Journal of Mathematical Economics 19 (1990) 95–106. North-Holland

A GEOMETRIC APPROACH TO A CLASS OF EQUILIBRIUM EXISTENCE THEOREMS*

M.D. HIRSCH

University of California, Berkeley, CA 94720, USA

M. MAGILL

University of Southern California, Los Angeles, CA 90089, USA

A. MAS-COLELL

Harvard University, Cambridge, MA 02138, USA

Submitted January 1988, accepted March 1989

In this paper we propose a general mathematical approach to existence problems in economics based on the geometry of vector bundles and the methods of intersection theory. The existence problem is formulated as: Does an approximate vector bundle admit a non-zero continuous section? The motivation and major application comes from incomplete market theory, where the appropriate vector bundle has, as one of the components of its base space, a Grassmanian manifold. Several general existence results are offered. One infinite dimensional generalization is included.

1. Introduction

Recent research on the economics of incomplete markets [Magill and Shafer (1985), Duffie and Shafer (1985); Husseini, Lasry and Magill (1990)] has yielded the following surprise: Brouwer's fixed-point theorem (and its variants, e.g. Kakutani's theorem) which traditionally had proved quite sufficient for the existence theory of economic equilibrium is too weak to establish a general existence theorem in the non-classical context of incomplete markets. More powerful theorems are required. These have been

*We are indebted to D. Gale, M.W. Hirsch and S. Husseini for helpful discussions. The paper was first conceived in the summer of 1986 while the second author was at the Laboratoire d'Econometrie de l'Ecole Polytechnique (Paris) and the last author was at MSRI, Berkeley and the Economics Department of the University of California, Berkeley. Both wish to acknowledge the wonderful environment provided by these institutions. Both also acknowledge the financial support of NSF and the latter author of a Guggenheim Fellowship.

provided by Duffie and Shafer (1985), using degree theory, and by Husseini, Lasry and Magill (1990), using the methods of algebraic topology.

In this paper we propose a general mathematical approach to existence problems in economics, based on the geometry of vector bundles and the methods of intersection theory [see Guillemin and Pollack (1974) and M.W. Hirsch (1976)]. More specifically, we suggest formalizing the economic problem via a section of an appropriate vector bundle in such a way that the equilibria correspond to the intersections of this section with the zero section. This is no more than a generalization of the already familiar procedure of formalizing the equilibria as the zeroes of a vector field or a system of equations. Proving the existence of equilibrium then reduces to the geometric question: *Does the vector bundle admit a non-zero continuous section?* For the vector bundles that we consider this question can be answered by the use of simple homotopy and transversality techniques (section 2).

In section 3 we describe briefly the particular economic motivation for this approach which arose from the theory of incomplete markets. In section 4 we apply the vector-bundle-intersection-theory approach to this problem and prove the appropriate fixed-point-like theorem. This result is less general than the theorem of Husseini, Lasry and Magill (1990) but it is sufficient for the economic application and it allows us to replace cohomology theory by intersection theory. It should be noted that the result (and proof) is very close in spirit to the original result of Duffie and Shafer (1985). Our approach in essence distills the purely mathematical facts from the economic problem and reduces it to an abstract mathematical theorem for which we provide a simple geometric proof.

The vector bundle appropriate for the theory of incomplete markets has as one of the components of its base space the *Grassmanian* manifold $G^{k,n}$ of k-dimensional subspaces of R^n. It is the presence of this manifold which distinguishes the incomplete markets problem from the classical complete market theory and which requires mathematical tools stronger than Brouwer's theorem. Section 5 focuses on this aspect by discussing the purest form of the fixed-point-like theorems involving Grassmanians. Section 6 presents an infinite-dimensional generalization of the theorem in section 5.

It is traditional in economic theory to transform the search for solutions of the equations of economic equilibrium into a fixed-point problem on the simplex and then to invoke the fixed-point property. While this approach has proved convenient and indeed fruitful, it has been noted that this transformation is not especially natural [see Smale (1981)]. This observation has special significance for the incomplete markets problem: *in contrast to the simplex, the Grassmanian manifold lacks the fixed-point property* (see section 5). The problem can thus not even be posed as a standard fixed-point problem.

Finally, we mention the paper by Geanakoplos and Shafer (1990) which constitutes a parallel, but distinct, effort at clarification of the mathematical basis of economic equilibrium existence theorems.

2. Vector bundles

Our approach is to reduce the economic existence problem to a certain *geometric* question. The basic geometric object which permits the transformation is a *vector bundle.* Intuitively a vector bundle is a family of vector spaces (the fibers) attached to each point of a manifold which twists in a certain way as the base space is traversed. See M.W. Hirsch (1976) or Bröcker and Jänich (1982) for details.

A vector bundle is specified as a triple $\xi=(E,M,\pi)$ where E is called the *total space* over the m-dimensional manifold M, the *base space*, and $\pi\colon E\to M$ is a continuous function such that for every $x\in M$, $\pi^{-1}(x)=E_x$ is an n-dimensional vector space (called the *fiber* above $x\in M$). The total space E is itself a manifold of dimension $m+n$.

An important example of a vector bundle is the *tangent bundle* $\xi=\tau_M=(TM,M,\pi)$, where M is a smooth m-dimensional submanifold of a Euclidean space R^l, $E=TM=\{(x,v)\in M\times R^l\colon v\in T_xM\}$ is the tangent manifold, π is the natural projection and $E_x=T_xM$ is the tangent space to M at x, for each $x\in M$.

A *section* of the vector bundle ξ is a map $\sigma\colon M\to E$ such that $\sigma(x)\in E_x$ for all $x\in M$. The *zero section* σ_0 assigns the origin of E_x to every $x\in M$. A section σ such that $\sigma(x)\neq\sigma_0(x)$ for all $x\in M$ is called a *non-zero* section. This is also expressed as $\sigma\cap\sigma_0=\emptyset$. Note that a section of the tangent bundle is simply a vector field.

A basic question for a vector bundle is the following: *Does ξ admit a non-zero continuous section?*

If M is a compact smooth manifold and if ξ is a vector bundle whose fiber dimension (n) coincides with the dimension (m) of the base space M (those are maintained hypotheses from now on), then intersection theory can be used to answer the above question. For in this case if $\sigma\colon M\to E$ is a continuous section then the *mod 2 intersection number* of σ and σ_0, which we write as $\#_2(\sigma,\sigma_0)$, is well defined [see Hirsch (1976, pp. 132, 133)]. This number takes only the values 0 and 1. In the case where σ is smooth and transverse to $\sigma_0(\sigma\pitchfork\sigma_0)$ then $\sigma(M)$ and $\sigma_0(M)$ are both n-dimensional manifolds intersecting transversally in the $2n$-dimensional manifold E. Hence the intersection is a finite set and $\#_2(\sigma,\sigma_0)$ is simply the number of points mod 2 in this intersection, i.e., $\#_2(\sigma,\sigma_0)=\#_2(\sigma\cap\sigma_0)=0$ or 1 according as $\sigma(M)\cap\sigma_0(M)$ is even or odd. Clearly $\#_2(\sigma,\sigma_0)\neq0$ implies $\sigma\cap\sigma_0\neq\emptyset$.

There is a remarkably simple method for computing $\#_2(\sigma,\sigma_0)$. It is based on the fact that any two sections σ,σ' of a vector bundle are homotopic (via the linear homotopy $F(x,t)=(1-t)\sigma(x)+t\sigma'(x)$). We spell out the details for three important cases: (i) M is boundaryless, (ii) M has a non-empty boundary and $\xi = \tau_M$, the tangent bundle over M, and (iii) the vector bundle ξ is a (finite) product of vector bundles as in (i) and (ii).

(i) If $\partial M = \emptyset$ then any two homotopic sections have the same mod 2 intersection number with σ_0. Therefore we can associate with the vector bundle ξ a number $e_2(\xi)$, called the *mod 2 Euler number* of ξ such that $\#_2(\sigma, \sigma_0) = e_2(\xi)$ for all continuous σ. Thus if $e_2(\xi) \neq 0$ then there is no non-zero continuous section. Also, in order to show that $e_2(\xi) \neq 0$ it suffices to find a smooth σ such that $\sigma \pitchfork \sigma_0$ and $\#_2(\sigma \cap \sigma_0) = 1$; if σ and σ_0 intersect transversally at a single point then, of course, $e_2(\xi) = 1$.

(ii) Suppose that $\partial M \neq \emptyset$ and $\xi = \tau_M$. Now homotopic sections need not have the same mod 2 intersection number with σ_0. An additional condition is required. We say that a section $\sigma: M \to TM$ is *inward pointing* at the boundary of M if for every $x \in \partial M$ we have $s(x) \cdot g(x) < 0$ where $\sigma(x) = (x, s(x))$ and $g(x) \in T_x M$ is the outward unit normal to ∂M at x. It is then true that if σ and σ' are both inward pointing then $\#_2(\sigma, \sigma_0) = \#_2(\sigma', \sigma_0)$. Thus we can associate with the tangent bundle τ_M a number $e_2(\tau_M)$, called the *mod 2 Euler number* of τ_M, such that $\#_2(\sigma, \sigma_0) = e_2(\tau_M)$ for every continuous and inward pointing σ [see Hirsch (1976, p. 135)]. Thus if $e_2(\tau_M) \neq 0$ then there is no non-zero, continuous, inward pointing section. To show that $e_2(\xi) \neq 0$ it suffices to find a smooth, inward pointing σ such that $\sigma \pitchfork \sigma_0$ and $\#_2(\sigma \pitchfork \sigma_0) = 1$.

(iii) If ξ is a cartesian product of bundles as in (i) and (ii) then ξ has a mod 2 Euler number $e_2(\xi)$ that is simply the product of the corresponding Euler numbers of the component spaces [see Hirsch (1976, p. 135)]. Thus, for example, if $\xi_1 = (E_1, M_1, \pi_1)$, $\partial M_1 = \emptyset$, and $\xi_2 = \tau_{M_2}$ then the product bundle

$$\xi_1 \times \xi_2 = (E_1 \times E_2, M_1 \times M_2, \pi_1 \times \pi_2)$$

has mod 2 Euler number $e_2(\xi_1 \times \xi_2) = e_2(\xi_1) e_2(\xi_2)$. If $e_2(\xi_1 \times \xi_2) \neq 0$ then there is no non-zero, continuous, inward pointing section. Here inward pointing means that if $x_2 \in \partial M_2$ then $s_2(x_1, x_2) \cdot g(x_2) < 0$ where

$$\sigma(x_1, x_2) = (x_1, x_2, s_1(x_1, x_2), s_2(x_1, x_2))$$

and, as above, $g(x_2)$ is the unit outward normal to ∂M_2 at x_2.

3. Motivation: Incomplete markets theory

The analysis of this paper is motivated by the following problem posed by the theory of equilibrium with incomplete markets.

Let $S_+ = \{p \in R^l : p \gg 0 \text{ and } \|p\| = 1\}$ be the positive part of the $l-1$ sphere and denote by $G^{k,n}$ the (Grassmanian) manifold of k-dimensional subspaces of R^n. Here $l \geqq n$ is interpreted as the total number of spot markets, n as the number of states and k as the number of assets. The topology in $G^{k,n}$ is the obvious one. The dimension of $G^{k,n}$ is $k(n-k)$.

The economic situation is described by an excess demand function $z: S_+ \times G^{k,n} \to R^l$ and k asset return functions $f_j: S_+ \to R^n$, $1 \leqq j \leqq k$.

A continuous map $z: S_+ \times G^{k,n} \to R^l$ is an *excess demand function* if:

(i) $p \cdot z(p, L) = 0$ for all $p \in S_+$, $L \in G^{k,n}$ (Walras' Law);
(ii) there is $b \in R^l$ such that $z(S_+ \times G^{k,n}) \geqq b$;
(iii) if $(p_m, L_m) \in S_+ \times G^{k,n}$, $(p_m, L_m) \to (p, L)$ and $p \notin S_+$ then $\|z(p_m, L_m)\| \to \infty$.

An *asset return function* is any continuous function from S_+ to R^n.

Pseudo-equilibrium problem: Given the excess demand function $z: S_+ \times G^{k,n} \to R^l$ and the asset return functions $f_j: S_+ \to R^n$, $1 \leqq j \leqq k$, does there exist $(p, L) \in S_+ \times G^{k,n}$ such that $z(p, L) = 0$ and $f_j(p) \in L$ for every j?

For a true equilibrium we should require that $\{f_1(p), \dots, f_k(p)\}$ spans L. The point is, however, that standard techniques of the theory of regular economies allow one to show that in a certain sense (i.e., if endowments and return matrices can be perturbed) the linear dependence of $\{f_1(p), \dots, f_k(p)\}$ at a pseudo-equilibrium is a non-generic property. Hence a positive answer to the pseudo-equilibrium problem implies the generic existence of equilibrium.

4. Existence of pseudo-equilibrium

In this section we illustrate the power of the vector bundle approach of section 2 by providing a positive solution to the pseudo-equilibrium problem of section 3.

We are given an excess demand function $z: S_+ \times G^{k,n} \to R^l$ and k asset return functions $f_j: S_+ \to R^n$, $1 \leqq j \leqq k$. Let $S = \{p \in R^l : p \geqq 0 \text{ and } \|p\| = 1\}$ denote the non-negative part of the $l-1$ sphere. By a familiar argument z can be modified to a function $\tilde{z}: S \times G^{k,n} \to R^l$ which by properties (i)–(iii) in the definition of the excess demand function z defines a continuous vector field on S which for each $L \in G^{k,n}$ is inward pointing on the boundary of S and has the same zeros as z. For elaborations on similar details see Husseini, Lasry and Magill (1990, appendix) or Mas-Colell (1985, sections 5.5 and 5.6). Without loss of generality we can allow L to be an argument of each function f_j and assume that $f_j: S \times G^{k,n} \to R^n$ is continuous. We now transform the problem of the existence of a pseudo-equilibrium into the problem of non-existence of a non-zero section of a vector bundle.

Consider first the tangent bundle of the non-negative part of the $l-1$

sphere S, namely $\tau_S=(TS,S,\pi_1)$. Secondly consider the vector bundle over the Grassmanian

$$\gamma^{k,n}=(\Gamma^{k,n},G^{k,n},\pi_2)$$

with total space

$$\Gamma^{k,n}=\{(L,v_1,\dots,v_k)\in G^{k,n}\times R^{kn}: v_j\in L^\perp \text{ for every } 1\leqq j\leqq k\},$$

π_2 being the natural projection. Each fiber $\pi_2^{-1}(L)$ is the product of k copies of $L^\perp$ and thus has dimension $k(n-k)$, which coincides with the dimension of the base space $G^{k,n}$. Finally, form the cartesian product bundle $\xi=\tau_S\times\gamma^{k,n}$. Define a section σ of ξ as follows:

$$\sigma(p,L)=(p,L,z(p,L),P_{L^\perp}f_1(p,L),\dots,P_{L^\perp}f_k(p,L)),$$

where $P_{L^\perp}$ is the perpendicular projection map onto $L^\perp$, This is a section because $P_{L^\perp}f_j(p,L)\in L^\perp$ for every j and $z(p,L)\in T_pS$ by Walras' law. Clearly, $\sigma(p,L)=\sigma_0(p,L)$ if and only if (p,L) is a pseudo-equilibrium. Therefore, the pseudo-equilibrium problem reduces to the geometric question: *Can ξ admit a non-zero continuous section?*

To show that there is no non-zero, inward-pointing section it suffices to establish that (i) $e_2(\tau_S)\neq 0$ and (ii) $e_2(\gamma^{k,n})\neq 0$. We do this in turn.

(i) The fact that $e_2(\tau_S)\neq 0$ is well known. Recall that S is homeomorphic to the $(l-1)$ unit ball. Hence $e_2(\tau_S)\neq 0$ simply follows from the fact that any inward-pointing vector field on the ball has a zero. This is in essence Brouwer's fixed-point theorem [see Dierker (1974), or Mas-Colell (1985, sections 5.5 and 5.6)]. To prove the result pick any $\bar p\in S_+$ and consider the inward-pointing section $\bar\sigma$ of τ_S defined by $\bar\sigma(p)=(p,(1/p\cdot\bar p)\bar p-p)$. Then $\bar\sigma \pitchfork \sigma_0$ and $\bar\sigma(p)=\sigma_0(p)$ if and only if $p=\bar p$. Hence $e_2(\tau_S)=\#_2(\bar\sigma,\sigma_0)=1$.
(ii) Pick any $\bar L\in G^{k,n}$ and let $u_1,\dots,u_k$ denote k orthonormal vectors in R^n such that $\bar L=\operatorname{span}\{u_1,\dots,u_k\}$. Consider the section $\bar\sigma$ of $\gamma^{k,n}$ defined by $\bar\sigma(L)=(L,P_{L^\perp}u_1,\dots,P_{L^\perp}u_k)$. Clearly $\bar\sigma(L)=\sigma_0(L)$ if and only if $L=\bar L$.

It remains to show that $\bar\sigma\pitchfork\sigma_0$. Consider the neighborhood in $G^{k,n}$ about $\bar L$ defined by $U_{\bar L}=\{L\in G^{k,n}: L\oplus\bar L^\perp=R^n\}$. Let $\mathscr{L}(\bar L,\bar L^\perp)$ denote the vector space of linear transformations from $\bar L$ to $\bar L^\perp$. For each $L\in U_{\bar L}$ there exists $A\in\mathscr{L}(\bar L,\bar L^\perp)$ such that L is the graph of A. Thus we can write $\bar\sigma(A)=(A,\bar\sigma_1(A),\dots,\bar\sigma_k(A))$. By definition $\bar\sigma_i(A)=u_i-w_i$ where $w_i=\lambda_i(u_i+Au_i)$. Thus

$$\cos\alpha_i=\frac{\|u_i\|}{\|u_i+Au_i\|}=\frac{\|w_i\|}{\|u_i\|}\quad\text{(see fig. 1) implies}\quad \lambda_i=\frac{1}{\|u_i+Au_i\|^2}$$

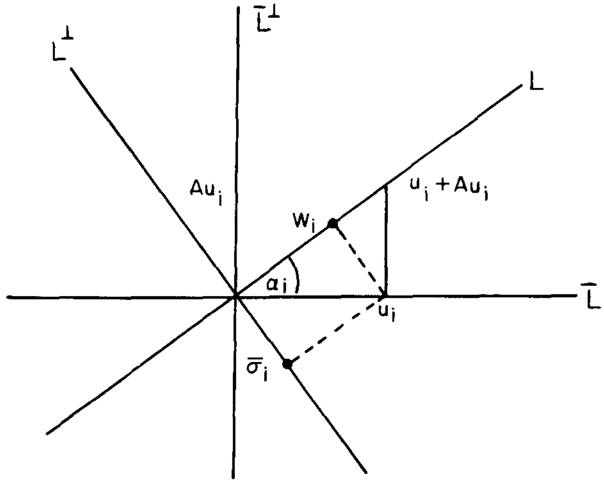

Fig. 1

$$\text{so that}\quad \bar{\sigma}_i(A) = u_i - \frac{(u_i + Au_i)}{\|u_i + Au_i\|^2}, \qquad i = 1, \dots, k.$$

For each $A \in \mathscr{L}(\bar{L}, \bar{L}^{\perp})$ the directional derivative of $\bar{\sigma}_i$ in the direction A is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\sigma}_i(tA)\bigg|_{t=0} = -Au_i,$$

where $(Au_i, u_i) = 0$ since $u_i \in \bar{L}$, $Au_i \in \bar{L}^{\perp}$. Thus $\bar{\sigma} \pitchfork \sigma_0$ and $e_2(\gamma^{k,n}) = 1$.

Part (i) is just the mathematical tool needed to prove existence with complete markets. This is the classical theory. Part (ii), a fixed-point-like result on Grassmanians, is the new result required by the incompleteness of markets. Because this is not familiar it will be discussed in more detail in the next section. But first we summarize:

Theorem 1. *Let $z: S \times G^{k,n} \to R^l$ be a continuous function which, for every $L \in G^{k,n}$, defines a vector field on S which is inward pointing on the boundary of S and let $f_j: S \times G^{k,n} \to R^n$ be continuous functions, $1 \leq j \leq k$. Then there exists $(p, L) \in S_+ \times G^{k,n}$ such that $z(p, L) = 0$, $f_j(p, L) \in L$, for every j.*

Corollary 1. *Under the conditions of section 3 a pseudo-equilibrium exists.*

5. A fixed-point-like theorem on subspaces

As we have seen the property $e_2(\gamma^{k,n}) \neq 0$ yields the following result:

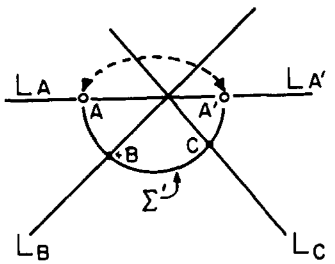

Fig. 2(a)

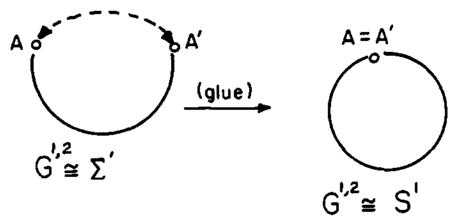

Fig. 2(b)

Theorem 2. *Let* $f_j: G^{k,n} \to R^n$, $1 \leqq j \leqq k$, *be continuous functions. Then there is* $L \in G^{k,n}$ *such that* $f_j(L) \in L$ *for every* j.

Note that this is not a fixed-point theorem for maps of $G^{k,n}$ into itself. For example, for $k=1, n=2$, $G^{k,n}$ is just the one-dimensional real projective space (the manifold of all lines passing through the origin). This space does not have the fixed-point property (just consider a rotation). An implication of Theorem 2 is that a map $F: G^{k,n} \to G^{k,n}$ will have a fixed point if it is possible to find k continuous function $f_j: G^{k,n} \to R^n$, $1 \leqq j \leqq k$, such that for every $L \in G^{k,n}$ the set $\{f_1(L), \ldots, f_k(L)\}$ constitutes a base for $F(L)$.

As already indicated, Theorem 2 or the equivalent property $e_2(\gamma^{k,n}) \neq 0$, is the key new tool underlying the existence theorem for pseudo-equilibria. It will be useful to discuss the geometry of the result in the simplest case of the one-dimensional real projective space, i.e., when $k=1, n=2$.

One way of representing $G^{1,2}$ is by the semicircle Σ^1 in figs. 2(a), (b). Each point $L \in G^{1,2}$ corresponds to a point in Σ^1: L_A, L_B, L_C correspond to A, B, C in Σ^1. Since the same line L_A passes through A and A' in Σ^1, we identify A and A'. If we actually glue A to A' then we obtain an equivalent representation of $G^{1,2}$ as the circle S^1 [fig. 2(b)].

Let us determine the total space of the vector bundle $\gamma^{1,2}$. Recall that $\gamma^{1,2} = \{(L, v) \in G^{1,2} \times R^2 : v \in L^{\perp}\}$ so that over each point L_A, L_B, L_C in $G^{1,2}$, i.e., over each point A, B, C in Σ^1, we have the fiber $L_A^{\perp}, L_B^{\perp}, L_C^{\perp}$. Referring to fig. 3 it is clear that since $L_A = L_{A'}$ the fiber must rotate (twist) by 180° in order that the vector $\bar{v} \in L_A^{\perp}$ coincides with $\bar{v} \in L_{A'}^{\perp}$. If we now glue A and A' together so that $G^{1,2}$ becomes the circle S^1 then the total space $\Gamma^{1,2}$ over S^1 becomes the *open Mobius band* $\mathscr{M}$ in fig. 4. It is the 180° twist in the fiber as we move once completely around the base which makes the mod 2 Euler number of $\gamma^{1,2}$ non-zero. Indeed, because of this twist no continuous section of $\gamma^{1,2}$ can ever be pulled apart from the zero section. Note that in this case the theorem is a consequence of the intermediate value theorem: We have only to cut the total space at some $\hat{L} \in G^{1,2}$ and then *untwist* $\Gamma^{1,2}$ to obtain fig. 5.

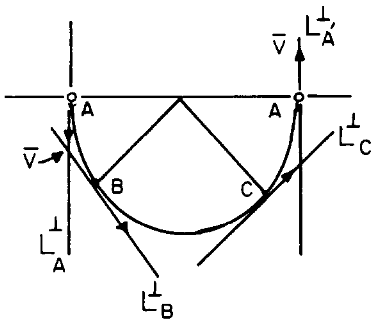

Fig. 3

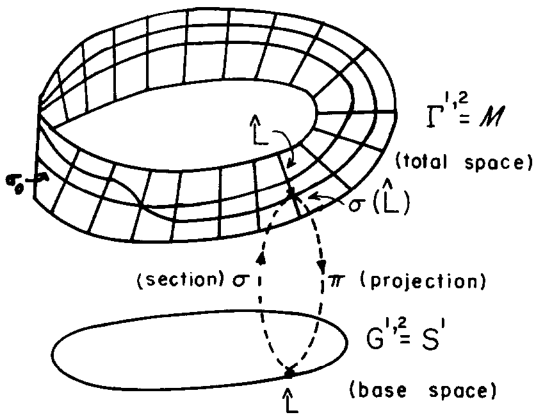

Fig. 4

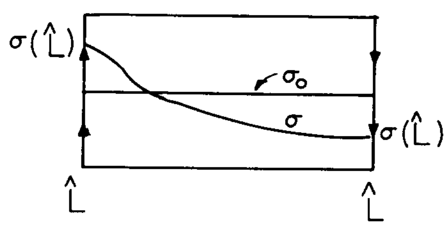

Fig. 5

To illustrate the power of theorem 2, we show that it admits as immediate corollaries the Brouwer's fixed-point theorem and the Borsuk–Ulam theorem. Since the latter is a more advanced result than the former, this also shows that Theorem 2 is strictly stronger than Brouwer's.

Corollary 2 (*Brouwer fixed-point theorem*). *Any continuous map f from the $(n-1)$-unit simplex Δ to itself has a fixed point.*

Proof. Denote $E=\{x\in R^n: \sum_i x^i=1\}$. Take an arbitrary vector $z\in\Delta\subset E$ and let $g: E\to\Delta$ be a continuous function with the properties that $g|\Delta$ is the identity and $\{x\in E: g(x)\neq z)$ is bounded. Such a function can easily be constructed.

Define now $\Psi: G^{1,n}\to\Delta$ by $\Psi(L)=f(g(L\cap E))$ if $L\cap E\neq\emptyset$ and $\Psi(L)=z$ otherwise. This function is continuous. Hence, by Theorem 1 there is $L\in G^{1,n}$ with $\Psi(L)\in L$. Because $\Psi(L)\in\Delta$ this means that $L\cap\Delta=\{\Psi(L)\}$. Therefore

$$\Psi(L)=f(g(L\cap E))=f(g(L\cap\Delta))=f(g(\Psi(L)))=f(\Psi(L)).$$

Hence $\Psi(L)$ provides the desired fixed point. □

Corollary 3 (Borsuk–Ulam). If $\Psi: S^{n-1}\to R^{n-1}$ is a continuous function satisfying $\Psi(-x)=-\Psi(x)$ for all $x\in S^{n-1}$ then there is $x\in S^{n-1}$ such that $\Psi(x)=0$.

Proof. Consider $G^{n-1,n}$ and the $n-1$ functions $f_j: G^{n-1,n}\to R^n$ defined by $f_j(L)=\Psi_j(x_L)x_L$ where x_L is a unit vector perpendicular to L. Note that because $\Psi_j(x_L)x_L=-\Psi_j(-x_L)x_L$ the functions f_i are well defined and continuous. By Theorem 2 there is $L\in G^{n-1,n}$ such that $f_j(L)\in L$ for all j. But this means that $\Psi_j(x_L)=0$ for all j, i.e., $\Psi(x_L)=0$. □

In Husseini, Lasry and Magill (1990) it is shown that Theorem 2 is equivalent to the following generalization of the Borsuk–Ulam theorem, established earlier by them. Let $O^{k,n}$ denote the *Stiefel manifold* consisting of k orthogonal vectors in R^n and let O_k denote the orthogonal group of $k\times k$ matrices. Then the quotient space $O^{k,n}/O_k$ is homeomorphic to $G^{k,n}$ and we have:

Theorem 2′. If $\Psi: O^{k,n}\to(R^k)^{n-k}$ is a continuous function satisfying $\Psi(Ax)=A\Psi(x)$ for all $x\in O^{k,n}$ and $A\in O_k$ then there exists $x\in O^{k,n}$ such that $\Psi(x)=0$.

The more general results of Husseini, Lasry and Magill (1990) can also be established by using the methodology proposed in this paper. This is done in Hirsch and Magill (1987).

6. An infinite-dimensional generalization

In this section we show how Theorem 2 can be generalized to an infinite-dimensional case.

Consider a Banach space V. We assume that V is the dual of a separable Banach space and endow V with the weak-star topology. Let $B=\{x\in V: \|x\|\leq 1\}$ denote the unit ball in V. Then B is a compact metric space. Let d denote a metric and let $\mathscr{C}(B)$ be the (compact) space of non-empty closed subsets of B endowed with the Hausdorff metric induced by d.

Denote by $\tilde{G}^k(V)$ the space of linear subspaces of V of dimension *less than or equal* to k. Identifying every L with $L \cap B$ we can view $\tilde{G}^k(V)$ as a subset of $\mathscr{C}(B)$. It is easy to see that as such it is closed, hence compact.

Theorem 3. *Let $f_j: \tilde{G}^k(V) \to V$, $1 \leqq j \leqq k$, be continuous functions. Then there is $L \in \tilde{G}^k(V)$ such that $f_j(L) \in L$ for every j.*

Proof. Let $K \subset V$ be a compact set such that $f_j(\tilde{G}^k(V)) \subset K$ for every j. Since K is compact for every n there exists

$$\{x_1, \ldots, x_{m(n)}\} \quad \text{such that} \quad K \subset \bigcup_{i=1}^{m(n)} (\{x_i\} + (1/n)B).$$

Let $\alpha_j: K \to [0,1]$, $j = 1, \ldots, m(n)$, be a partition of unity corresponding to this covering and denote by E_n the linear subspace spanned by $\{x_1, \ldots, x_{m(n)}\}$. We can assume that $\dim E_n \geqq k$ for all n. Consider now the map $\beta_n: K \to E_n$ defined by $\beta_n(x) = \sum_{i=1}^{m(n)} \alpha_i(x) x_i$. Then $\|\beta_n(x) - x\| \leqq (1/n)$ for all $x \in K$. Define $f_j^n: \tilde{G}^k(V) \to E_n$ by $f_j^n(L) = \beta_n(f_j(L))$. By Theorem 2 there exists $L_n \in \tilde{G}^k(V)$ such that $f_j^n(L_n) \in L_n$ for all j. Because $\tilde{G}^k(V)$ is compact we can assume that $L_n \to L$. But then we also have $f_j^n(L_n) \to f_j(L)$ for all j. Hence L is as desired. □

Note that a difference between Theorems 3 and 2 is that we now need to include the subspaces of dimension lower than k. This is to insure compactness. In the infinite-dimensional case it is possible for a sequence of k-dimensional subspaces to converge to a subspace of lower dimension. For example, if we consider a sequence $\{x_n\}$ lying in the unit ball of V and such that $x_n \neq 0$ but $x_n \to 0$ weakly then the sequence of one-dimensional subspaces spanned by x_n converges to $\{0\}$.

Theorem 3 should be a basic tool in order to obtain a pseudo-equilibrium existence theorem for economies with infinitely many commodities and a finite number of assets. It is only fair to point out, however, that in the infinite-dimensional case going from pseudo-equilibrium to equilibrium becomes more problematical than in the finite-dimensional case.

References

Bröcker, T. and K. Jänich, Introduction to differential topology (Cambridge University Press, Cambridge).

Dierker, E., 1974, Topological methods in Walrasian economics, Lecture notes in economics and mathematical systems, no. 92 (Springer-Verlag, Berlin).

Duffie, D. and W. Shafer, 1985, Equilibrium in incomplete markets: I. A basic model of generic existence, Journal of Mathematical Economics 14, 285–300.

Geanakoplos, J. and W. Shafer, 1990, Solving systems of simultaneous equations in economics, Journal of Mathematical Economics 19, this issue.

Guillemin, V. and A. Pollack, 1974, Differential topology (Prentice-Hall, Englewood Cliffs, NJ).

Hirsch, M.W., 1976, Differential topology (Springer-Verlag, New York).

Hirsch, M.D. and M. Magill, 1987, A generalization of the Borsuk–Ulam theorem, Manuscript.

Husseini, S.Y., J.-M. Lasry and M.J.P. Magill, 1990, Existence of equilibrium with incomplete markets, Journal of Mathematical Economics 19, this issue.

Magill, M.J.P. and W. Shafer, 1985, Equilibrium and efficiency in a canonical asset market model, MRG working paper (University of Southern California, Los Angeles, CA).

Mas-Colell, A., 1985, The theory of general economic equilibrium – A differentiable approach (Cambridge University Press, Cambridge, MA).

Smale, S., 1981, Global analysis and economics, in: K. Arrow and M. Intriligator, eds., Handbook of mathematical economics (North-Holland, Amsterdam).