsaur-Laroque (1981), and McLennan (1982). In order to obtain Walrasian net trades, they have showed the sufficiency of the following three types of conditions: (i) the support of the characteristics' distribution is *connected* (in a precise technical sense); (ii) all preferences are *smooth;* and (iii) the net trade under investigation is *interior* (or, alternatively, one can impose boundary conditions on preferences). The first is a clear richness condition, but all three are indispensable. This is an interesting result, and we have no refinement to offer. In this chapter we shall pursue a different approach, namely, we shall require that the support of the characteristics' distribution be large and, of course, varied. More precisely (and leaving some technical conditions aside), we should be able to prescribe arbitrarily a consumption vector, a supporting price vector, and (within some limits) an endowment vector, and find some agent in the economy with compatible characteristics. Finite dimensional parameterizations of charactertistics are not ruled out, but the number of parameters must be of the same order as the number of commodities. Note also that our richness condition is of a global nature in that it involves large, not just local, variation.

Our research on strict anonymity can be viewed as a natural extension of Schmeidler and Vind's (1972) work. We find that in order to get the desired result for efficient, strictly anonymous net trades, we will not have to require, as with anonymous trades, a condition of global richness. A minimal requirement of local variation of endowment vectors will suffice.

The work presented here should also be compared with Hurwicz (1979). The conclusions are of a similar nature, that is, both works establish the Walrasian character of allocations that are optima and equilibria of certain noncooperative mechanisms. The differences in the models are, however, substantial. Hurwicz's key hypothesis is made on the mechanism. We make no assumption on the mechanism. On the other hand, our results are valid only for *anonymous* allocations in *continuum* economies. There are no such restrictions in Hurwicz's approach. Incidentally, this is perhaps the place to say what everybody knows, namely, that Hurwicz's work has had a seminal and pervasive influence in this entire area of research.

Section 2 describes the model and basic definitions. Section 3 (resp. Section 4) presents a theorem for efficient, anonymous (resp. strictly anonymous) net trades, and Section 5 (resp. Section 6) contains the proofs.

2. Basic Concepts

2.1. The Economic Environment

There are $l \geqq 1$ commodities and a population of agents characterized by an endowment vector $\omega \in R^l_+$ and a continuous, convex, monotone preference relation $\succcurlyeq$ on the consumption set R^l_+. The space of such preference relations is denoted P. By using the closed convergence on P (remember that each $\succcurlyeq$ is a closed subset of $R^l_+ \times R^l_+$), we endow the space of agents characteristics $\mathscr{A} = P \times R^l_+$ with a topology and a corresponding σ-field. (See Hildenbrand 1974 for further discussion.)

A number of subsets of P will be important at different points of this chapter. We record them here. First, we let P_b be the preferences $\succcurlyeq$ with compact indifference classes or, equivalently, satisfying the conditions: "if $y, z, v \geqq 0$, $v > 0$, then $y + \alpha v > z$ provided α is large enough." This is a form of strong desirability. Formally, $P_b = \{\succcurlyeq \in P\colon$ for each $z \in R^l_+$ the set $\{v \in R^l_+\colon z \succcurlyeq v\}$ is compact$\}$.

If u is a C_2 utility function for $\succcurlyeq$, the bordered Hessian at $v \in R^l_+$ is the determinant:

$$\begin{vmatrix} \partial^2 u(v) & (\partial u(v))^T \\ \partial u(v) & 0 \end{vmatrix}.$$

We let

$P_s = \{\succcurlyeq \in P\colon \succcurlyeq$ admits a C^2 utility function u with nonvanishing bordered Hessian at each $v \in R^l_+\}$.

In other words, P_s is the usual space of C^2 preferences with indifference classes displaying nonzero Gaussian curvature everywhere (see Debreu, 1972).

For each $\succcurlyeq \in P$, $p \gg 0$, and $w \geqq 0$ denote by $\phi(\succcurlyeq,p,w)$ the (nonempty) set of maximizers of $\succcurlyeq$ on $\{v \in R^l_+\colon p.v \leqq w\}$. This defines the demand correspondence on $P \times R^l_{++} \times R_+$. By an abuse of language we say that $\succcurlyeq$ is strictly convex whenever $\phi(\succcurlyeq,p,w)$ is a singleton for all $p \gg 0$. We let, finally,

$P_n = \{\succcurlyeq \in P\colon \succcurlyeq$ is strictly convex, admits a C^1 utility function u and has normal demand, that is, for all $p \gg 0$ and $w' \geqq w \geqq 0$, we have $\phi(\succcurlyeq,p,w') \geqq \phi(\succcurlyeq,p,w)\}$.

The economy is composed by a continuum of agents. The set of agents' names is the unit interval $I = [0,1]$ equipped with Lebesgue measure, denoted λ. The following definitions and concepts are standard.

DEFINITION 2.1. An economy is a (Borelian) map $\mathscr{E}: I \rightarrow \mathscr{A}$ such that, denoting $\mathscr{E}(t) = (\succcurlyeq_t, \omega(t))$, $\int \omega < \infty$.

For the rest of the chapter, we have given a fixed reference economy $\mathscr{E}$.

DEFINITION 2.2. A net trade is a (Borelian) map $x: I \rightarrow R^l$ such that:

(i) $x(t) + \omega(t) \geqq 0$ for *a.e.* $t \in I$
(ii) $\int x \leqq 0$.

The distribution of agents' characteristics induced by $\mathscr{E}$, that is, the probability measure $\lambda \circ \mathscr{E}^{-1}$ on $\mathscr{A}$, is denoted ν. The support of ν, denoted supp $\nu \subset \mathscr{A}$, is the smallest closed set that has full measure.

Every net trade x induces a distribution $\lambda \circ x^{-1}$ on R^l. We denote by $B_x \subset R^l$ the support of this distribution.

2.2. *Efficient and Walrasian Net Trades*

Again the definitions of this section are standard.

DEFINITION 2.3. The net trade x is efficient if there is no other net trade x' such that:

(i) $x'(t) + \omega(t) \succcurlyeq_t x(t) + \omega(t)$ for *a.e.* $t \in I$,
(ii) $\lambda\{t \in I: x'(t) + \omega(t) \succ_t x(t) + \omega(t)\} > 0$.

DEFINITION 2.4. The net trade x is Walrasian if there is a $p \in R^l$ such that, for *a.e.* $t \in I$:

(i) $p.\, x(t) \leqq 0$, and
(ii) $x(t) + \omega(t)$ is $\succcurlyeq_t$ − maximal on $\{v \in R^l_+: p.v \leqq p.\omega(t)\}$.

It is well known that every Walrasian net trade is efficient. Conversely, if x is efficient, then there is a vector $p > 0$ (called an efficiency price vector) such that, for *a.e.* $t \in I$, $x(t)$ minimizes $p.v$ on $\{z \in R^l: z + \omega(t) \succcurlyeq_t x(t) + \omega(t)\}$. The quantities $p.x(t)$ can be interpreted as the imputed net wealth transfers at x. If $p.x(t) = 0$ for *a.e.* t and $p >> 0$, then x is, in fact, Walrasian.

2.3. *Anonymous Net Trades*

DEFINITION 2.5. A net trade x is anonymous if for *a.e.* $t \in I$, $x(t) + \omega(t)$ is $\succcurlyeq_t$ − maximal on $(B_x + \omega(t)) \cap R^l_+$.

If x is Walrasian, then $p.z \leqq 0$ for each $z \in B_x$. Therefore Walrasian net trades are anonymous. We have seen that they are also efficient. In Section 3 we shall give conditions for the properties of efficiency and anonymity to characterize Walrasian net trades or, loosely speaking, for the anonymity property to imply, given efficiency, uniform (hence, zero) transfers.

REMARK 2.1. The interpretation of the anonymity property that we wish to emphasize views it as a property necessarily satisfied, in continuum economies, by the (noncooperative) equilibria of anonymous allocation mechanisms (see Dubey, Mas-Colell, and Shubik 1980 for a precise model of the latter). We briefly describe the situation. Because the mechanism is anonymous, the net trades attainable to each agent depend only on his action and the distribution of actions taken by all the agents. Because, with a continuum of agents, this distribution does not depend on the actions of any single individual, it must follow that at equilibrium (almost) every agent maximizes preferences on the same set $K \subset R^l$ of individual net trades. If K is closed, which is a reasonable hypothesis, then $B_x \subset K$, and therefore the equilibrium must be anonymous according to Definition 2.5. For present purposes the content of this definition provides a sufficiently powerful property. Hence, we apply Occam's razor and dispense with the formal description of a mechanism. (For an explicit treatment of mechanisms see the previous reference and Hammond 1979.)

REMARK 2.2. The notion of anonymity for a net trade, which does, of course, make sense for economies with a finite number of agents, is conceptually the same as the no-envy property introduced by Foley (1967) and extensively studied since (see, for example, Schmeidler and Vind 1972; Varian 1976). We prefer the term *anonymous* because it is less judgmental and serves as a reminder of the basic property of the underlying, and hypothetical, allocation mechanism.

REMARK 2.3. For the case of a continuum of agents, our definition of anonymous net trades is slightly stronger than the corresponding no-envy definition used by Kleinberg (1980) and Champsaur-Laroque (1981a). According to them, x has the no-envy property, if for $a.e.$ $t \in I$, $x(t) + \omega(t) \succcurlyeq_t v + \omega(t)$ for $(\lambda \circ x^{-1}) - a.e.$ $v \in B_x$, while we require that this be true for every $v \in B_x$. In other words, according to the weaker definition, for an agent to be envious, he must envy a set of agents of positive measure, whereas according to the stronger definition, it suffices that the agent envies a net trade that is a limit of the set of net trades of positive-measure sets of agents. It is not difficult to produce examples showing that our definition is, indeed, strictly stronger. Nevertheless, the difference is conceptually small and even more so if the mechanism point of view of Remark 2.1 is kept in mind. We may add that while it simplifies the exposition, the theorems of Sections 3 and 4 remain valid for the weaker definitions with only minor adjustments of the proofs.

2.4. *Strictly Anonymous Net Trades*

Given a net trade x, let B_x^* be the set of finite sums of elements of B_x, that is,

$$B_x^* = \bigcup_{m=1}^{\infty} \{v_1 + \ldots + v_m : v_1 \in B_x, i = 1,\ldots,m\}.$$

DEFINITION 2.6. A net trade x is strictly anonymous if for *a.e.* $t \in I$, $x(t) + \omega(t)$ is $\succcurlyeq_t$ − maximal on $(B_x^* + \omega(t)) \cap R_+^l$.

As with the anonymity property, Walrasian net trades are strictly anonymous. In Section 4 we shall give conditions for the properties of efficiency and strict anonymity to characterize Walrasian net trades.

REMARK 2.4. With a finite number of agents, the concept of strictly anonymous trades was introduced, under the name of "strongly fair" by Schmeidler and Vind (1972) (see also Vind 1977). With a continuum of traders, Dubey, Mas-Colell, and Shubik (1980) presented a concept of strict noncooperative equilibrium (for an anonymous mechanism) which stands relative to the notion of strictly anonymous net trades as the noncooperative equilibrium stands relative to the anonymity of net trades. The motivation for considering strict noncooperative equilibria then and strictly anonymous net trades now should be clear. In an anonymous market in which individual agents are negligible, it may be difficult to avoid the situation in which agents enter the market several times (or, simply, that they use proxies). If this is so, then the strict concept is the appropriate solution notion.

REMARK 2.5. As with anonymous net trades (Remark 2.3), we can think of a formally weaker definition. Let μ be the measure $\lambda \circ x^{-1}$ on R^l. For each n, endow R^{ln} with the product measure $\mu \times \ldots \times \mu$. The map $v_1 + \ldots + v_n$ from R^{ln} to R^l induces a measure μ^n on R^l. We could simply require that, for *a.e.* $t \in I$, $x(t) \succcurlyeq_t v$ for μ^n − *a.e.* $v \in R^l$ and each n. Again, the conclusions of Section 4 remain valid; however, it is not now clear to us if this really is a weaker definition. As in the previous case, the conceptual difference is so minor that we shall not concern ourselves with this matter.

3. Efficient and Anonymous Net Trades

3.1. Motivation by Examples

Even with the continuum of agents, efficient and anonymous net trades need not be Walrasian. We illustrate this by three simple and well-known examples.

EXAMPLE 3.1. The measure $\nu = \lambda \circ \mathscr{C}^{-1}$ gives equal weight to the two characteristics represented in Figure 9.1. The net trade value for the two types are also indicated. It is obvious from Figure 9.1 that the net trade considered is efficient, anonymous, and not Walrasian. This example shows that the continuum alone does not make for a situation qualitatively different from a two-agent economy. To generate more interesting phenomena, we will need at the very least some dispersion of characteristics.

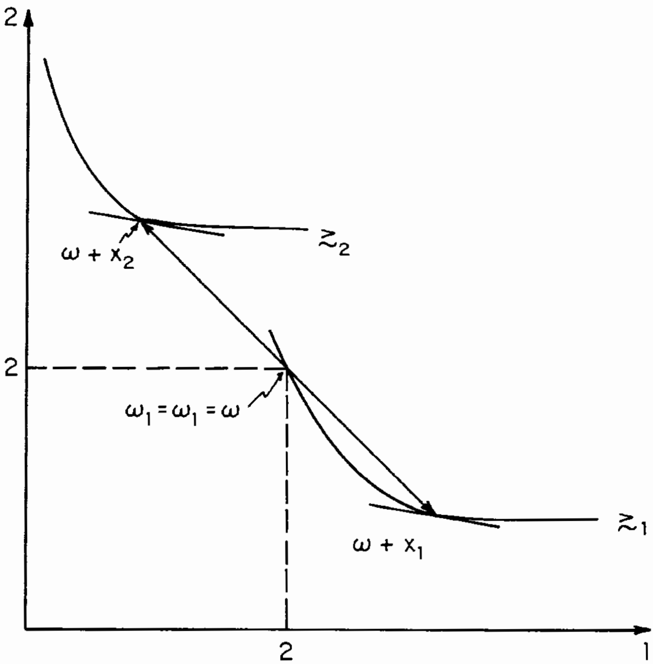

Figure 9.1. Illustration for Example 3.1.

EXAMPLE 3.2. The functions $\mathscr{E}$ and x are represented in Figure 9.2. Because $\omega \in \operatorname{Int}\operatorname{co}\{x(t)\colon t \in I\}$, the assignment $t \mapsto x(t)$ can be made so that $\int x = 0$. Again, x is efficient, anonymous, and not Walrasian. Further, $x(t) \gg 0$ for *a.e.* $t \in I$. In contrast with Example 3.1, supp ν is now connected. However, it is not very large, that is, we have gone from 0- to 1-dimensional, and preferences in supp ν are not smooth. In particular, the efficient allocations can be supported by more than one normalized price vector.

EXAMPLE 3.3. Let $l = 2$. Using utility functions, we describe the characteristics as follows:

(i) for $0 \leqq t \leqq 1/3$, $\omega(t) = (4 - 9t, 4 - 9t)$ and $u_t(x) = x^2$,
(ii) for $1/3 \leqq t \leqq 2/3$, $\omega(t) = (1,1)$ and $u_t(x) = (t - 1/3)(x^1 + 1)^{1/2} + (2/3 - t)(x^2 + 1)^{1/2}$,
(iii) for $2/3 \leqq t \leqq 1$, $\omega(t) = (9t - 5, 9t - 5)$ and $u_t(x) = x^1$.

Let $B \subset R^2$ be the set described in Figure 9.3. For each $t \in I$, let $x(t)$ be such that $x(t) + \omega(t)$ is $\succcurlyeq_t$ − maximal on $B + \omega(t)$. In Figure 9.3 the relative position of the $x(t)$'s are indicated. Because every point of B is taken

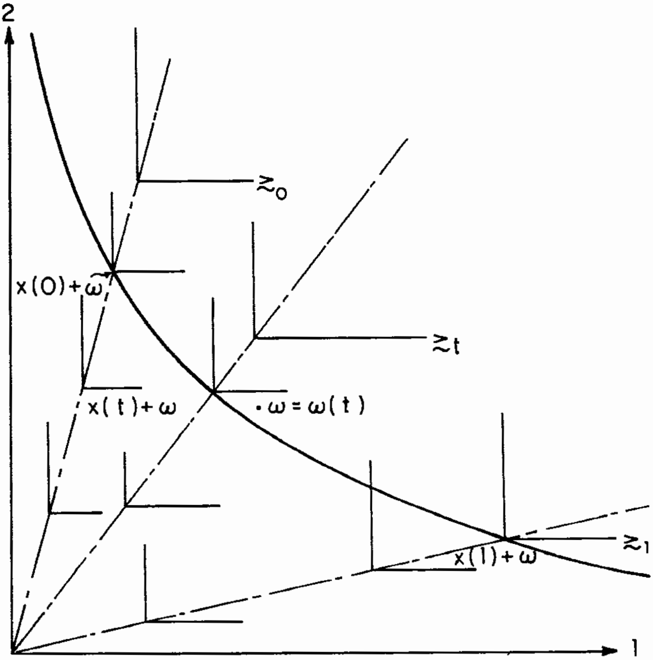

Figure 9.2. Illustration for Example 3.2.

and $0 \in \text{Int co } B$, we can reparameterize $t \mapsto (\succcurlyeq_t, \omega(t), x(t))$ so as to make x a net trade.

By construction, x is anonymous. To verify that it is efficient, note that it is supported by the price vector $p = (1,1)$. In fact, up to normalization, this is the only supporting price vector. Clearly, however, x is not Walrasian.

As in the previous example, supp ν is parameterized by a single variable. Preferences are now smooth and x is supported by a single price vector, but the net trade is not interior for all agents, that is,

$$\lambda(\{t \in I: x(t) + \omega(t) \in \text{Bdry } R^2_+\}) = \frac{2}{3} > 0.$$

3.2 Boundedness of Anonymous Net Trades

Denote by μ the distribution of initial endowments on R^l_+, that is, $\mu = \lambda \circ \omega^{-1}$. It shall be convenient (but not essential) to assume that supp μ is not too thin. This shall take the form of requiring $\mu(\text{Bdry supp } \mu) = 0$. Then,

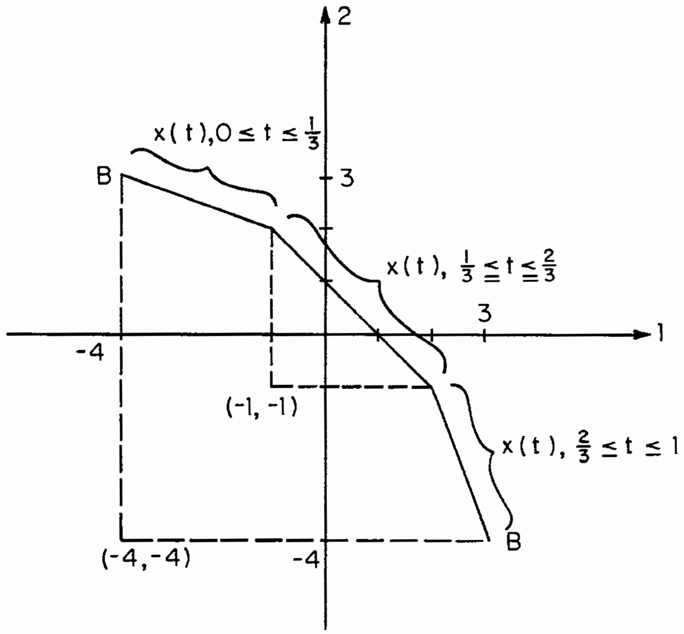

Figure 9.3. Illustration for Example 3.3.

in particular, supp μ is the closure of its interior, and for each $\omega \in$ Int supp μ, $\nu(P \times R^l_+ + \omega)) > 0$.

Even if supp μ is bounded, a net trade x may not be (even when sets of measure zero are disregarded). This is most unpleasant, and in order to rule it out we shall impose the following weak hypothesis on the economy $\mathscr{E}$:

(i) μ (Bdry supp μ) = 0
(ii) for all $\omega \in$ Int supp μ, $\nu(P_b \times (R^l_+ + \omega)) > 0$ (A.1)

This roughly says that every commodity is strongly desirable to a representative group of agents. Of course, part (ii) is satisfied if $\nu(P_b \times R^l_+) = 1$.

REMARK 3.1. Champsaur-Laroque (1981a) also encountered the boundedness problem and dealt with it differently. As will be seen in the next two sections, the method we use, that is, hypothesis (A.1), is in tune with the general approach of this chapter of contemplating a large supp ν.

3.3 A Theorem

In various degrees of generality Kleinberg (1980), Hammond (1979), Champsaur-Laroque (1981), and McLennan (1982) have established the

Walrasian character of an (bounded) efficient and anonymous net trade x under three kinds of hypotheses:

(i) every preference in supp ν is smooth, that is, supp $\nu \subset P_s \times R^l_+$. This is the condition violated by Example 3.2.
(ii) supp ν satisfies a (Lipschitzian) arcconnectedness property. This is the condition violated by Example 3.1.
(iii) For *a.e.* $t \in I$, $x(t) + \omega(t) \gg 0$. This is the condition violated by Example 3.3. We should remark that it is possible to state conditions directly on $\mathscr{E}$ implying that every efficient net trade satisfies (iii) (see Champsaur-Laroque 1981).

In this chapter we shall pursue a different approach. A common feature of the three previous examples is that supp ν is small: what we shall do here is to require that supp ν be large. Take the limit case where the economy $\mathscr{E}$, assumed to satisfy (A.1), is such that supp $\nu = \mathscr{A}$. Then it is implied by the theorem below that every efficient and anonymous net trade is Walrasian. A property such as supp $\nu = \mathscr{A}$ is one of global richness. In the economy there are agents of every possible kind, although nothing is said about their frequency.

Of course, much less than supp $\nu = \mathscr{A}$ will do. In particular, the condition to be given is entirely compatible with supp ν being finite-dimensional (although the number of dimensions cannot be smaller than $2l$).

THEOREM 1. Suppose that $\mathscr{E}: I \to \mathscr{A}$ satisfies (A.1). Let $Q \subset P$ satisfy:

(i) Q is compact and $Q \subset P_n$
(ii) for every $z \in R^l_+$ and $p \in R^l_{++}$, there is a $\succsim \in Q$ such that $p = \partial u(z)$, where u is a C^1 utility for $\succsim$.

If $Q \times$ supp $\mu \subset$ supp ν, then every efficient and anonymous net trade is Walrasian.

REMARK 3.2. The hypothesis Theorem 1 is a richness condition on supp ν. Put into words, and somewhat loosely, it says that given an initial endowment vector ω present in the economy, there is some agent with endowment ω and nice preferences displaying arbitrarily prescribed marginal rates of substitution at an arbitrarily prescribed point.

REMARK 3.3. Note that supp ν is required neither to be connected nor to be contained on $P_s \times R^l_+$. Even if supp μ and Q are connected, supp ν may not be. Thus, our hypotheses are neither weaker nor stronger than the ones associated with the Kleinberg-Hammond-Champsaur-Laroque-McLennan theorem.

REMARK 3.4. It is obvious from Examples 3.1–3.3 that part (ii) of the hypothesis of Theorem 1 is essential. Concerning part (i), Example 3.4 (resp. Example 3.5) below will show that the strict convexity of the elements of

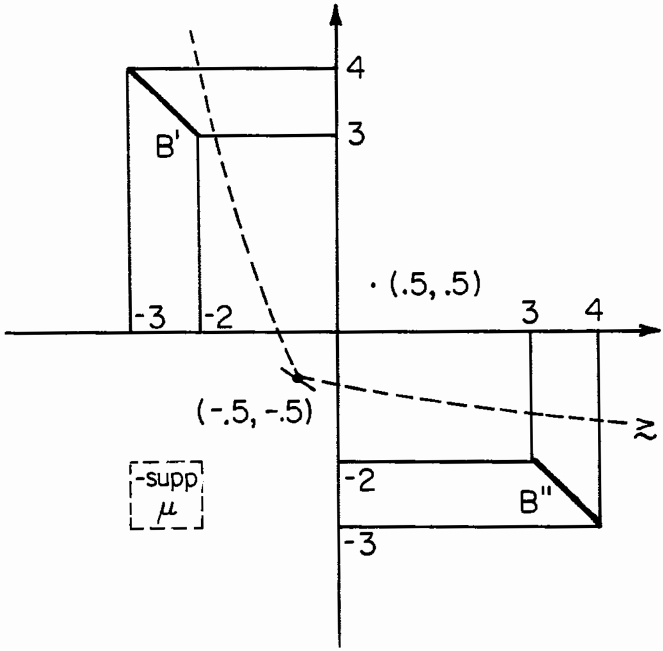

Figure 9.4. Illustration for Example 3.4.

Q (resp. the compactness of Q) cannot be dispensed with. We do not know if the normality of demand for the elements of Q is dispensable. At any rate, it is a weak hypothesis.

EXAMPLE 3.4. We take $l = 2$. Initial endowment vectors are concentrated on the square $[2,3]^2$. Half the total mass of agents have linear utility functions, which we identify with a gradient vector $q \in (0,1)^2$. More specifically, half the total mass are uniformly distributed on $(0,1)^2 \times [2,3]^2$. For these agents we let $x(t)$ be such that $x(t) + \omega(t)$ maximizes $q(t).z$ on the set $((B' \cup B'') + \omega(t)) \cap R^2_+$, where B', B'' are as in Figure 9.4. As is clear by reason of symmetry, the mean net trade of these agents is the vector (.5,.5). The other half of the total mass of agents have common characteristics $(\omega, \gtrsim)$, where $\omega = (2.5,2.5)$ and $\gtrsim$ admits a C^1 utility function u with $\partial u(2,2) = (1,1)$ and $u(2,2) > u(.5,5.5) = u(5.5,.5)$. For these agents we put $x(t) = (-.5,-.5)$. Clearly, we end up with a feasible net trade with $B_x = B' \cup B'' \cup \{(-.5,-.5)\}$ and which is, therefore, anonymous and, because supported by $p = (1,1)$, efficient. See Figure 9.4.

If we require $0 \in \text{supp}\ \mu$, then an example such as 3.4 cannot be constructed for $l = 2$; however, there are analogous examples with $l \geqq 3$.

EXAMPLE 3.5. We take $l = 2$ and the same endowment vector ω for all

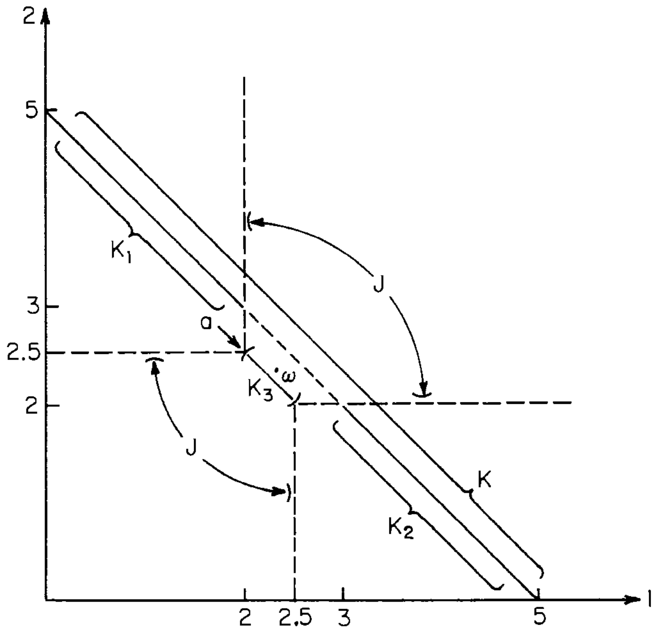

Figure 9.5. Illustration for Example 3.5.

agents (the example can be modified easily to satisfy (A.1)). To every $(z,q) \in R^l_+ \times R^l_{++}$ we assign an arbitrary $\succsim_{z,q} \in P_n$ representable by a utility function u with $\partial u(z) = q$. However, with reference to Figure 9.5, the assignment $(z,q) \mapsto \succsim_{z,q}$ can be made "measurably" and so that (i) if $z \notin J$ or "$z \in J$ and $q^1 \neq q^2$," then the maximizer of $\succsim_{z,q}$ on K, denoted $y(z,q)$, belongs to $K_1 \cup K_2$, and (ii) if $z \in J$ and $q^1 = q^2$, then there is a $y(z,q) \in K_3$ that maximizes $\succsim_{z,q}$ on $K_1 \cup K_2 \cup K_3$. Now let f: $[0,1] \rightarrow R^l_+ \times R^l_{++}$ be an arbitrary, measurable map with $\text{supp}(\lambda \circ f^{-1}) = R^l_+ \times R^l_{++}$ and put $\succsim_t = \succsim_{f(t)}$, $z(t) = y(f(t))$, $\omega = \int z$, $\omega(t) = \omega$, $x(t) = z(t) - \omega$. The net trade x is then anonymous and efficient, but not Walrasian. A moment's reflection will reveal that the closure of $\{\succsim_t: t \in I\}$ cannot be a compact subset of P_n. Indeed, if the closure is compact, then it should contain some preference $\succsim^*$ with an indifference map exhibiting a kink at point a.

REMARK 3.5. The global character of the richness hypothesis should be emphasized. A condition of local richness would be, for example, that $\nu(\text{Bdry supp } \nu) = 0$. Then, for *a.e.* $t \in I$, every $(\succsim,\omega)$ sufficiently near $(\succsim_t,\omega_t)$ would (nearly) correspond to the characteristics of some agent in the economy. This is, however, too weak for the conclusion of the theorem. It is not difficult to modify Example 3.1 to illustrate this fact. Because it will be useful in Section 4, we record the example explicitly.

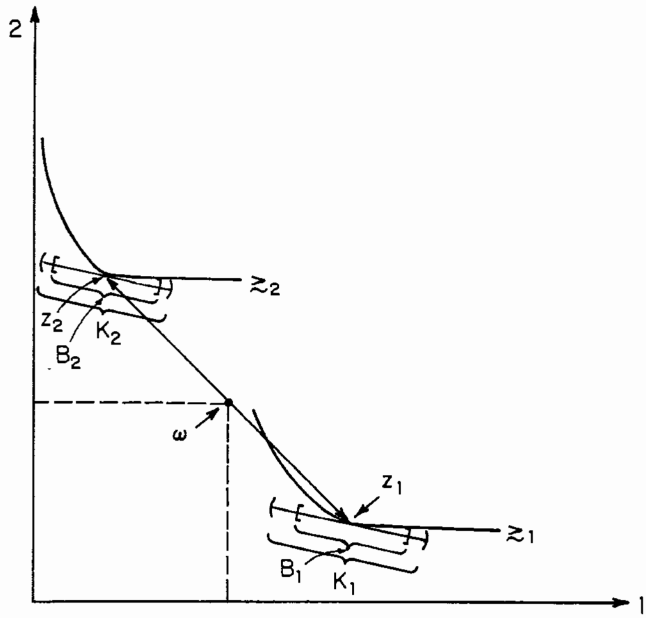

Figure 9.6. Illustration for Example 3.6.

EXAMPLE 3.6. We take $l = 2$. The support of the measure ν is constituted by two small neighborhoods of the characteristics $(\succcurlyeq_1,\omega)$, $(\succcurlyeq_2,\omega)$ indicated in Figure 9.6, and it gives equal weight to both of them. Also, $\nu(\text{Bdry supp } \nu) = 0$. We assign to the agents t with characteristics near $(\succcurlyeq_1,\omega)$ (resp. $(\succcurlyeq_2,\omega)$) the net trade $x(t)$ such that $x(t) + \omega$ maximizes preferences on K_1 (resp. K_2). If supp ν is sufficiently small, these individual net trades are well defined and supp $\lambda \circ x^{-1}$ is constituted by two small segments B_1, B_2 containing, respectively, z_1 and z_2 in their relative interiors. Then $0 \in \text{Int co}(B_1 \cup B_2)$ and, therefore, the distribution ν can be fixed up so that we have a feasible net trade that, clearly, is anonymous and efficient but not Walrasian.

REMARK 3.6. It is not clear to us if the conditions (i) $\nu(\text{Bdry supp } \nu) = 0$ and (ii) Int supp ν is arcconnected, would suffice for the conclusion of the theorem. Remember that the closed convergence topology on P is very coarse; thus the requirement that supp ν has a nonempty interior is strong. In particular, the topology on P imposes no tight restrictions on supporting hyperplanes to $\{z: z \succcurlyeq v\}$ near the boundary of R^l_+. A sensible result should not depend on this peculiarity. If supp $\nu \subset P_s \times R^l_+$, then it is natural to consider C^1-type topologies. With them, conditions such as (i) and (ii) above do not yield the desired conclusions. It is straightforward how to modify Example 3.3 so as to make supp ν the closure of its interior in a C^1 topology

(C^1 uniform covergence on compacta of a normalized family of utility functions, for example).

4. Efficient and Strictly Anonymous Net Trades

In this section we shall investigate conditions under which efficient, strictly anonymous net trades are Walrasian. Relevant references on previous research are Schmeidler and Vind (1972) and Vind (1977). Working with a finite number of agents, they gave a condition that, when fulfilled by a strictly anonymous net trade, implied that it must be Walrasian. Confining ourselves to efficient, strictly anonymous net trades, we look for conditions on the data of the problem, that is, the distribution of agents' characteristics. Schmeidler and Vind's work, however, is important for ours both because we draw on some of their techniques of proof and because one could interpret our hypotheses as guaranteeing that, in the presence of efficiency, any strictly anonymous net trade must satisfy the Schmeidler-Vind condition.

We begin by pointing out that some condition is needed, and for this Example 3.1 will suffice. Figure 9.7 represents B_x^*, which is a discrete set.

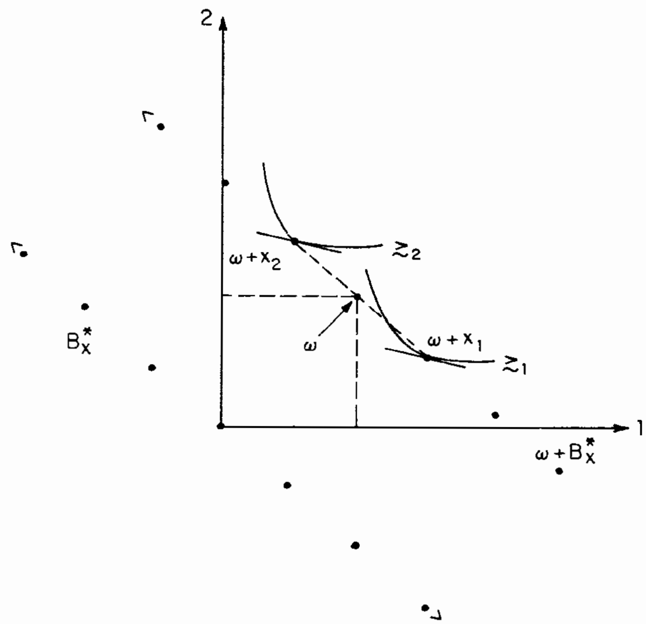

Figure 9.7. Variation on Example 3.1 for strict anonymity.

It is clear that if $\succsim_1$ and $\succsim_2$ are chosen appropriately, the same net trade x of Example 3.1 will be not only anonymous but also strictly anonymous.

On the other hand, the strictness requirement is very strong, so it must make a substantial difference. To illustrate this point, consider Example 3.6. Figure 9.8 represents B_x^* (it is instructive to consider why this is so). Obviously, no agent now is maximizing preferences, and thus x is not strictly anonymous. The difference between Examples 3.1 and 3.6 is that in the latter the distribution of characteristics is not concentrated in two points, but rather it is spread over two, possibly small, open neighborhoods. We saw that this did not matter much for the analysis of anonymous net trades, but it is crucial for the strictly anonymous case. Indeed, the contrast between the two examples suggests that a condition of local variation of characteristics may suffice to make every efficient, strictly anonymous net trade Walrasian. We shall see that this is so in a particularly strong manner. First, the local richness of characteristics will not be needed over the entire support of the distribution but only at some point. Second, at that point, and provided the preference relation is nice, the local variability of initial endowment vectors will suffice. Precisely, we will require that supp ν contains a set of the form $\{\succsim\} \times V$ where $\succsim$ is smooth (i.e., $\succsim \in P_s$) and $V \subset R^l_{++}$ is open.

As with anonymous net trades (see 3.2) we also need in this section some

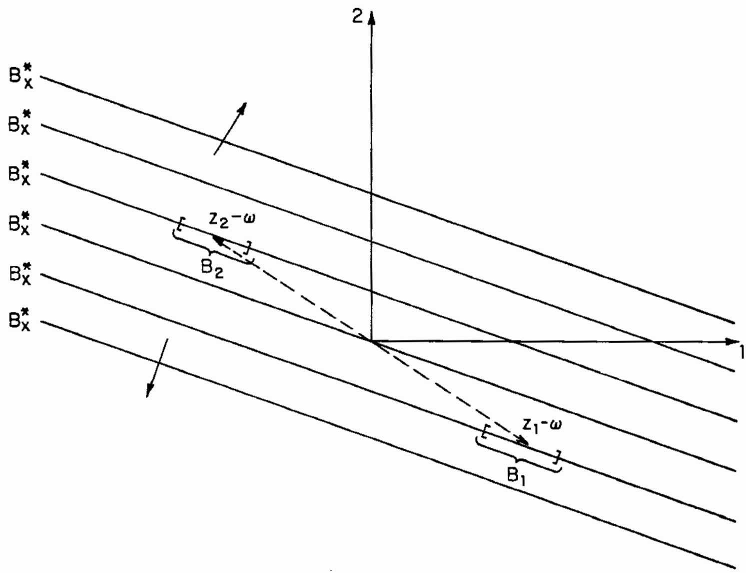

Figure 9.8. Example 3.6 is not strictly anonymous.

condition yielding, when supp ν is compact, the boundedness of strictly anonymous net trades. As even weaker desirability hypothesis than that in Section 3 will suffice:

$$\nu(P_b \times R^l_{++}) > 0. \tag{A.2}$$

We are now ready to state the second theorem.

THEOREM 2. Suppose that $\mathscr{E}: I \to \mathscr{A}$ satisfies (A.2). Let $\{\succcurlyeq\} \times V \subset P \times R^l_{++}$ satisfy:

(i) $\succcurlyeq \in P_s$, and
(ii) V is open.

If $\{\succcurlyeq\} \times V \subset$ supp ν, then every efficient and strictly anonymous net trade is Walrasian.

REMARK 4.1. It goes without saying that the local richness condition of Theorem 2 is not the only possible one. If so desired, the variation could be in preferences, for example. We have given the formulation of the theorem because it is simple to state and also because it appears minimal in the sense of using no more than l parameters. In general, we cannot hope to obtain the conclusion of Theorem 2 with an n-dimensional local variation condition if n is less than l or $l - 1$.

5. Proof of Theorem 1

The proof shall proceed in five steps, the key ones being the second and the fourth.

Step 1

Let x be the given efficient and anonymous net trade. With $\mu = \lambda \circ x^{-1}$ we put $J =$ supp $\mu \subset R^l$ and call $B = B_x$. By (A.1), J is the closure of its interior. Hence, Int $J \neq 0$.

LEMMA 1. $(B + \bar{\omega}) \cap R^l_+$ is nonempty and bounded for every $\omega \in$ Int J.

PROOF. Because $x(t) + \omega(t) \in (B + \omega(t)) \cap R^l_+$ for *a.e.* $t \in I$ and $\lambda\{t: \omega(t) \leqq \bar{\omega}\} > 0$ (otherwise $\bar{\omega} \notin$ Int J), we get $(B + \bar{\omega}) \cap R^l_+ \neq \varnothing$. Suppose now that $(B + \bar{\omega}) \cap R^l_+$ is unbounded. Then $(B + \omega) \cap R^l_+$ is unbounded for all $\omega \geqq \bar{\omega}$. If $\succcurlyeq_t \in P_b$ and $\omega(t) \geqq \bar{\omega}$, then $\{z: x(t) \succcurlyeq_t z\}$ is compact, and so $x(t)$ cannot maximize $\succcurlyeq_t$ on $(B + \omega(t)) \cap R^l_+$. Hence $\lambda\{t: \succcurlyeq_t \in P_b$ and $\omega(t) \geqq \bar{\omega}\} = 0$, which contradicts hypothesis (A.1). Therefore, $(B + \bar{\omega}) \cap R^l_+$ is bounded. QED

By Lemma 1 the correspondence $\omega \mapsto (B + \omega) \cap R^l_+$ is upper hemicontinuous (*u.h.c.*) on Int J. Therefore, by Fort's theorem (see, for example, Dierker 1973), $\omega \mapsto (B + \omega) \cap R^l_+$ is continuous as a correspondence on a set $J^* \subset$ Int J dense in Int J.

For each $(\succcurlyeq,\omega) \in P \times \text{Int } J$, let $\phi(\succcurlyeq,\omega) \subset R^l_+$ be the set of maximizers of $\succcurlyeq$ on $(B + \omega) \cap R^l_+$. By Lemma 1, $\phi(\succcurlyeq,\omega) \neq \emptyset$ and by the Maximum Theorem (see, for example, Debreu 1959), ϕ is an u.h.c. correspondence on $P \times J^*$.

By (A.1), $\omega(t) \in \text{Int } J$ for *a.e.* $t \in I$. Therefore, because x is anonymous, we should have $x(t) + \omega(t) \in \phi(\succcurlyeq_t,\omega_t)$ for *a.e.* $t \in I$.

We say that $p > 0$ is a supporting price vector for x if, for *a.e.* $t \in I$, $p.v \geqq p.(x(t) + \omega(t))$ whenever $v \succcurlyeq_t x(t) + \omega(t)$. Because x is efficient it has some supporting price vector.

Denote by ρ the measure on $\mathcal{A} \times R^l_+$ defined by $\rho = \lambda \circ (\mathscr{E},x + \omega)^{-1}$.

LEMMA 2.

(i) If $(\succcurlyeq,\omega) \in \text{supp } \nu$ and $\omega \in J^*$, then $(\succcurlyeq,\omega,y) \in \text{supp } p$ for some $y \in R^l_+$, and $y \in \phi(\succcurlyeq,\omega)$ for any such y.

(ii) If x is supported by the price vector $p > 0$ and $(\succcurlyeq,\omega,y) \in \text{supp } \rho$, then $p.v \geqq p.y$ whenever $v \succcurlyeq y$.

PROOF. (i) Let $(\succcurlyeq,\omega) \in \text{supp } \nu$ and $\omega \in J^*$. To prove that $(\succcurlyeq,\omega,y) \in \text{supp } \rho$ for some $y \in R^l_+$, it suffices to show that there is $\beta > 0$ such that if U is a (sufficiently small) open neighborhood of $(\succcurlyeq,\omega)$, then, for *a.e.* $t \in x^{-1}(U)$, $x(t) + \omega(t)$ belongs to the compact ball $\{z\colon \|z\| \leqq \beta\}$. This β is easily obtained. Just let $\omega' \gg \omega$, $\omega' \in \text{Int } J$ and pick β so that $\|z\| < \beta$ whenever $z \in (B + \bar{\omega}) \cap R^l_+$. We can do so by Lemma 1. Of course, if $z \in (B + \omega'') \cap R^l_+$, $\omega'' \leqq \omega'$, then also $\|z\| < \beta$. Remember too that $x(t) + \omega(t) \in (B + \omega(t)) \cap R^l_+$ for a.e. $t \in I$.

Suppose now that $(\succcurlyeq,\omega,y) \in \text{supp } \rho$. For each open neighborhood $U \subset \mathcal{A}$ of $(\succcurlyeq,\omega)$ and $\varepsilon > 0$, we have $\lambda\{t\colon \mathscr{E}(t) \in U \text{ and } \|(x(t) + \omega(t)) - y\| \leqq \varepsilon\} > 0$. Hence, $\lambda\{t\colon \mathscr{E}(t) \in U, \|x(t) + \omega(t) - y\| \leqq \varepsilon \text{ and } x(t) + \omega(t) \in \phi(\succcurlyeq_t,\omega(t))\} > 0$, and so we can find a net (in fact, a sequence) $(\succcurlyeq_\alpha,\omega_\alpha,y_\alpha)$ such that $\succcurlyeq_\alpha \to \succcurlyeq$, $\omega_\alpha \to \omega$, $y_\alpha \to y$ and $y_\alpha \in \phi(\succcurlyeq_\alpha,\omega_\alpha)$. Because $\omega \in J^*$, ϕ is u.h.c. at $(\succcurlyeq,\omega)$. Hence, $y \in \phi(\succcurlyeq,\omega)$.

(ii) By monotonicity of preferences it suffices to show that $p \cdot v \geqq p \cdot y$ whenever $v > y$. We argue by contradiction. Suppose that $p \cdot v < p \cdot y$ and $v > y$. Because $(\succcurlyeq,\omega,y) \in \text{supp } \rho$, we have $\succcurlyeq_\alpha \to \succcurlyeq$, $\omega_\alpha \to \omega$, $y_\alpha = x_\alpha + \omega_\alpha \to y$ where p supports $\succcurlyeq_\alpha$ at y_α, that is, if $z \succcurlyeq_\alpha y_\alpha$, then $p \cdot z \geqq p \cdot y_\alpha$. But by continuity, we eventually have $p \cdot v < p \cdot y_\alpha$ and $v \succ_\alpha y_\alpha$. Hence, we get a contradiction. QED

For the next three steps we fix an arbitrary $\bar{\omega} \in J^*$.

Step 2

Denote $K = (B + \bar{\omega}) \cap R^l_+$.

LEMMA 3. There is $p \gg 0$ and $c > 0$ such that the hyperplane $H_{p,c} = \{z\colon p \cdot z = c\}$ leaves K below it and $H_{p,c} \cap R^l_+ = H_{p,c} \cap \text{co } K$ (see Figure 9.9a).

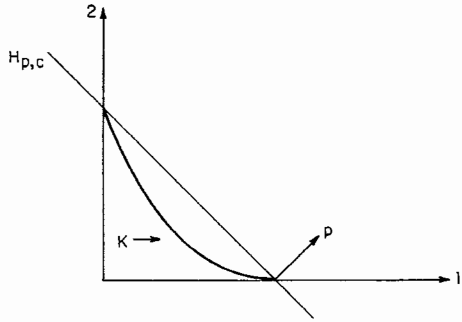

Figure 9.9a. Illustration for Lemma 3.

PROOF. We begin by noting that K cannot be contained in any (strict) coordinate subspace of R^l. This is because if $\varepsilon > 0$ is small, then, for all i, $\bar{\omega} - \varepsilon e_i \in \text{Int } J$, and so, by Lemma 1, $(B + (\bar{\omega} - \varepsilon e_i)) \cap R^l_+ \neq \emptyset$.

For every $p > 0$ let $\xi(p)$ be the set of maximizers of $p \cdot v$ on co K. The correspondence $p \mapsto \xi(p)$ is u.h.c., and by the observation of the previous paragraph $p \cdot \xi(p) > 0$ for all $p > 0$. Of course, $\xi(p)$ is a compact, convex set whose extreme points belong to K.

It is clear that our problem can be reformulated as that of finding a $p \gg 0$ such that $\xi(p)$ intersects all the coordinate axes of R^l. We argue by contradiction. The contradiction hypothesis is "for all $p \gg 0$, $\xi(p)$ does not intersect some axis." The essence of the proof argument is illustrated in Figure 9.9b.

We first show that for some $p \gg 0$, $y \in \xi(p) \cap K$, and i, $\xi(p)$ misses the i-th axis and $y^i > 0$. Because K is not contained in any strict coordinate subspace, co K intersects R^l_{++}. Therefore, we can find $z \in \text{Bdry}(\text{co } K - R^l_+) \cap R^l_{++}$. Let $q > 0$ support co $K - R^l_+$ at z. If $q \gg 0$, then we take p

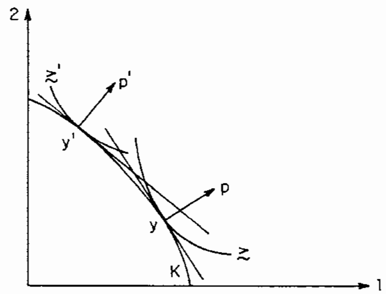

Figure 9.9b. Illustration for Lemma 3.

$= q$. By hypothesis, $\xi(p)$ misses some axis, say the i-th. Because $0 \ll z \leqq z' \in \xi(p)$, some extreme point $y \in \xi(p)$ must have $y^i > 0$. Of course, $y \in K$. Suppose now that $q^i = 0$ for some i. Because $q \cdot \xi(q) > 0$, $\xi(q)$ misses any axis i with $q^i = 0$. Let q_ε be given by $q_\varepsilon^i = q^i$ if $q^i > 0$ and $q_\varepsilon^i = \varepsilon > 0$ otherwise. Then $q_\varepsilon \gg 0$ and, by continuity, if ε is small, then $\xi(q_\varepsilon)$ is disjoint from any axis i with $q^i = 0$. Suppose that $v \in \text{co}\, K - R^l_+$ and $v^i = 0$ for all i with $q^i = 0$. Then $q_\varepsilon \cdot v = q \cdot v \leqq q \cdot \xi(q) = q \cdot z < q_\varepsilon \cdot z \leqq q_\varepsilon \cdot \xi(q_\varepsilon)$, and so $v \notin \xi(q_\varepsilon)$. Therefore, for any $y \in \xi(q_\varepsilon)$ (hence, for any extreme point), we have that $y^i > 0$ for some i with $q^i = 0$. Take $p = q_\varepsilon$ for ε small and we are done.

Let p and y be as above. Say that $i = 1$. Put $p_\varepsilon = (p^1, p^2 - \varepsilon, \ldots, p^l - \varepsilon)$. If ε is small, then $\xi(p_\varepsilon)$ should contain some v with $v^i > 0$ for some $i \neq 1$. Otherwise, $\xi(p_\varepsilon)$ intersects the first axis for arbitrarily small ε and, by the u.h.c. of ξ, so does $\xi(p)$. Clearly, v can be taken to be an extreme point.

In summary, relabeling commodities if necessary, we can find $p, p' \gg 0$ and $y, y' \in K$ such that

(i) y (resp. y') maximizes $p \cdot v$ (resp. $p' \cdot v$) on K
(ii) $p^1 = p'^1$, $p^2 > p'^2$
(iii) $y^1 > 0$, $y'^2 > 0$.

According to the hypothesis of the theorem, there are strictly convex $\succcurlyeq$, $\succcurlyeq' \in P$ representable by C^1 utility functions u, u' such that $\partial u(y) = p$, $\partial u(y') = p'$, and $(\succcurlyeq, \bar\omega)$, $(\succcurlyeq', \bar\omega) \in \text{supp}\, \nu$. Then $\phi(\succcurlyeq, \bar\omega) = \{y\}$, $\phi(\succcurlyeq', \bar\omega) = \{y'\}$. By Lemma 2(i), $(\succcurlyeq, \bar\omega, y)$, $(\succcurlyeq', \bar\omega, y') \in \text{supp}\, \rho$. Hence, by Lemma 2(ii), there is some price vector q supporting $\succcurlyeq$, $\succcurlyeq'$ at y and y', respectively. But this is impossible because $(p^1/p^2) < (p'^1/p'^2)$, $y^1 > 0$, and $y'^2 > 0$, that is, there is a feasible favorable trade of the first commodity for the second between $(\succcurlyeq, \bar\omega)$ and $(\succcurlyeq', \bar\omega)$. This contradiction proves the lemma. QED

Step 3

LEMMA 4. If q is a supporting price vector for x, then $q = \alpha p$ where p is as in Lemma 3.

PROOF. With e_i the i-th unit vector, let $y_i = (c/p^i)e_i$. The vectors $\{y_1, \ldots, y_l\}$ are the extreme points of $H_{p,c} \cap R^l_+$. Let $\succcurlyeq_1, \ldots, \succcurlyeq_l \in P$ be strictly convex and representable by C^1 functions $u_1, \ldots, u_n$. Suppose that $p = \partial u_i(y_i)$ and $(\succcurlyeq_i, \bar\omega) \in \text{supp}\, \nu$ for all i. These preference relations exist by the hypothesis of the theorem. Clearly, up to a positive multiplicative constant, p is the only price vector that simultaneously supports each $\succcurlyeq_i$ at y_i (i.e., for all i, $v \succcurlyeq_i y_i$ implies $p \cdot v \geqq p \cdot y_i$). By Lemma 3, for each i, $y_i \in K$ and, because $\succcurlyeq_i$ is strictly convex and K lies below $H_{p,c}$, we have $y_i = \phi(\succcurlyeq_i, \bar\omega)$. By Lemma 2(i), $(\succcurlyeq_i, \bar\omega, y_i) \in \text{supp}\, \rho$ for all i. Therefore, by Lemma 2(ii), q supports each $\succcurlyeq_i$ at y_i. Hence $q = \alpha p$ for some $\alpha > 0$. QED

Note that because x is efficient, it has a supporting price vector q. Therefore, it is a consequence of Lemma 4 that we can chose $p = q$ in Lemma 3 (in particular, p can be chosen to be independent of $\bar{\omega}$). It is also a consequence of Lemma 4 that, up to positive scalar multiplication, q is unique.

Step 4

With $Q \subset P$ as the statement of the theorem, let $M = \{y \in R^l_+ : (\succcurlyeq, \bar{\omega}, y) \in \text{supp}\ \rho$ for some $\succcurlyeq\ \in Q\}$. By Lemma 2(i), $M \subset K$.

LEMMA 5. *M is compact.*

PROOF. M is the projection on its third coordinate of $C = \text{supp}\ \rho \cap (Q \times \{\bar{\omega}\} \times [(B + \bar{\omega}) \cap R^l_+])$. Because supp ρ is closed, $(B + \bar{\omega}) \cap R^l_+$ bounded (Lemma 1), and Q compact, the set C is compact and, being that the projection is continuous, so is M. QED

If p and c are as in Lemma 3, then K, and therefore M, lie below, or on, $H_{p,c}$. Lemma 6 proves that M lies above, or on, $H_{p,c}$. Combining the two Lemmas we get $M \subset H_{p,c}$. In fact, we show in Lemma 7 that equality holds, that is, $M = H_{p,c} \cap R^l_+$.

LEMMA 6. *$M \subset H_{p,c}$.*

PROOF. For any $q > 0$, let $\xi(q)$ minimize $q.v.$ for $v \in M$. Because M is compact, $\xi(q) \neq \emptyset$ and $q \mapsto \xi(q)$ is u.h.c.

Suppose, by way of contradiction, that M is not contained in $H_{p,c}$. The subsequent arguments are illustrated in Figure 9.10.

By Lemma 3 and the contradiction hypothesis, we have $a = p \cdot \xi(p) < c$.

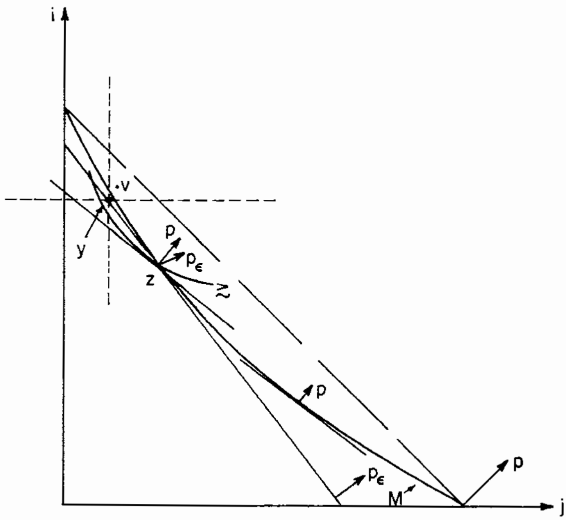

Figure 9.10. Illustration for Lemma 6.

We argue, first, that $\xi(p)$ cannot intersect any coordinate axis of R^l. Indeed, suppose that $v = (a_1/p^1,0,\ldots,0) \in \xi(p)$. By Lemma 3, $v' = (c/p^1,0,\ldots,0) \in K$. Therefore, $v' > v$ for some $v' \in K$ and $v \in M$. Let $(\succcurlyeq,\bar{\omega},v) \in \text{supp } \rho$, $\succcurlyeq \in Q$.

By the monotonicity of $\succcurlyeq$, $v' \succcurlyeq v$. By Lemmas 2 and 4, the price vector p supports $\succcurlyeq$ at v. Hence, if u is a C^1 utility for $\succcurlyeq$, $\partial u(v) \leqq \alpha p$ and $(\alpha p - \partial u(v))v = 0$ for some α. If $a > 0$, then $\partial_1 u(v) > 0$, and so $\partial u(v)(v' - v) > 0$. Hence, $v' \succ v$, which contradicts Lemma 2(i). If $a = 0$, then $0 = v \in M$. With $\succcurlyeq$ and u as above, we have $\partial u(0) \neq 0$ by hypothesis. Let $\partial_i u(0) > 0$ and put $v' = (0,\ldots,c/p^i,\ldots,0) \in K$. Then $v' \succ v$ and we have again a contradiction. We conclude that $\xi(p)$, does not intersect any axis.

Let $\varepsilon < 1$. Define p_ε by $p^i_\varepsilon = p^i_\varepsilon(1 - \varepsilon^i)$, where ε^i is the i-th power of ε. By continuity, if ε is small, then $\xi(p_\varepsilon)$ does not intersect any axis. Hence, there is $z \in \xi(p_\varepsilon)$ such that z^i, $z^j > 0$ for some $i < j$. Note that $(p^i/p^j) > (p^i_\varepsilon/p^j_\varepsilon)$. Pick $\succcurlyeq \in Q$ with $(\succcurlyeq,\bar{\omega},z) \in \text{supp } \rho$. By Lemmas 2(ii) and 4, p supports $\succcurlyeq$ at z, that is, for some $\alpha > 0$, we have $\partial_i u(z) = \alpha p_i$, $\partial_j u(z) = \alpha p_j$, where u is a C^1 utility representing $\succcurlyeq$. Because z_i, $z_j > 0$ and $(\partial_i u(z)/\partial_j u(z)) > (p^i_\varepsilon/p^j_\varepsilon)$, we can pick $y \in R^l_+$ such that $y \succ z$ and $p_\varepsilon \cdot y = p_\varepsilon \cdot \xi(p_\varepsilon)$.

By hypothesis there is $\succcurlyeq' \in Q$, representable by a C^1 utility u', such that $\partial u'(y) = p$. By Lemma 2(i), there is $v \in R^l_+$ such that $(\succcurlyeq,\bar{\omega},v) \in \text{supp } \rho$. By Lemmas 2(ii) and 4 $\succcurlyeq'$ is supported at v by p. Therefore, p supports $\succcurlyeq'$ both at y and at v. By the Normality hypothesis on the elements of Q (this is the only time it is used!), we have either $v < y$ or $v \geqq y$. Because $v \in M$, $p_\varepsilon \cdot v \geqq p_\varepsilon \cdot y$. Hence, $v < y$ is impossible. Therefore $v \geqq y$.

In summary, we have found $(\succcurlyeq,\bar{\omega},z) \in \text{supp } \rho$, $y \succ z$ and $v \in M \subset K$ such that $v \geqq y$. By monotonicity, $v \succ z$ and we have obtained our final contradiction because, by Lemma 2(i), z should maximize $\succcurlyeq$ on K. QED

LEMMA 7. $M = H_{p,c} \cap R^l_+$.

PROOF. This is simple enough. Let $y \in H_{p,c} \cap R^l_+$. By hypothesis there is $\succcurlyeq \in Q$ such that p supports $\succcurlyeq$ at y. Because $(\succcurlyeq,\bar{\omega}) \in \text{supp } \nu$ there is, by Lemma 2(ii), a z such that $(\succcurlyeq,\bar{\omega},z) \in \text{supp } \rho$. By Lemmas 2(ii) and 4, $\succcurlyeq$ is supported at z by p. Because $z \in M \subset H_{p,c}$ (Lemma 6), we have $p \cdot z = c$. Of course, $p \cdot y = c$. Therefore, with the budget defined by wealth c and price vector p, the preferences are maximized at y and z. By hypothesis, this maximizer is unique. Hence $y = z \in M$ and we are done. QED

The next lemma follows simply from Lemma 7.

LEMMA 8. $K = H_{p,c} \cap R^l_+$.

PROOF. Because $M \subset K$, it suffices to show that $y \in K$ and $p \cdot y < c$ is not possible. Suppose, by way of contradiction, that this was the case. By Lemma 7, there is then $v \gg 0$ such that $y - \bar{\omega} + v \in B$. If $\varepsilon > 0$ is small, then $\|y - \bar{\omega} - x(t)\| < \varepsilon$ implies $x(t) \ll y - \bar{\omega} + v$. If, moreover,

$x(t) + \omega(t) \geqq 0$, we have $0 \leqq x(t) + \omega(t) \ll y - \bar{\omega} + v + \omega(t) \in (B + \omega(t)) \cap R^l_+$. By the monotonicity of preferences, this yields $x(t) + \omega(t) \notin \phi(\succcurlyeq_t, \omega_t)$. But $x(t) + \omega(t) \in \phi(\succcurlyeq_t, \omega_t)$ for *a.e.* $t \in I$ and, by the definition of B and $y - \bar{\omega} \in B$, $\lambda\{t: \|y - \bar{\omega} - x(t)\| < \varepsilon\} > 0$. This contradiction proves the lemma. QED

Step 5

We are now ready to argue that the price vector p obtained in Lemma 3 and which, by Lemma 4, can be taken to be independent of $\bar{\omega}$ is a Walrasian price vector. Because p supports x and $p \gg 0$, it suffices to show that $p \cdot x(t) = 0$ for *a.e.* $t \in I$ or, simply, that $p \cdot y \geqq p \cdot \omega$ whenever $y \in (B + \omega) \cap R^l_+$ and $\omega \in J^*$. (Remember that J^* is dense in Int J; so, for each $\omega \in$ Int J, there is $\omega' \in J^*$ with $\omega' \ll \omega$.)

Let $c(\omega)$ be as in Lemma 2 for $\omega = \bar{\omega}$. By Lemma 8, $p \cdot y = c(\omega)$ for all $y \in (B + \omega) \cap R^l_+$. We shall show that $c(\omega) \geqq p \cdot \omega$ for all $\omega \in J^*$.

If $c(\omega) < p \cdot \omega$ for all $\omega \in J^*$, then $p \cdot x(t) < 0$ for *a.e.* $t \in I$. Therefore, $p \cdot (\int x) < 0$, which contradicts $\int x = 0$ (because of (A.1), efficiency precludes $\int x \leqq 0$ and $\int x \neq 0$). Hence, $c(\bar{\omega}) \geq p \cdot \bar{\omega}$ for some $\bar{\omega} \in$ Int J^*. Take $\bar{v} = (c(\bar{\omega})/p \cdot e)e - \bar{\omega} \geqq 0$. By Lemma 7, $\bar{v} \in B$, and therefore $\bar{v} + \omega \in (B + \omega) \cap R^l_+$ for all $\omega \in J$. By Lemma 8, we get $p \cdot (\bar{v} + \omega) = c(\omega)$ for all $\omega \in J^*$, which yields $c(\omega) \geqq p \cdot \omega$ for all $\omega \in J^*$ and finishes our proof.

6. Proof of Theorem 2

The given efficient and strictly anonymous net trade is denoted x. We let ρ be the measure on $\mathcal{A} \times R^l$ induced by $(\mathscr{E}, x + \omega)$, that is, $\rho = \lambda \circ (\mathscr{E}, x + \omega)^{-1}$. Put $B = B_x$, $B^* = B^*_x$, $T =$ closure B^*. The generic symbols for a coordinate subspace of R^l and its strictly positive orthant are, respectively, M and M_{++}.

The proof proceeds by a sequence of five lemmas. Lemma 1 shows that T always has the form indicated in Figures 9.7 and 9.8, that is, a discrete union of translates of a linear subspace L. The proof relies on some algebraic facts already used by Schmeidler and Vind (1972). Lemma 2 uses (A.2) and shows that T is contained in an hyperplane with strictly positive normal. The next three lemmas exploit the efficiency of x and obtain dim $L = l - 1$. This, then, yields the theorem. Each lemma presupposes the previous ones.

LEMMA 1. There is a linear subspace $L \subset R^l$ and vectors $v_1, \ldots, v_k \in R^l$ such that $\{L, v_1, \ldots, v_k\}$ are linearly independent and $T = \{y \in R^l: y = v + \sum_{j=1}^k n_j v_j,\ v \in L \text{ and } n_j \text{ integer for all } j\}$.

PROOF. The usual operation of vector addition makes R^l into an algebraic group. We shall demonstrate that T is a subgroup of this additive group. Then we need only appeal to the fact that any closed subgroup of R^l can be expressed in the form required by the lemma (see Bourbaki 1947, chapter 7, p. 65, Theorem 2 and its corollary).

Denote by B_m the sum of m copies of B. Then $B^* = \bigcup_{m=1}^{\infty} B_m$.

It is clear that T is closed under addition. Let $z, y \in T$. Then $z_n \to z$, $y_n \to y$ for $z_n, y_n \in B$. Of course, $z_n + y_n \to z + y$. Suppose that $z_n \in B_m$, $y_n \in B_h$. Then $z_n + y_n \in B_{m+h} \subset B^*$. Hence, $z + y \in T$.

To show that T is a subgroup we must prove that $z \in B^*$ implies $-z \in T$. This is more delicate. Let $z \in B_m$ and $\varepsilon > 0$. We shall exhibit a $v \in B^*$ such that $\|z + v\| \leqq \varepsilon$. Because ε is arbitrary, this will suffice.

We begin by proving that 0 belongs to the relative interior of co B, denoted J. If not, we can find $q \in R^l$ such that $q \cdot y > 0$ whenever $y \in J$. We should have $\lambda\{t: q \cdot x(t) > 0\} > 0$. Otherwise the support of the measure induced by x, which is B, and its convex hull, would be contained on $\{y: q \cdot y \leqq 0\}$. Henceforth, $\int q \cdot x(t) > 0$, which contradicts $\int x = 0$ (efficiency precludes $\int x < 0$). Because co B_m is a sum of m copies of co B, we also have that 0 belongs to the relative interior of B_m. What this gives us, by Caratheodory's theorem, is that we always can express $0 = z + \sum_{j=1}^{k} \alpha_j z_j$ for some $\alpha_j > 0$ and $z_j \in B_m$. Let $c > \max_j \|v_j\|$.

By elementary number theoretic facts (see Hardy and Wright 1954, p. 170), there are positive integers r and $r_1, \ldots, r_k$ such that $|r\alpha_j - r_j| < \varepsilon/kc$ for every j. Because $-rz = \sum_{j=1}^{k} r\alpha_j z_j$ if we put $y = \sum_{j=1}^{k} r_j z_j$, we get $\|y + rz\| < \varepsilon$. Finally, put $v = y + (r - 1)z$. Then $\|z + v\| < \varepsilon$ and $v \in \sum_{j=1}^{k} B_{mr_j} + B_{m(r-1)} \subset B^*$ as we wanted. QED

LEMMA 2. There is $p \gg 0$ such that $p \cdot T = 0$.

PROOF. The lemma holds if co $T \cap R_+^l = \{0\}$. Suppose, by way of contradiction, that $v > 0$, $v \in$ co T.

Consider any $t \in \mathscr{E}^{-1}(P_b \times R_{++}^l)$. Then there is $\delta > 0$ and $\bar{\beta} > 0$ such that $\omega(t) + y >_t x(t)$ whenever $\|\beta v - y\| \leqq \delta$ for some $\beta > \bar{\beta}$. Also, $\omega(t) + y \gg 0$ if δ is small enough.

We can express v as $v = z + \sum_{j=1}^{k} \alpha_j v_j$, where $z \in L$ and the v_j's are as in the statement of Lemma 1. Let $c > \max_j \|v_j\|$. By elementary number theoretic facts (see Hardy and Wright 1954, p. 170), there are integers $m > \bar{\beta}$ and $m_1, \ldots, m_k$ such that $|m\alpha_j - m_j| < \delta/kc$ for all j. Denoting $y = mz + \sum_{j=1}^{k} m_j v_j \in T$, we then have $\|mv - y\| \leqq \delta$. Hence $\omega(t) + y >_t x(t)$ and $\omega(t) + y \gg 0$. If $y' \in B^*$ approximates y close enough, then these properties are preserved. Therefore, $x(t)$ does not $\geqslant_t$ − maximize on $B_+^* + \omega(t)$. So $v(P_b \times R_{++}^l) = 0$, which contradicts (A.2). QED

LEMMA 3. There is an open and dense set $J \subset R^l$ such that if $M \cap (T + \omega) \neq \emptyset$, $\omega \in J$, then $M \cup L$ spans R^l, that is, L is transversal to every coordinate subspace.

PROOF. If dim $L = l$, we are done. So let dim $L = l - s < l$ and A be a $s \times l$ matrix such that $L = \{v: Av = 0\}$. Of course, rank $A = s$. We let

A' be the generic symbol for $s \times m$ matrices whose columns are taken from A.

With $v_1, \ldots, v_k$ as in Lemma 1, let $\{c_i\}$ be an enumeration of the vectors of the form $c_i = \sum_{j=1}^{k} n_j A v_j$, where the n_j are integers. Define $J = \{\omega \in R^l$: if rank $A' < s$, then $A'y' = c_i + A\omega$ has no solution for any $c_i\}$. The set J is open and dense. Indeed, if rank $A' < s$, then $S = \cup_i(\text{Range } A' - c_i) \subset R^s$ is a discrete union of translates of a lower-dimensional subspace. Hence if $A\omega \notin S$, then $A\omega' \notin S$ for all ω' near ω, and if $A\omega \in S$, then, because rank $A = s$, $A\omega' \notin S$ for some ω' near ω.

Now let $z \in M \cap (T + \omega)$, $\omega \in J$. Say that $M = \{y: y^{m+1} = \ldots = y^l = 0\}$ and put $A = [A', A'']$ where A' are the first m columns. For some $n_1, \ldots, n_k$ and $v \in L$, we have $z = v + \sum_{j=1}^{k} n_j v_j + \omega$ or, premultiplying by A, $Az = c_i + A\omega$ for some c_i. Denoting by z' the first m entries of z, we have $Az = A'z'$. Hence, $A'z' = c_i + A\omega$. Because $\omega \in J$, this yields rank $A' = s$. The kernel of A' is $M \cap L$. Therefore, $\dim(M \cap L) = m - s$, and so $\dim(M \cup L) = \dim M + \dim L - \dim(M \cap L) = m + l - s - (m - s) = l$, which is what we wanted. QED

LEMMA 4. There is $\succsim \in P_s$, an open $V \subset R^l_{++}$, $b \in R^l$, a continuous function $f: V \to R^l_+$, and coordinate subspace M transversal to L such that, for all $\omega \in V$:

(i) $(\succsim, \omega, f(\omega)) \in \text{supp } \rho$,
(ii) $f(\omega)$ is $\succsim$ − maximal on $(L + b + \omega) \cap R^l_+$,
(iii) $f(\omega) \in M_{++}$.

PROOF. Let $\succsim$ and $V \subset J$ satisfy the hypothesis of the theorem.

For each $\omega \in V$, define $G(\omega) = (T + \omega) \cap R^l$ and $g(\omega) = \{z \in R^l$: $(\succsim, \omega, z) \in \text{supp } \rho\}$. Of course, $g(\omega) \in G(\omega)$. By Lemma 2, G, and therefore g, is an u.h.c. correspondence. Also, $g(\omega) \neq \varnothing$ for all $\omega \in V$ (see the proof of Lemma 2(i) in Section 5). Every u.h.c. correspondence is continuous somewhere. (This is Fort's theorem; see Dierker 1973.) Let g be continuous at $\bar{\omega}$ and pick up $y \in g(\bar{\omega})$. Then $y = v + \sum_{j=1}^{k} n_j v_j + \bar{\omega}$ for $v \in L$ and integers v_j. Call $b = \sum_{j=1}^{k} n_j v_j$. Because if $b' = \sum_{j=1}^{k} n'_j v_j$ is different from b, then $L + b + \bar{\omega}$ is at a finite distance from $L + b' + \bar{\omega}$ and because g is continuous at $\bar{\omega}$, there must be an u.h.c. selection $f(\omega) \in g(\omega)$ such that $f(\omega) \in L + b + \omega$ for all $\omega \in V$ (if needed, V is replaced by a smaller open set).

Lemma 3 guarantees that locally around $\bar{\omega}$, we have $\omega \mapsto (L + b + \omega) \cap R^l_+$ continuous. (Let M be the minimal coordinate subspace to which $y \in f(\bar{\omega})$ belongs and M^c its complementary. By Lemma 3, L projects onto M^c_{++}. Hence $(L + b + \omega) \cap R^l_{++} \neq \varnothing$, and this yields continuity straightforwardly.) Therefore, if V is sufficiently small and $\omega \in V$, every $y \in f(\omega)$ maximizes $\succsim$ on $(L + b + \omega) \cap R^l_+$ (see the proof of Lemma 2(i) in Section

5). But the set $(L + b + \omega) \cap R^l_+$ is convex, and $\succcurlyeq$ is strictly convex. Hence, there is a unique maximizer, and we conclude that $f: V \to R^l_+$ is in fact a continuous function.

The function $c(\omega) = \#\{j: f^j(\omega) = 0\}$ is lower semicontinuous, integer-valued, and bounded above by l. Therefore, by reducing V, if necessary, we can assume that $c(\omega)$ is constant on the connected V or, equivalently, that $\{j: f^j(\omega) = 0\}$ is independent of ω. Let M be the minimal coordinate subspace containing $f(\omega)$ and we are done. QED

LEMMA 5. With M as in Lemma 4, $\dim M \cap L = \dim M - 1$.

PROOF. Let $u: M_{++} \to R$ be a C^2 utility with nonvanishing gradient for $\succcurlyeq$ on M_{++}. For each $\omega \in V$, $\partial u(f(\omega))$ is perpendicular to $M \cap L$.

The first observation is that the efficiency of x implies the collinearity of any $\partial u(f(\omega))$, $\partial u(f(\omega'))$ for $\omega, \omega' \in V$. Indeed, $(\succcurlyeq,\omega,f(\omega))$, $(\succcurlyeq,\omega',f(\omega'))$ $\in$ supp ρ, and Lemma 2(ii) of Section 5 applies here without modification.

Suppose now that $\dim(M \cap L) < \dim M - 1$. Fix $\bar{\omega} \in V$, $y = f(\omega)$. Because the surface of $\{z \in M: z \succcurlyeq y\}$ has nonzero curvature at y (which ensures that $y' \mapsto (1/\|\partial u(y')\|)\partial u(y')$ covers locally a neighborhood of $\partial u(y)$ in the sphere), we can find $y' \in M_{++}$ arbitrarily near y and such that $\partial u(y')$ is perpendicular to $M \cap L$ but not collinear with $\partial u(y)$. Let A be a full rank matrix whose kernel is L. Because $y \in L + b + \bar{\omega}$, we have $Ay = Ab + A\bar{\omega}$. We can always find a solution ω' to $Ay' = Ab + A\omega'$. Moreover, if y' is close to y, ω' can be chosen close to ω. Notice that $y' \in (L + b + \omega') \cap M_{++}$ and $\partial u(y')$ is perpendicular to $M \cap L$. Because $\succcurlyeq$ is strictly convex, y' is the only vector that is $\succcurlyeq -$ maximal on $(L + b + \omega') \cap M_{++}$. Therefore, $f(\omega') = y'$ and we have obtained a contradiction because $\omega, \omega' \in V$ and $\partial u(f(\omega))$, $\partial u(f(\omega'))$ are not collinear. Hence, $\dim(M \cap L) = \dim M - 1$. QED

The proof is now almost complete. Because M and L are transversal, that is, $M \cup L$ spans R^l, we have $\dim L + \dim M = l + \dim(M \cap L)$. Hence, from Lemma 5, $\dim L = l - 1$ and, therefore, from Lemma 2, $T = \{z \in R^l: p \cdot z = 0\}$. Let $x(t)$ be $\succcurlyeq_t -$ maximal on $(B^* + \omega(t)) \cap R^l_+$. If $\omega(t) = 0$, then $x(t) \in B^* \cap R^l_+ \subset T \cap R^l_+ = \{0\}$, that is, $x(t) = 0$, which is, of course, $\succcurlyeq_t -$ maximal on $T \cap R^l_+$. If $\omega(t) > 0$, then $(T + \omega) \cap R^l_+$ is the closure of its interior relative to T, that is, $(B^* + \omega) \cap R^l_+$ is dense in $(T + \omega) \cap R^l_+$, and so, by the continuity of $\succcurlyeq_t$, $x(t)$ is again $\succcurlyeq_t -$ maximal on $(T + \omega) \cap R^l_+$. Because preferences are monotone, p is a Walrasian price vector.

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