CHAPTER 8

Infinite-dimensional equilibrium theory: discussion of Jones*

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There is little I can possibly add to the thorough account of the current state of equilibrium theory with infinitely many commodities that has been given in the chapter of L. Jones. But since the duty of the commentator is to offer comments I will oblige by touching on two points. The first concerns a contrast between the analytical treatment of exchange and of production economies, the second refers to the difficulties to extend the theory to environments where the first fundamental theorem fails and, in particular, to incomplete markets economies.

But before plunging into my specific comments perhaps I will be allowed a word on the purposes of the theory. The following observation is evident enough but it nonetheless bears repeating. While from a purely formal point of view an equilibrium model with infinitely many commodities includes, and therefore generalizes, the standard model with finitely many commodities, the justification of the theory is not generality but, on the contrary, concreteness. If all we wanted to know is, say, that an equilibrium exists we would do as well with the finite number of commodities model. After all the number of commodities can be arbitrarily large and since large can be very large indeed there can be no loss of substance in doing so. The problem, and the *raison d'être* of the infinite dimensional approach, is that proceeding in this way we neglect any preexisting regularities among the set of commodities (e.g., neighboring relations) and treat it as a finite, but otherwise undifferentiated, set. A considerable amount of information (on the structure of equilibrium prices, for example) is lost on the way. The point of view of infinite dimensional theory is that if there is more than a few commodities (I will not now commit myself to attach a number to this "few") it may altogether be more instructive to look at the infinite limit and proceed by deriving properties of equilibrium prices (e.g., summability, continuity in characteristics, stationarity, etc.) that, in the first place, exploit features of preferences, endowments, and technology best formalized in the limit case and, in the second place, would remain informative for the finite approximation if only one had the patience to carry out the ε,δ exercise. In summary: the infinite dimensional approach requires a heavier mathematical input than its finite counterpart. But it also gets answers to conceptually sharper questions.

1 EXCHANGE VERSUS PRODUCTION ECONOMIES

One of the reasons that so much of equilibrium theory has traditionally concentrated on the examination of exchange economies is that the incorporation of production (at least in the manner done by general equilibrium theory) presents no particular difficulty. Or perhaps it would be more precise to say that this is so once a few standard tricks have been learned.

It is possible that this will also be the eventual picture with infinite dimensional equilibrium theory. But for the sake of a discussion let me argue that exchange economies have one feature that from an infinite dimensional viewpoint makes them very particular. This feature is that the entire set of feasible allocations is contained in an order interval $\{(x_1, \ldots, x_N): 0 \leq x_i \leq \omega, \text{ all } i\}$, i.e., in a "box." The existence of an upper bound to the set of feasible allocations is given to us (by the initial endowments) in the exchange case but it may easily fail with production, except of course in a finite dimensional world where the boundedness of the feasible set does automatically imply that we can contain it in a box.

The order boundedness of the feasible set facilitates the analysis in at least three (related) respects.

The first is that it allows us to invoke the mathematics of vector lattices (or Riesz spaces) which seem, therefore, ideally fitted for the study of exchange economies. The order structure did not play any major role in the technical development of the finite dimensional theory. The reason is that the full power of the theory could be displayed by using only convexity structures (see, e.g., Debreu, 1962). To the extent that this does not appear to be so in general it has been most convenient to have at hand an alternative, and effective, tool. I refer to Aliprantis, Brown, and Burkinshaw (1989) that in all certainty will become the standard reference in economics for Riesz space applications.

The second respect is more specific. Knowing that the feasible set is contained in an order interval helps to establish its compactness for appropriate topologies. Some form of compactness is essential if, for example, we hope to prove existence.

The third respect is more conceptual and probably more important. The order boundedness property allows the neat separation of the equilibrium problem into two parts. First one searches for prices defined on the ideal generated by the total endowments. This is the program effectively pursued by Aliprantis, Brown, and Burkinshaw (1989). It turns out that the problem is then quite parallel to the classical contribution of Bewley (1972) (it can in fact be argued that the commodity space of Bewley is implictly nothing but the ideal of a larger universal commodity space). Second one then attempts to extend prices from the ideal to the, possibly much larger, entire space. I do not now want to discuss to what extent it is important to actually carry out the second step or if we can rest content with pricing any commodity bundle that could conceivably be marketed given the total endowments. I want to emphasize simply that in the exchange case the possibility for this clean separation in two steps exists and it is most useful.

2 DIFFICULTIES IN NON-CLASSICAL SET-UPS: INCOMPLETE MARKETS

The chapter by Larry Jones makes it patently clear that the extension of the classical Arrow–Debreu–McKenzie model to an infinite dimensional setting has been essentially achieved and that the mathematical techniques of functional analysis have proven to be tools extraordinarily effective for the task.

Equilibrium theory, however, is not exhausted by the classical model. Its practical use has required the consideration of many departures and the incorporation of many forms of "imperfections" and of market failure. Thus, there are equilibrium models with taxes, with externalities, without complete markets, with imperfect competition, etc. It would be comforting if we could assert that once the infinite-dimensional extension of the classical model is well understood the extension of the non-classical theory presents no particular difficulty. Unfortunately, this is not so. The point has already been made in Jones' chapter with reference to an equilibrium problem with externalities. I will discuss it further in the context of a model with incomplete markets. The issues raised are similar. In fact, they are likely to be similar in any situation where the first welfare theorem breaks down.

For concreteness take the simplest infinite dimensional incomplete market model (see the chapters of D. Cass and of D. Duffie). There is a set of states $S=[0,1]$. At each $s\in S$ there are spot markets for $l+1$ commodities (having $l>0$ is a way to capture some of the phenomena that can occur with more than two periods. We take the first commodity to be a numeraire). Final allocations $x=(x_1, \ldots, x_N), x_i \in (L_\infty^+([0,1])^{l+1}$ and price functionals p are determined from utility functions $u_i(\cdot)$ and endowments ω_i in the usual way. The key restriction, reflecting the incompleteness of markets, is that there is a certain linear space (even possibly finite dimensional) of feasible net wealth transfers $L \subset L_\infty([0,1])$, that is, every individual optimization problem is subject to the constraint: "the function $s \mapsto p(s)\cdot(x_i(s) - \omega_i(s))$ belongs to L." This is a highly reduced description. It corresponds to the real numeraire asset case of Geanokoplos–Polemarchakis (1986) (the spot price of the numeraire being fixed to one).

For a state space S of finite cardinality a more general version of the above model was first formulated and studied by Radner (1972) who gave sufficient conditions for the existence of equilibrium prices p. He established existence by using the standard fixed point techniques of the classical theory. We will now argue, by means of two observations, that an attempt to follow a parallel approach for the case $S = [0,1]$ and to apply the functional analytical techniques of, say, Bewley (1972) or Aliprantis, Brown, and Burkinshaw (1989), to the existence problem will not work.

The first observation, already made in Mas-Colell and Zame (1988), has to do with the joint continuity of the evaluation maps. Typically, in attacking the problem along the suggested functional analytical lines one finds oneself at some stage with a double sequence (or net) x_n, p_n which, by appealing to suitably weak topologies, can be assumed to converge $x_n \to x, p_n \to p$. Moreover, by the nature of the sequences, the limits x,p would constitute an equilibrium if only $p_n \cdot x_{ni} \to p \cdot x_i$ for every i. There is no general mathematical reason for this joint continuity to hold for topologies weak enough to guarantee the existence of limits. But, as it was first shown by Bewley (1972), we may be rescued by the fact that the sequences x_n, p_n will not be arbitrary. In the first place, x_n will be alocations, that is $\Sigma_i x_{ni} = \Sigma_i \omega_i \equiv \omega$. Because the right-hand side does not depend on n this can be made to yield $\Sigma_i p_n \cdot x_{ni} \to p \cdot \omega = \Sigma_i p \cdot x_i$. In the second place, p_n will support, in an appropriate sense, the preferred sets to x_{ni}. To be specific, by an upper semicontinuity property on utility which is normally satisfied we will have $u_i(x_i + \varepsilon\omega) > u_i(x_{ni})$ for n large enough. If from this we could conclude that $p_n \cdot (x_i + \varepsilon\omega) \geq p_n \cdot x_{ni}$ we would be done since then $\limsup p_n \cdot x_{ni} \leq p \cdot x_i$ for every i and therefore $p_n \cdot x_{ni} \to p \cdot x_i$. The supporting property $p_n \cdot (x_i + \varepsilon\omega) \geq p_n \cdot x_{ni}$ obtains if markets are complete but may well fail if markets are incomplete because the net wealth trade function $s \mapsto p_n(s)\cdot(x_i(s) - (1-\varepsilon)\omega_i(s))$ may not belong to L (compare with the externality model in Jones' chapter).

The second observation goes more to the heart of the matter because it shows that the joint continuity problem cannot be easily sidestepped. It has substantive implications. Indeed we will show that in the incomplete case with state space [0,1] the equilibrium price set fails to be sequentially compact in the same sense that it is in the complete market case (the same remark can be made for the upper hemicontinuity of the equilibrium correspondence). Consider the following trivial example. Preferences (which are of the expected utility type) and endowments are state independent. Markets are as incomplete as they can be, i.e., $L(p)=\{0\}$ at all p. Suppose that spot markets clear at any of three (normalized) linearly independent price vectors $p',p'',p'''\in R_+^{l+1}$. Then an equilibrium of the incomplete market model is any price function $p:[0,1]\to\{p',p'',p'''\}$. There is no sense in which this set of equilibrium functions can be sequentially compact. Let p_n be the function such that $p_n(s)=p'$ (resp., $p_n(s)=p''$) if $\frac{j}{2n}\leq s\leq\frac{j+1}{2n}$ and j is even (resp. odd). Then, depending on the topology, either p_n has no accumulation point or if it has a limit function it cannot be other than $p(s)=\frac{1}{2}p'+\frac{1}{2}p''$, which is not an equilibrium price function. It should be evident that this spells trouble for any brute force attempt to prove the existence of an equilibrium by taking limits from finite state space approximations. Note that things are very different in the complete case. In the example the only equilibrium price functions would be the three constant functions. Thus the equilibrium price set, being finite, is necessarily compact.

What to do? One possibility is to recover the finite approximation techniques (more generally, the upper hemicontinuity of the equilibrium correspondence). There are at least two ways to do so. The first is to restrict the analysis to a countable state space. Existence theorems have then been obtained by Green and Spear (1987) and Zame (1988). The second is to extend the notion of equilibrium by allowing random prices (and allocations). We would now require as equilibrium conditions on p,x that for every s, $p(s),x(s)$ be a spot price equilibrium with probability one and that given the price expectations p,x_i be the random variable (jointly distributed with p) induced by utility maximization subject to the constraint: "for every s, $p(s)\cdot x(s)$ is non-stochastic (this is because asset trade comes before the realization of the random price) and the function $s\mapsto p(s)\cdot(x_i(s)-\omega_i(s))$ belongs to L." With this approach if we look again at the example of the previous paragraph then we see that the existence of equilibrium is restored (at any state randomize with equal weight between p' and p''). This is quite general. It is well known (at least since Hart, Hildenbrand, and Kohlberg, 1974) that equilibrium defined as distributions is very resistant to taking limits and it is thus more than likely that a distribution approach applied to the current problem will allow a comparatively simple extension of the Radner style proofs. However, this does not seem to have been done to date (see Duffie, Geanakoplos, Mas-Colell, and McLennan, 1988, for an application of distribution-like ideas to a related problem).

There are limitations to the two suggestions. Non-discrete, uncountable state spaces are non-pathological and are common in applications (e.g., in finance). As for the introduction of extraneous noise (sunspots after all) it is of course interesting to verify once more that it emerges "endogenously" as a natural closure of the equilibrium set. But one should realize that it brings with it a host of new problems. One is indeterminacy (the following is an informal conjecture: for the "typical" case either the model does not admit any equilibrium with random prices or there is a continuum of them). Another is suboptimality. Even if the underlying model is complete random prices can be self-sustaining and consequently Pareto optimality is not reached.

But perhaps the most serious limitation is that neither restrictions on the state space nor extensions of the equilibrium concept should be really necessary. It is the mathematical machinery, not the validity of the theory, that fails on us. It is in fact the strength of the functional analytical approach that proves at the end to be its shortcoming. That strength is abstraction. The time and uncertainty structures of the specific problems (the recursivities in time or the additivity properties implied by expected utility behavior) are all excised from models that once reduced to its bare essentials and free of "cluttering detail," are dealt with in a full blow. This works beautifully when applied to the classical Arrow–Debreu–McKenzie model. But, as we have seen, it does not in non-classical worlds. The way to proceed is clear. We should abstract less (or abstract differently). In particular, for the incomplete market existence problem it can be supposed that a positive solution will be obtained by not just remembering but actually building on time recursivities (broadly defined, no suggestion of stationarity) and the expected utility hypothesis. We will know in the next few years.

Note

* I thank L. Jones for his comments.

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