17 Capital Theory Paradoxes: Anything Goes

Andreu Mas-Colell*

1 INTRODUCTION

A first approximation to the study of the intertemporal aspects of resource allocation (capital theory from now on), consists of concentrating on the steady states (the rest points) of associated dynamical systems. Provided one does not lose sight from the fact that this is not an end in itself, there is much useful information to be gleaned from steady-state analysis – indeed, one of the prime tools of economics. No doubt, the pervasiveness of the notion of the stationary state in classical economics and the fact that there is so much one can do without invoking powerful mathematics, have also added to its popularity.

An excellent, but advanced, introduction to steady-state capital theory in the multigood case (the focus of this paper) is von Weizsäcker (1971). The books by Bliss (1975) and Burmeister (1980), which are general surveys of capital theory, also have good accounts of steady-state aspects.

In the early 1950s, Joan Robinson started the systematic comparative static analysis of the steady states of economies with several capital goods (see Robinson, 1953). Although the fundamental, and slow to sink in, contributions of von Neumann (1945–6) and Malinvaud (1953) come earlier, the emphasis on comparative statics was distinctive from the tradition she helped to launch.

The line of research initiated by Robinson culminated in the 1960s with the realization that many of the comparative statics theorems valid for the one capital good case do not generalize to the multigood case. To a sensibility educated on the former, the heterogeneous capital good case admitted models with behavior that appeared 'bizarre', 'exotic', 'paradoxical' ... I should be quick to add that this was no disappointment to J. Robinson and her followers. It was rather their point and in this they were perceptive. Many controversies and much noise accompanied the process of intellectual discussion. The story has been told many times (see the Bliss or Burmeister references; or, for a militant Robinsonian point of view, Harcourt, 1972) and I will not repeat it here.

Consider the central example with which I will concern myself in this

* The research for this paper has been helped by the support of a Guggenheim Fellowship.

paper: the dependence of steady-state consumption (say, that there is only one consumption good) on the rate of interest. For simplicity I restrict myself to a world without technological change or population growth. A basic and quite general theorem, the golden rule, asserts that the maximum consumption level is associated with a rate of interest equal to the rate of population growth, that is, zero. In addition, the standard one capital good case displays a monotonically decreasing relationship between consumption levels and the rate of interest (as in Fig. 17.1a). This does not generalize and it is now well understood that even with only two capital goods, a non-monotone association such as in Figure 17.1b is possible (see, for example, Burmeister, 1980, ch.4).

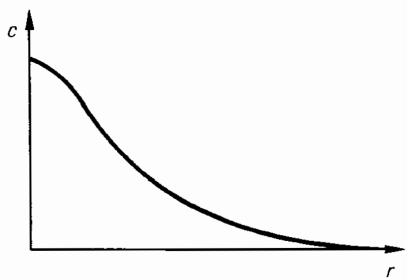

Figure 17.1a

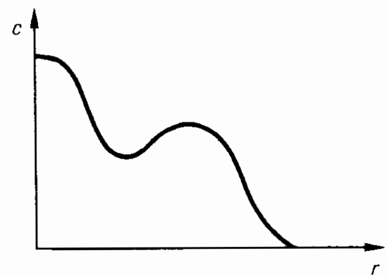

Figure 17.1b

The purpose of this chapter is to bring the above point to its logical conclusion (to out-Cambridge Cambridge, so to speak) by showing that in general (i.e. two capital goods but no specific restriction on the technology beyond convexity) there is no other theorem on the association across steady states of consumption levels and rates of interest than a slightly strengthened version of the golden rule. The result of this chapter says, roughly, that given any set of consumption and rate of interest pairs, say the one in Figure 17.2, a well-behaved technology can be found having precisely this set as the steady-state comparative static locus. The uncovering of a slight strengthening of the golden rule is a minor positive pay-off of this research strategy.

The informed reader will notice that the project carried out in this chapter follows a parallel development in general equilibrium theory. As in capital theory, intensive research by general equilibrium theorists on comparative statics, and the closely related topic of stability, did not yield many general results and few that measured up to early hopes (the story is well told in Arrow and Hahn, 1971). The research culminated in the results by, among others, Sonnenschein, Mantel and Debreu (see Shafer and Sonnenschein, 1982, for a good survey) showing that but for the obvious restrictions (e.g. Walras Law) literally anything can be the excess demand of a well-behaved exchange economy. My intention in this paper is thus to make Joan

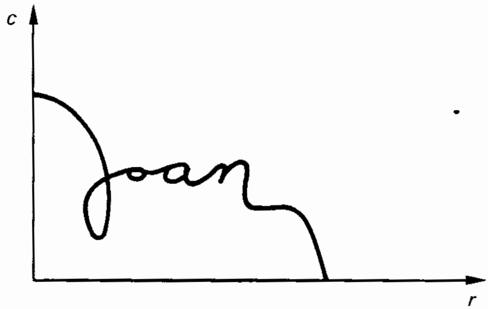

Figure 17.2

Robinson's point in the manner of the Sonnenschein–Mantel–Debreu theorem.

It is interesting to ponder why two analytically similar developments gave rise to such different styles of scientific debate: placid and consensual in general equilibrium theory; tumultuous and passionate in capital theory. I will make a couple of comments on this although I do not think that they constitute much of an explanation for the difference. More likely the reasons are to be found in the sociology of knowledge and in the fact that, perhaps because of the Marxian tradition, capital theory has tended to be more ideologically charged than other areas of economics. There has been a distressing propensity, on all sides, to draft analytical results for service at the trenches.

My first comment touches on a superficial aspect, but it may be psychologically significant. It has to do with pictures. The Edgeworth box can be made to display quite a lot of complex behavior in front of our very eyes (e.g. multiplicity of equilibria, transfer paradoxes, . . .). The ground may therefore be prepared for bad news in the n-good case. On the contrary, in capital theory only the one capital good case is amenable to self-evident graphical analysis. The interesting complexities arise, however, only with more than one capital good.

My second comment goes deeper and reveals, probably, my bias to view capital theory as subsumed in general equilibrium analysis. While undoubtedly general equilibrium theorists would wish to have general comparative static theorems, the news that they are not available is not looked at as the deathknell of the theory. The reason is that general foundational theorems abound (e.g. the welfare theorems) and the strength of the theory rests assured on them. The interpretation of the negative results is rather that comparative statics is not an area for armchair thinking but for empirical assessment of parameters. The 'real world' may or may not be simple. If it is, so much the better. If, as is more likely, it is not, then we still need, and have, sophisticated analytical tools to study it. But the values of the parameters determining the comparative static predictions have to be calibrated by an empirical appeal.

I believe that a similar point can be made for steady-state capital theory. There may not be general comparative static theorems (although the golden rule is general and not a result to be dismissed) but there are general theorems (e.g. short-run efficient steady state can be competitively supported by proportional price systems, a short-run efficient steady state is long-run efficient if and only if the rate of interest is not smaller than the rate of population growth). What the 'paradoxical' comparative statics has taught us is simply that modelling the world as having a single capital good is not *a priori* justified. So be it.

2 THE GOLDEN RULE THEOREM

To focus on essentials I will only consider a stationary economic situation with no technical change or population growth.

There are n capital goods. Vectors of stocks of capital goods are denoted $x,y,z \in R^n$.

There is a single consumption good, denoted c.

The intertemporal technological possibilities are described in a familiar but highly reduced form. Namely, there is a function $v(x,y)$ which gives the largest amount of consumption possible today if the capital stock vector today is x and the capital stock available tomorrow is constrained to be y.

The following standard hypotheses are made on the technology. Let $P = \{\Delta \in R^n\colon z \gg 0\}$. Then *$v$ is defined on $P \times P$, it is strictly concave, continuously differentiable and satisfies*

$$v(0,y) \leqslant 0,\ \frac{\partial v}{\partial x_i}(x,y) > 0,\ \frac{\partial v}{\partial y_i}(x,y) < 0$$

for any $(x,y) \in P \times P$.

The assumption that the concavity is strict and partial derivatives exist are just a matter of simplicity. Nothing of economic substance depends on it. The same is true for the permitted possibility of negative consumption. The neglect of stock vectors with some zero component is more important. Allowing for them would require some minor qualification to our conclusion.

A *steady state* is a configuration where $x = y$. Typically we denote by z this common value.

We are not interested in any state but only in those for which their corresponding stationary trajectories are dynamically efficient. (See the references in the introduction for the relevant concepts and results on the characterization of efficiency. An excellent treatment is Gale (1973).)

A steady state (z,z) is efficient if and only if it satisfies two conditions having, respectively, a short-run and a long-run character.

The short-run condition is that the steady state be supportable by a proportional price system. More precisely, there should be an interest rate factor r_z such that

$$\nabla_y v(z,z) = -(1-r)\nabla_x v(z,z).$$

The long-run condition is that the interest factor r associated with (z,z) be at least as large as the rate of population growth. In our case, $r \geqslant 0$.

The purpose of this paper is to investigate the properties of the locus $L \subset R_+ \times R$ formed by the pairs $(r_z, v(z,z))$ of rates of interest and consumption levels as (z,z) runs over all possible efficient state states.

A first restriction on L follows from the well known, and very general,

Golden Rule Theorem: The steady states that maximize consumption among all steady states are characterized by being efficient and having an associated rate of interest equal to the rate of population growth, in our case zero.

Thus, *a fortiori* (maximization over all steady states implies maximization over the efficient subset), we have that $c > c'$ whenever $(0,c) \in L$, $(r,c') \in L$ for some $r \neq 0$, and $(0,c) \neq (r,c')$. The locus represented in Figure 17.3a is inadmissible.

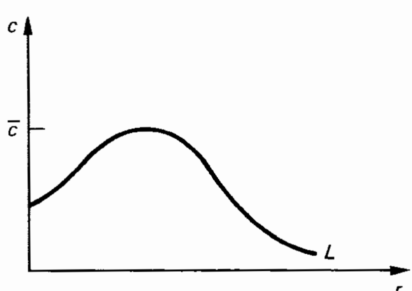

Figure 17.3a

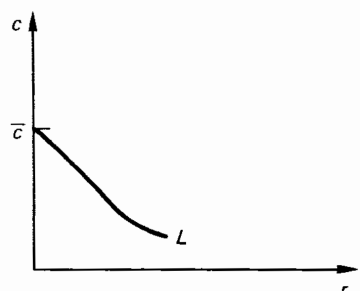

Figure 17.3b

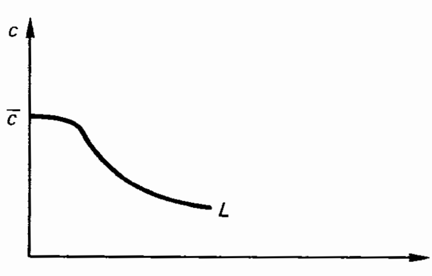

Figure 17.3c

The Golden Rule Theorem is very easy to prove. We should maximize $v(z,z)$ over $z > 0$. The first-order condition for this problem is:

$$\nabla_x v(\bar{z},\bar{z}) = -\nabla_y v(\bar{z},\bar{z}).$$

Hence, $(\bar{z},\bar{z})$ has an associated rate of interest which equals zero. Figure 17.4 provides a graphical illustration in the case $n = 1$.

Note that the Golden Rule Theorem would remain valid if the technology function v is merely quasiconcave (thus allowing for some degree of increasing returns in the production of the consumption good). Indeed, only the convexity of the sets $\{(x',y'): v(x',y') \geq v(x,y)\}$ matter to the proof. However, I shall now show that if the technology is in fact concave then the Golden Rule Theorem can be slightly strengthened. Not only is the maximum consumption reached at $r = 0$ but the consumption loss from a first-order increase in r (from $r = 0$) is of the second order. Geometrically, the top of the locus L as it intersects the vertical axis must be flat. Thus, the locus of Figure 17.3c is admissible but the one in Figure 17.3b is inadmissible. An examination of Figure 17.4 will convince the reader that Figure 17.3b could be generated from a quasiconcave v. This is because we are free to assign consumption values to the isoquants in the figure as we please.

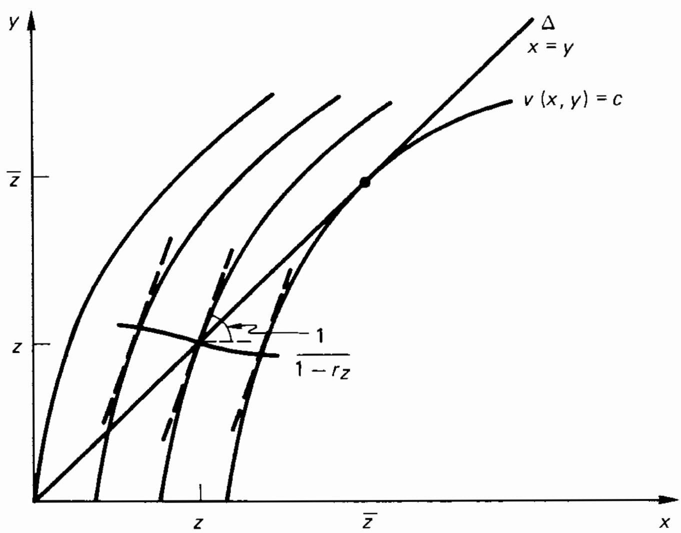

Figure 17.4

Formally the claimed property is:

Fact: Suppose that $\bar{c}$ is a golden rule consumption. If $r_m > 0$, $r_m \to 0$ and $(r_m, c_m) \in L$ then $(\bar{c} - c_m)/(r_m) \to 0$.

Proof: Let $\bar{z}, z_m$ be the capital stock associated, respectively, with the golden rule and (r_m, c_m). Denote $w(z) = v(z,z)$. Then

$$\nabla_z w(z_m) = \nabla_x v(z_m, z_m) + \nabla_y v(z_m, z_m) = r_m \nabla_x v(z_m, z_m).$$

By concavity,

$$\bar{c} = c_m = w(\bar{z}) - w(z_m) \leqslant \nabla_z w(z_m) \cdot (\bar{z} - z_m) = r_m \nabla_x w(z_m) \cdot (\bar{z} - z_m).$$

By the strictly concavity of $w(\cdot)$ we have have $z_m \to \bar{z}$. Therefore,

$$\nabla_x w(z_m) \cdot (\bar{z} - z_m) \to 0$$

and so

$$\frac{\bar{c} - c_m}{r_m} \to 0.$$

Technical Remark: I suspect, although I have only proved it for the case $n = 1$, that the above property can be strengthened to:

There is a continuous function α: $(-\infty, c] \to R$ such that

(i) $\alpha(c) \leqslant r$ whenever $(r,c) \in L$ and

(ii) $\int_{-\infty}^{\bar{c}} \frac{1}{\alpha(c)} dc < \infty.$

3 A CONVERSE TO THE GOLDEN RULE THEOREM

The central issue to be discussed in this section is: given a set $L \subset [0,\infty) \times R$, when can L be generated as the interest rate consumption locus of a technology v?

We will always take L to be closed and bounded above. We postulate also that for some $\bar{r} > 0$ we cannot have $(r,c) \in L$, $r \geqslant \bar{r}$ and $c \geqslant 0$. Further, we shall only care about non-negative consumption. Summarizing: we can assume without further loss of generality, that L is a closed subset of a rectangle $[0,\bar{r}] \times [0,\bar{c}]$.

Let $(0,\bar{c}) \in L$ be the golden rule point. We know from Section 2 that L should intersect the vertical axis perpendicularly and only at this point. Essentially, this will turn out to be the only restriction on L and thus the (strengthened) golden rule the only general comparative statics theorem.

The term 'essentially' in the previous paragraph is required because I will impose an extra, weak technical condition controlling the local behavior of L at $(0,\bar{c})$. The technical condition (or, for that matter, the strengthened golden rule itself) *places no restriction whatsoever on the shape of* L *over* $[\epsilon,\bar{r}] \times [0,\bar{c}]$.

Golden Rule Restriction: There is a continuous function α: $(-\infty,\bar{c}]\to R$ with $\alpha(\bar{c})=0$ and $\alpha(c)>0$ for $c<\bar{c}$ such that:

(i) for a constant $k>0$, $\alpha(c)\leqslant r\leqslant k\alpha(c)$ whenever $(r,c)\in L$, and

(ii) $\int_{-\infty}^{\bar{c}} \frac{1}{\alpha(c)}dc<\infty$. See Figure 17.5.

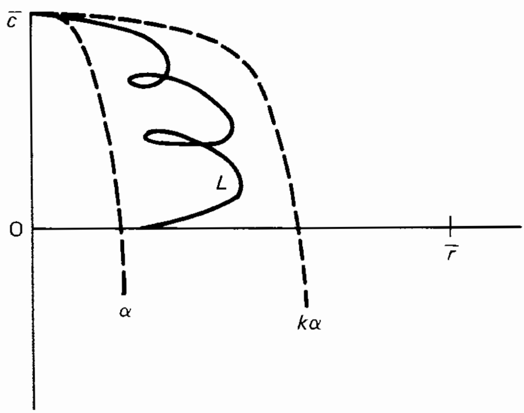

Figure 17.5

I wish to emphasize that, granted the golden rule, the technical golden rule restriction is purely local, i.e. depends only on the form of L near $(0,\bar{c})$. The condition has two parts. The first asserts the existence of a continuous function α: $(-\infty,\bar{c}]\to R$ with $[1]/[\alpha(c)]$ integrable and such that its graph leaves L to its right. As indicated in the remark at the end of the previous section, this implies the perpendicularity of L at $(0,\bar{c})$ and it is only slightly stronger than this property. It may also be a necessary implication of the concavity of v (this is unsettled). The second part is that L be captured between the graphs of α and $k\alpha$. This is necessitated by our method of proof and it could be violated by the locus generated by an admissible technology. Nevertheless, this part of the condition is extremely weak. Its violation appears quite pathological. Indeed, we must be able to find $c_m\to\bar{c}$, $(r_m,c_m),(r'_m,c_m)\in L$ but with r_m,r'_m approaching zero at different rates. In particular, if for c close to $\bar{c}$ we can solve uniquely for the rate of interest $r(c)$ such that $(r(c),c)\in L$, then this second part is automatically satisfied.

After this technical digression, we are able to state the main result.

Proposition: Suppose that $L\subset[0,\bar{r}]\times[0,\bar{c}]$ is a closed set satisfying the golden rule restriction. Then L can be generated by an admissible technology function $v(x,y)$.

The proof of the proposition will be given in the next section. Only two capital goods are needed for the technology.

One could ask if the construction underlying a given L is robust in the sense that a small perturbation of the (production, and productivities) of the generating technology will change L only slightly. Informally, the answer is that, in general, the construction cannot be robust because if it is then the set L itself must belong to the 'generic class' formed by the sets which are one-dimensional smooth manifolds. In less technical terms, if the construction is robust then L should look around any of its points as a little (curved) segment, i.e. as in Figure 17.6a but not as in Figure 17.6b where the manifold condition fails at the three indicated crossing points. On the other hand, we shall observe in the next section, while carrying out the key construction of the proof, that if the set L we start with is a one-dimensional smooth manifold, then the construction is in fact robust.

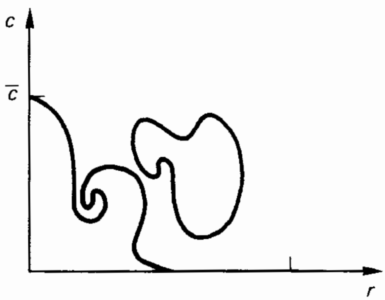

Figure 17.6a

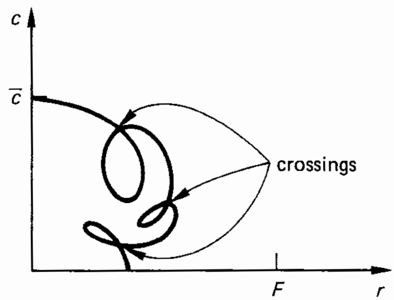

Figure 17.6b

What can we accomplish if we are constrained to a single capital good? Actually, the proposition remains valid provided we add to the restrictions on L that for every $c \leqslant \bar{c}$ there be a unique r_c with $(r_c, c) \in L$. That this restriction is necessary is obvious from Figure 17.4. We see there that fixing c and requiring a steady state determines z uniquely and so there can be at most one rate of interest. On the other hand, in the one-dimensional case, prices across two periods are always proportional. Hence, every steady-state z has associated with it a rate of interest which equals one minus the inverse of the slope to the isoquant through z. Still referring to Figure 17.4, the basic intuition of our proof, reduced to the case $n=1$, consists in observing that in the segment $[0,(\bar{z},\bar{z})]$ we can assign the slopes in essentially any manner we wish. There is a slightly delicate point (which accounts for the term 'essentially'): the assignment of isoquants to slopes should be compatible with the isoquant map being generated from a concave function. It is for this that the more refined golden rule restriction (i.e. the existence of the function α) is required.

We have just argued that a set L such as the one in Figure 17.7 can be generated from a one capital good model. This seems to be in conflict with the conventional claim that the one capital good case is well behaved and yields a monotone consumption–interest rate locus. But there is no contradiction. Suppose we impose on the isoquant maps of $v(x,y)$ the following 'normality' condition: if the marginal rates of transformation of x for y are fixed at any arbitrary value, then the resulting one-dimensional curve is increasing with consumption; a higher consumption level is possible only if there is at least as much current capital and less capital requirement for the next period. With this condition, the assignment of slopes to steady state is monotone (see Fig. 17.4 again) and so will be the set L. A sufficient condition for the normality property is that $v(x,y)$ have the 'quasilinear' form $v(x,y)=f(x)-y$. In this case, the isoquants of v are generated from each other by vertical displacement. The form $v(x,y)=f(x)-y$, i.e. the capital and consumption goods are perfect substitutes, is what is usually understood by the one capital good case and for it we have seen that the claim of well behavedness does hold.

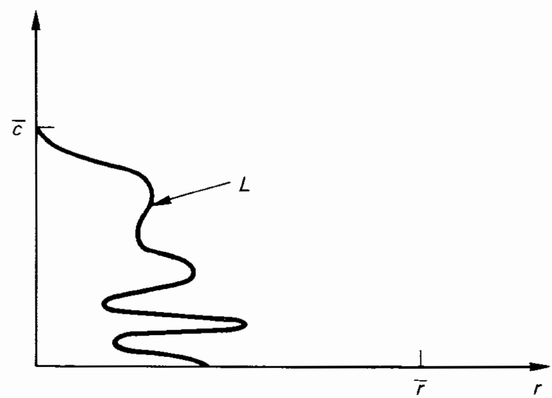

Figure 17.7

4 PROOF OF THE PROPOSITION

The proof consists of six steps. Also, we shall rest content with having $v(x,y)$ continuous and concave. The function can be made continuously differentiable and strictly concave at the cost of two extra steps. Because they are rather messy and not very interesting we skip them.

The technology function $v(x,y)$ is constructed in step 4 after preliminary work in steps 1 to 3. Steps 5 and 6 verify that v has the desired properties. Step 1 constructs v on the steady-state set $\Delta=\{(x,y)\in P\times P\colon x=y\}$. Steps 2 and 3 associate to every steady state an affine function on R^4 which supports v

over Δ. Step 4 then extends v to the entire $P \times P$ by taking the infimum of all the affine functions.

Without loss of generality we take $\bar{c}=1$, $\bar{r}=1$.

Step 1

Take $n=2$.

There is a considerable freedom in choosing the technology function over the steady-state set $\Delta=\{(x,y)\in P \times P\colon x=y\}$. We shall choose one which is particularly convenient for latter constructions.

Let $\beta\colon(-\infty,1]$ be an arbitrary, continuously differentiable, strictly increasing, strictly convex function with

(i) $\beta(0)=0$,

(ii) $\beta(1)=1$ and

(iii) there is a constant $a>0$ such that $\beta'(c)=\dfrac{a}{\alpha(c)}$.

Such a function exists because by the golden rule restriction

$$\int_{-\infty}^{\bar{c}} \frac{1}{\alpha(c)}\,dc<\infty.$$

By putting $\beta^{-1}(t)=1$ for $t>1$ we can view β^{-1} as defined on $[0,\infty)$. (See Figure 17.8.)

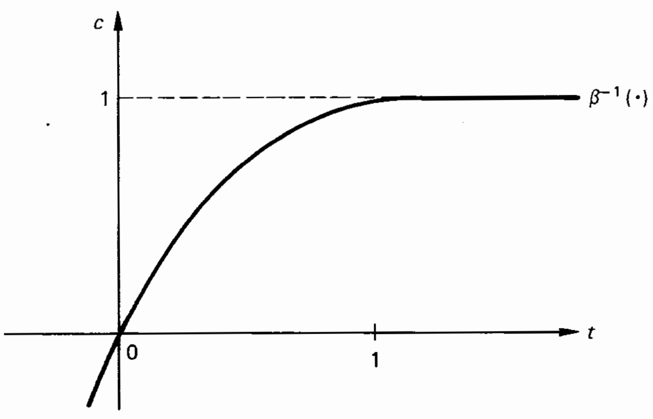

Figure 17.8

Let $e=(1,1)$ and $g\colon R_+^2 \to R_+$ be an increasing concave function with non-vanishing gradient and Hessian determinant. Suppose also that:

(i) $g(e)=1$,

(ii) $g((2,0))<0$, $g((0,2))<0$ and

(iii) there is a constant $h>0$ such that $\frac{1}{h}\|\nabla\partial g(z)\|\leqslant h$ for all $z\geqslant 0$.

Finally, define w: $\Delta\rightarrow(-\infty,1]$ by the order preserving transformation $w(z)=\beta^{-1}(g(z))$. (See Figure 17.9.)

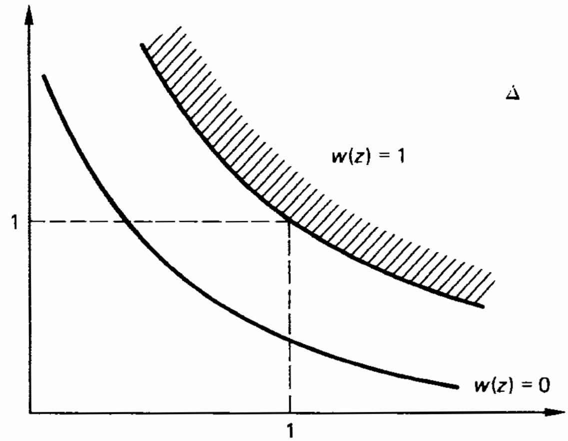

Figure 17.9

Note that for $w(z)<1$ we have

$$\nabla w(z)=\frac{a}{\beta'(w(z))}\nabla g(z)=\alpha(w(z))\nabla g(z).$$

Hence,

$$\frac{a}{h}\alpha(w(z))\leqslant\|\nabla w(z)\|\leqslant ah\alpha(w(z)).$$

Step 2

The task here is to inject L into Δ in such a manner that the injection is one-to-one (note that this could *not* be done if Δ was not at least two dimensional) and the isoconsumption sections of L get mapped into the corresponding isoconsumption curves of w. See Figure 17.10.

Parameterize the region $Q=\{z\in\Delta\colon z\leqslant e, w(z)\geqslant 0\}$ by identifying every $z\in Q$ with $(w(z),\theta(z))$ where $\theta(z)$ is the angle in grades of the vector $z-e$. Obviously $180\leqslant\theta(z)\leqslant 270$.

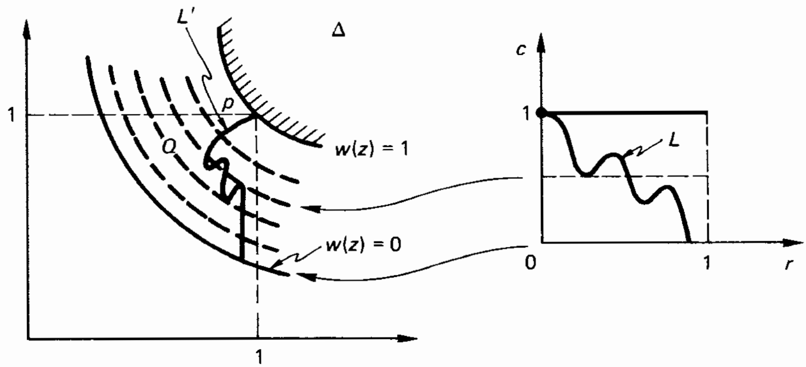

Figure 17.10

To any $(r,c) \in [0,1]^2$ associate the point $z(r,c) \in Q$ given by $(c, 200+50r)$. Note that, indeed, $w(z(c,r))=c$ and that the function $z(\cdot)$ is one-to-one on L. Denote the L' the image of L.

Step 3

We now proceed to associate with every $z \in \Delta$, $0 \leqslant w(z) < 1$, a certain affine function q_z on R^4 majorizing w on Δ. But first we need to pick two auxiliary functions whose existence is a consequence of the closedness of L'.

Let $r\colon \Delta \to [0,1]$ be a continuous function such that:

(i) $\alpha(w(z)) \leqslant r(z) \leqslant k\alpha(w(z))$ for any $z \in \Delta$ and

(ii) if $z \in L'$, then $r(z)$ is the correct rate of interest associated with z, i.e. $z(r(z), w(z)) = z$.

Let $\gamma\colon \Delta \to (-\frac{1}{2}, \frac{1}{2})$ be an arbitrary continuously differentiable function such that, whenever $0 \leqslant w(x) \leqslant 1$, we have $\gamma(z)=0$ if and only if $z \in L'$. In addition, if L (and therefore L') happens to be a smooth one-dimensional, then γ can be chosen so that $\nabla_z \gamma(z) \neq 0$ whenever $\gamma(z)=0$. It is this requirement which guarantees the robustness of our construction in the case that L is a smooth manifold.

Finally, for every z with $0 \leqslant w(z) < 1$, define $q_z\colon R^4 \to R$ by

$$q_z(x,y) = b_z^x \cdot x + b_z^y \cdot y + d_z$$

where:

$$b_z^x = \frac{1}{r(z)} \left([1+\gamma(z)] \frac{\partial w}{\partial z_1}(z),\ \frac{\partial w}{\partial z_2}(z) \right) \gg 0$$

$$b_z^y = -\frac{1-r(z)}{r(z)}\left(\left[1+\frac{\gamma(z)}{1-r(z)}\right]\frac{\partial w}{\partial z_1}(z), \frac{\partial w}{\partial z_2}(z)\right) \leqslant 0$$

$$d_z = w(z) - (b_z^x + b_z^y)\cdot z.$$

Observe that $b_z^x + b_z^y = \nabla w(z)$ and $q_z(z,z) = w(z)$. Therefore, if $x = y = v$, we have $q_z(v,v) = (b_z^x + b_z^y)\cdot v + d_z = \nabla w(z)\cdot(v-z) + w(z)$, i.e. on Δ q_z coincides with the gradient of w at z.

The vectors b_z^x, b_z^y have been designed so that they are proportional if and only if $z \in L'$, in which case $b_z^y = -(1-r(z))b_z^x$.

Finally

$$\|q_z\| = \|(b_z^x, b_z^y)\| \leqslant \frac{ah\alpha(w(z))}{r(z)} + (1+\gamma(z))\frac{ah\alpha(w(z))}{r(z)} \leqslant 3ah$$

and

$$\|q_z\| \geqslant \|b_x^z\| \geqslant \frac{1}{r(z)}\|\nabla w(z)\| \geqslant \frac{a}{h}\frac{\alpha(w(z))}{r(z)} \geqslant \frac{a}{hk}.$$

Therefore, the collection $\{\|q_z\|\colon 0 \leqslant w(z) < 1\}$ is uniformly bounded above and below.

Step 4

Define the technology function v: $P \times P \to R$ by:

$$v(x,y) = \operatorname{Inf}\{q_z(x,y)\colon z \in \Delta,\ 0 \leqslant w(z) < 1\}.$$

By construction, v is concave. Because of the boundedness property claimed at the end of the previous section, $v(x,y) > -\infty$ for all (x,y) and, also, every subgradient vector is non-trivial and has components of the right (weak) sign. We have in addition that $c(z,z) = w(z)$ on Δ.

To conclude the proof we should show, therefore, that a steady state (z,z) admits a supporting rate of interest $r \geqslant 0$ and consumption $c \geqslant 0$ if and only if $(r,c) \in L$. Step 5 (resp., step 6) establishes the if (resp., only if) part.

Step 5

The if part is trivial. Let $(r,c) \in L$. Take $z = z(r,c) \in L'$. Then $c = v(z(r,c))$ and by construction q_z, which supports $v(x,y)$ at (z,z), has

$$b_z^y = -(1-r(z))b_z^x = -(1-r)b_z^x.$$

Step 6

The only if part is slightly more delicate.

The key fact is that for any $0 \leqslant w(z) < 1$, the function $v(x,y)$ has q_z as the *unique* gradient at (z,z). This implies that if a (z,z) with $1 > v(z,z) \geqslant 0$ admits a rate of interest r, then this r can only come from q_z, i.e. we must have $b_z^y = -(1-r)b_z^x$. Hence, $z \in L'$ (otherwise b_z^x, b_z^y are not proportional) and therefore $r = r(z)$, which yields $(r, v(z,z)) \in L$, the desired conclusion. As for $c = 1$, since $w(\ )$ has a maximum value equal to one it follows from the Golden Rule Theorem that $c = 1$ can only be associated with the rate of interest $r = 0$.

We should therefore prove that if $0 \leqslant w(z) < 1$ then q_z is the unique supporting affine function to $v(x,y)$ at (z,z). We argue by contradiction and suppose that there was another one, denoted p.

Take $au \neq 0$ such that

$$p((z,z)+u) < q_z((z,z)+u).$$

Since $p(z,z) = q_z(z,z)$, this implies $p(u) < q_z(u)$.

By definition of v there is a sequence z_m such that

$$q_{z_m}\left((z,z)+\frac{1}{m}u\right) < v\left((z,z)+\frac{1}{m}u\right) + \frac{1}{2^m} \leqslant p\left((z,z)+\frac{1}{m}u\right)$$
$$+\frac{1}{2^m} < q_z\left((z,z)+\frac{1}{m}u\right) + \frac{1}{2^m}.$$

By the strict convexity of w on a neighborhood of z and the boundedness of $\{q_z\!: 0 \leqslant w(x) < 1\}$, we can find $\delta_n \to 0$ such that if $\|z' - z\| > 1/n$, then $q_{z'}((z,z)+\varepsilon v) > q_z((z,z)+\varepsilon v) + \delta_n$ for $\varepsilon < \delta_n$.

Combining the last two paragraphs, we conclude $z_m \to z$.

Rearranging terms and multiplying by m, we get:

$$m(q_{z_m}(z,z) - p(z,z)) + q_{z_m}(u) - p(u) \leqslant \frac{m}{2^m}.$$

The left-hand side goes to zero. The first term of the right-hand side is non-negative because $q_{z_m}(z,z) \geqslant q_z(z,z) = w(z) - p(z,z)$. Therefore, $q_{z_m}(u) \to p(u)$. But $z_m \to z$ implies $q_{z_m} \to q_z$. Therefore, $q_z(u) = p(u)$, which constitutes the desired contradiction.

QED

References

Arrow, K. and F. Hahn (1971) *General Competitive Analysis* (San Francisco: Holden-Day).
Bliss, C. (1975) *Capital Theory and the Distribution of Income* (Amsterdam: North-Holland).
Burmeister, E. (1980) *Capital Theory and Dynamics* (Cambridge: Cambridge Univeristy Press).
Gale, D. (1973) 'On the Theory of Interest', *American Mathematical Monthly*, 80: 8.
Harcourt, G. C. (1972) *Some Cambridge Controversies in the Theory of Capital* (Cambridge: Cambridge University Press).
Malinvaud, E. (1953) 'Capital Accumulation and Efficient Allocation of Resources', *Econometrica*, 21.
Neumann, J. von (1945–6) 'A Model of General Economic Equilibrium', *Review of Economic Studies*, 13.
Robinson, J. (1953–4) 'The Production Functions and the Theory of Capital', *Review of Economic Studies*, 21.
Shafer, W. and H. Sonnenschein (1982) 'Market Demand and Excess Demand Functions', ch.14 in *Handbook of Mathematical Economics*, Vol. II, K. Arrow and M. Intriligator (eds) (Amsterdam: North-Holland).
Weizsäcker, C. C. von (1971) *Steady State Capital Theory* (New York: Springer-Verlag).