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A New Approach to the Existence of Equilibria in Vector Lattices*

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Motivated by recent work of Huang and Kreps it is shown that the existence of a general equilibrium in an exchange economy whose commodity space is a vector lattice is guaranteed if (in addition to standard hypotheses) the price space is also a lattice. This is very weak in contrast with the usual requirement that the topology of the commodity space be locally solid (i.e., continuity of the lattice operations), a condition violated in models of Jones and of Huang and Kreps. The key to our proof is a disaggregated approach to the construction of supporting prices. *Journal of Economic Literature* Classification Numbers: 021, 022.

1. INTRODUCTION

In a recent and quite stimulating paper C. Huang and D. Kreps [8] have proposed a new model of intertemporal preferences designed to capture the possibility of lumpy consumption. Huang and Kreps subjected their model to several soundness tests but one issue they did not explore is to what extent their model lends itself to establishing the existence of a general equilibrium.

It turns out that the Huang–Kreps model is not covered by available equilibrium existence theory for an interesting reason. All the current

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results (e.g., Mas-Colell [14]; see Aliprantis, Brown, and Burkinshaw [2] and Mas-Colell and Zame [15] for general overviews) require that the commodity space be a vector lattice (Huang and Kreps's is) which moreover satisfies the condition that the lattice operations be (uniformly) continuous, i.e., that the space be a *topological* vector lattice or, equivalently, that the topology be locally solid. Huang and Kreps' topologies are not locally solid.

We were certainly aware that the continuity of the lattice operations was not a minor hypothesis. But in fact most of the infinite dimensional models that had arisen in applications could be treated by the topological vector lattice theory. The exception was L. Jones' [9] commodity differentiation model (see also Mas-Colell [12]), to which Huang and Kreps is closely related, but even there the general methods were partially useful and at any rate Jones had succeeded in proving a general existence theorem by space specific arguments. So we had left well enough alone. It is the Huang and Kreps model that has convinced us that well enough is not good enough and that a reexamination of the basic existence theorem in warranted. Perhaps surprisingly, given the mathematical centrality of the concept of locally-solid topologies, it turns out that this topological requirement can be largely dispensed with.

Specifically what we do in this paper is to free the existence theorem in Mas-Colell [14] from the requirement that the topology be locally solid. Only a minor restriction is retained: *we require that the price space itself be a lattice.* (Informally, that the pointwise maximum of two price functions be an admissible price function.) A well-known example by Jones establishes that this is indispensable. But it is a weak condition satisfied both by Jones' model and by Huang and Kreps' [8], which does thus pass the existence test. In every other respect the hypotheses of this paper are as in Mas-Colell [14]. Although the general strategy of our proof is as in the latter paper, it goes without saying that the essential steps (construction of supporting prices to weak optima and verification of upper hemicontinuity and convex valuedness properties) are very different. Nonetheless we still rely heavily on lattice theoretic arguments on the space and, a novelty of this paper, its dual.

Section 2 states the result. Section 3 is devoted to comment. The proof is in Section 4.

2. Model and Main Result

We consider economies in which the commodity space, L, is a vector lattice (or Riesz space) endowed with a Hausdorff, locally convex topology τ. We assume that the positive cone of the space, $L_+ = \{x \in L : x \geqslant 0\}$, is

τ-closed. Note that we do not require that L be a topological vector lattice; i.e., it is not necessary that the order operations be uniformly continuous. (See Aliprantis and Burkinshaw [1].)

We require that prices be chosen in the topological dual of L, denoted by L^*, which is the set of continuous linear functionals on L. Furthermore, we assume that for any $p, q \in L^*$ the functionals $p \vee q$, $p \wedge q$ are continuous. More precisely, with the positive cone $L^*_+ = \{q \in L: q \cdot z \geqslant 0$ for all $z \geqslant 0\}$, we require that L^* be a sublattice of the order dual. In particular, for $p, q \in L^*$ and $z \geqslant 0$, we have $(p \vee q)(z) = \sup\{p \cdot z_1 + q \cdot z_2 : z_1 + z_2 = z,$ $z_1, \ z_2 \geqslant 0\}$. (See Aliprantis and Burkinshaw [1].) Typically, if p, q are functions then $p \vee q$ is simply their pointwise maximum.

Consider a trade economy with a set of consumers $N = \{1, ..., n\}$. Consumer i is endowed with $\omega_i \in L_+$ and a preference relation $\succsim_i$ on L_+ which is convex, monotone, and continuous on L_+. Let $\omega = \sum_{i \in N} \omega_i$. We assume $\omega \neq 0$. We say that $\succsim_i$ is *ω-uniformly proper* on L_+ if there is a neighborhood of zero, $W_i \subset L$, such that $y - \alpha\omega + z \succsim_i y$, $y \in L_+$ for $\alpha > 0$ implies that $z \notin \alpha W_i$ (see Mas-Colell [14] for more on this definition). We assume that $\succsim_i$ is ω-uniformly proper for all $i \in N$. It is immediate that this implies that $\succsim_i$ is ω-nonsatiated in the sense that $y + \alpha\omega \succ_i y$, $\alpha > 0$, $y \in L_+$. It has been established in Richard and Zame [16] that the uniform properness condition is implied by (and is essentially equivalent to) the extendibility of $\succsim_i$ to some set with nonempty interior.

We assume that the order interval $[0, \omega] = \{z \in L: 0 \leqslant z \leqslant \omega\}$ is $\sigma(L, L^*)$-compact. Of course, if the topology of the space guarantees that every order interval is $\sigma(L, L^*)$-compact then the hypothesis is automatically satisfied.

Let $X = \{x \in L^n_+ : \sum_{i \in N} x_i = \omega\}$ be the set of (exactly) feasible allocations. A quasi-equilibrium is a feasible allocation $x \in X$ and a price $\pi \in L^*_+$, $\pi \neq 0$, such that for every $i \in N$:

(a) if $y \succsim_i x_i$, then $\pi \cdot y \geqslant \pi \cdot x_i$; and

(b) $\pi \cdot x_i = \pi \cdot \omega_i$.

We now state our main theorem, but defer the proof until Section 4.

THEOREM. *Under the hypotheses that every $\succsim_i$ is ω-uniformly proper and that X is $\sigma(L, L^*)$-compact there exist an $x \in X$ and a $\pi \in L^*_+$ such that (x, π) is a quasi-equilibrium with $\pi \cdot \omega > 0$.*

3. Discussion

If L is a topological vector lattice then L^* is automatically a sublattice of the order dual (Aliprantis and Burkinshaw [1]). Therefore our theorem generalizes the results that rely on locally solid topologies which, in turn, constitute a generalization of the seminal work of Bewley [7] for the space L_∞.

To be specific we have required our compactness hypothesis to be the $\sigma(L, L^*)$-compactness of X. But as in Mas-Colell [14] a weaker condition suffices, namely the compactness of the utility set $\hat{U} = \{(u_1, ..., u_N): 0 \leqslant u_i \leqslant u_i(x_i)$ for some $x \in X\}$. Also, the careful reader will realize that without a word of the proof being changed, the theorem remains valid if "ω-uniform proper" is interpreted as "ω-uniform proper on $[0, \omega]$."

The most interesting applications of our theorem are, of course to examples not covered by previous theory. Generally speaking, those are likely to occur whenever L has a weak topology or when L_+ is small.

The commodity differentiation model studied by Jones [10] corresponds to $L = ca(K)$, K compact, $L_+ = \{z \in L: z(A) \geqslant 0$ for every Borel set $A \subset K\}$, and the weak-star topology on L. Here the topology is too weak for the lattice operations to be continuous. (For $K = [0, 1]$ consider $\delta_{1/k} - \delta_0$ as $k \to 0$.) Nonetheless, Jones' model is included in ours since $L^* = C(K)$, with the induced positive orthant, is a vector lattice (more precisely, a sublattice of the order dual). Also, our uniform properness condition (which has a Lipschitz-like character) is somewhat weaker than his smoothness-like hypothesis. For example, with $K = [0, 1]$ the utility function $u(x) = \min\{x([0, 1]), 3 + x([0, 1]) - \frac{1}{2} \int t\, dx(t)\}$ is admissible by us but not by him.

The family of intertemporal preferences models of Huang and Kreps provides another example. The simplest specimen of the family is $L = \{$right continuous functions of bounded variation on $[0, 1]\}$ with the $\|\cdot\|_1$ norm and positive cone $L_+ = \{x \in L: x$ is nonnegative and nondecreasing$\}$. Equivalently, $L = ca([0, 1])$ with the standard positive cone and the topology induced by the duality with $C_l(K)$, the Lipschitz continuous functions on $[0, 1]$. Here the positive orthant L_+ is too small for the lattice operations to be continuous. However, for this space the dual is $L_\infty([0, 1])$ (equivalently $L^* = C_l([0, 1])$) which with the induced order can be shown to be a vector lattice. (This is obvious for the equivalent form.) Moreover, order intervals are weakly compact. Hence all the structural conditions of our theorem are satisfied (see Huang and Kreps [8] for all this).

Our theorem also throws light on a well-known example of Jones. In it $L = L_\infty([0, 1])$ is endowed with the weak topology induced by $C^1([0, 1])$. There are two consumers with utility functions $u_1(x_1) = \int_0^1 t x_1(t)\, dt$, $u_2(x_2) = \int_0^1 (1 - t)\, x_2(t)\, dt$. Initial endowments are $\omega_1 = \frac{1}{2}e$, $\omega_2 = \frac{1}{2}e$, where e is a constant positive function. It is immediate that for this economy there is no equilibrium with continuous prices. Heuristically the failure of existence in the example is due to L_+^* not making L^* into a vector lattice.

(Recall that $L^* = C^1([0, 1])$ and $L^*_+ = \{x \in L^* : x(t) \geqslant 0, \text{ all } t\}$; since the supremum of two differentiable functions need not be differentiable, L^* is not a lattice.) The theorem makes this point sharply. Indeed, that L^* is a vector lattice is the only hypothesis of the theorem that Jones's example fails to satisfy. The example has a very pathological feel to it: the obvious candidate to a quasi-equilibrium price vector fails to qualify simply because it is decreed not to be continuous. It could be argued that as long as L is a vector lattice, any problem which is well-posed from the point of view of existence should have a price space that can be viewed itself as a lattice.

What about the hypothesis that the consumption set, denoted by X_i, is the positive orthant L_+ of a vector lattice? An example of Back [6] shows that it cannot simply be replaced by X_i being closed, convex, and satisfying $X_i + L_+ \subset X_i \subset L_+$. But we do not know how far it can be relaxed. At any rate, the hypothesis is central to our proof for the following reason: Let $p_i \in L^*$, $i = 1, ..., N$ be arbitrary and, for any $z \geqslant 0$, define $\pi(z) = \sup\{\sum_{i \in N} p_i \cdot z_i : z_i \in L_+, \sum_{i \in N} z_i = z\}$. If L_+ defines a lattice order then (because the order has the so-called decomposition property) $\pi(\cdot)$ will be linear. This is essential to us sice we construct supporting functionals in precisely this way.

If a preference relation is continuous and ω-uniformly proper in one topology then it will remain so in any stronger topology. This means that for a given duality (L, L^*) the sharpest theorem lets the topology of the space be the Mackey topology.

It is easy to generate preferences to which our theorem is applicable. For example, let $J \subset L^*_+$ be $\sigma(L^*, L)$-compact with every $q \in J$ ω-increasing. Let $v: R \times J \to R$ be jointly continuous with $v(\cdot, q)$ concave and increasing for every $q \in J$. Finally, let μ be a *regular* measure on J. Then the function $u(z) = \int v(q \cdot z, q)\, d\mu(q)$ defined on the entire L is Mackey continuous, concave, and ω-increasing. The restriction of u to L_+ yields a preference relation which satisfies all of our conditions.

Recall from Richard and Zame [16] that a sufficient condition for ω-uniform properness on $[0, \omega]$ is that $\succsim_i$ be representable by the restriction to the positive orthant of a continuous, quasiconcave function u_i increasing in the ω direction and defined on a convex set which contains L_+ and has nonempty interior. Hence, in particular, if L_+ has nonempty interior to start with, then ω-uniform properness on $[0, \omega]$ follows from ω-desirability.

Can uniform properness be weakened to pointwise properness (a necessary condition for individual supportability at any consumption vector)? An example of Richard and Zame [16] shows that the answer is no. Nonetheless, recent work by Araujo and Monteiro [4] suggests that in some particular contexts there may be room for weakening the uniformity requirement.

An extension of the results in this paper for an economy with production can be found in Richard [17].

For an investigation of the necessity for existence purposes of the lattice property on the dual see Aliprantis and Burkinshaw [3].

A problem (raised by a referee) which we have not studied is to what extent requiring the price vectors to belong to L^*, i.e., to be continuous, does or does not bias the allocations that can be obtained. The natural conjecture is that under the hypothesis made any allocation sustainable as a quasi-equilibrium with a price vector not in L^* can also be sustained with a price vector in L^* (note that this regularization statement is weaker than the false assertion that all quasi-equilibrium price vectors belong to L^*).

4. Proof of the Theorem

In this section we prove our theorem. First we establish a proposition stating that any weak optimum can be supported by a price vector which is the maximum of vectors supporting the individual preferred sets. A similar characterization of supporting prices was proved in Mas-Colell [13] for the simpler case where each $\succsim_i$ is representable by a utility function, concave, and continuous on L and having a gradient at each point $x_i \in L_+$. Then we use this proposition to prove the existence of a quasi-equilibrium. As in Mas-Colell [14] we follow the convenient Negishi approach to existence theory (also used by Arrow and Hahn [5], Bewley [7], and Magill [11]). The heart of the proof, however, is markedly different from Mas-Colell [14]. This is to be expected since there the continuity of the lattice oeprations is used in an essential way and in our weaker set-up this property is not available.

The hypotheses of the theorem are assumed throughout the proof.

An allocation $x \in X$ is a *weak optimum* if there is no other allocation $x' \in X$ such that $x'_i \succ_i x_i$ for every $i \in N$. A weak optimum, x, is price supported if there exists $\pi \in L^*$, $\pi \neq 0$, such that $x'_i \succsim_i x_i$ implies that $\pi \cdot x'_i \geqslant \pi \cdot x_i$ for every $i \in N$. The following lemma contains our basic separation argument. The novelty is that the argument is entirely disaggregated (i.e., upper contour sets are never added up). It is this which allows us to dispense with the use of the lattice decomposition property and the local solidness of the topology.

Lemma 1. *There is a $\sigma(L^{*n}, L^n)$-compact, convex set $K \subset L_+^{*n}$, such that $\sum_{i \in N} p_i \cdot \omega = 1$ for all $p \in K$, and for any weak optimum, x, the set*

$$P(x)=\left\{p\in K\colon \textit{for all } i,\ p_i\cdot x_i'\geqslant p_i\cdot x_i \textit{ whenever } x_i'\succsim_i x_i \right.$$
$$\left. \textit{and } \sum_{i\in N} p_i\cdot x_i\geqslant \sum_{i\in N} p_i\cdot x_i' \textit{ whenever } x'\in X\right\}$$

is nonempty, convex, and $\sigma(L^{*n}, L^n)$*-compact.*

Proof. Let $W=W_1\times\cdots\times W_n\subset L^n$ where W_i is from the definition of uniform properness. Without loss of generality $W\subset L^n$ can be taken to be a circled (i.e., $W=-W$), convex, open neighborhood of zero. Define $K=\{p\in L_+^{*n}\colon |\sum_{i\in N} p_i\cdot w_i|\leqslant 1,\ w\in \mathrm{cl}\, W, \text{ and } \sum_{i\in N} p_i\cdot\omega=1\}$. By Alaoglu's theorem (Aliprantis and Burkinshaw [1]) K is $\sigma(L^{*n}, L^n)$-compact.

Let $\Gamma_i\subset L$ be the open, convex cone spanned by $\{\omega\}+W_i$ and put $S_i=\{z_i\in L_+\colon z_i\succsim_i x_i\}$, $V_i=S_i+\Gamma_i$, and $V=V_1\times\cdots\times V_n\subset L^n$. Note that V_i is convex and open and that $x_i\in \mathrm{cl}\, V_i$. Because of uniform properness, if $z_i\in V_i\cap[0,\omega]$ then $z_i\succ_i x_i$. In particular, $x_i\notin V_i$.

We must have $X\cap V=\phi$ since otherwise there is $z\in X$ such that $z_i\succ_i x_i$ for all $i\in N$, contradicting the weak optimality of x. By the Separation Theorem (Aliprantis and Burkinshaw [1]) there is $p\in L^{*n}$, $p\neq 0$, such that $\sum_{i\in N} p_i\cdot x_i\geqslant \sum_{i\in N} p_i\cdot z_i$ for all $z\in X$, and $\sum_{i\in N} p_i\cdot v_i>\sum_{i\in N} p_i\cdot x_i$ for all $v\in V$ (recall that $x\in\mathrm{cl}\, V$). This implies that $p_i\cdot v_i\geqslant p_i\cdot x_i$ whenever $v_i\succsim_i x_i$. Since $L_++\{x_i\}\subset S_i$, we have $p\in L_+^{*n}$. Because $\{x_i+\omega\}+W_i\subset V_i$, we have $\sum_{i\in N} p_i\cdot(\omega+w_i)>0$ for all $w\in W$. In particular $\sum_{i\in N} p_i\cdot\omega>0$. We are therefore free to normalize p so that $\sum_{i\in N} p_i\cdot\omega=1$, in which case $|\sum_{i\in N} p_i\cdot w_i|\leqslant 1$, $w\in\mathrm{cl}\, W$. Hence $p\in K$. We conclude that $P(x)$ is nonempty; that it is convex and compact follows from the definition of the set. Q.E.D.

If $p\in L^{*n}$, then $\pi\equiv\bigvee_{i\in N} p_i$ is, by hypothesis, a linear functional on L. For $z\in L_+$ it takes the values $\pi\cdot z=\sup\{\sum_{i\in N} p_i\cdot z_i\colon z_i\geqslant 0 \text{ and } \sum_{i\in N} z_i=z\}$, $z\in L_+$. If for same $z\in L_+$ the order interval $[0,z]$ is $\sigma(L, L^*)$-compact, then the sup is realized because we are maximizing a $\sigma(L^n, L^{*n})$-continuous function on a $\sigma(L^n, L^{*n})$-compact set.

We are now ready to state and prove our proposition.

PROPOSITION. *Let* $x\in X$ *be a weak optimum. If* $\pi=\bigvee_{i\in N} p_i$ *for* $p\in P(x)$, *then* π *supports* x. *Moreover,* $\pi\in L_+^*$, $\pi\cdot\omega\geqslant 1/n$, $\pi\cdot\omega=\sum_{i\in N} p_i\cdot x_i$, *and* $\pi\cdot x_i=p_i\cdot x_i$ *for all* i.

Proof. Since $p\in P(x)$, we have $p_i\geqslant 0$ for all $i\in N$, implying $\pi\geqslant 0$. Furthermore, the definition of π and $p\in P(x)$ immediately yield that $\pi\cdot(n\omega)\geqslant\sum_{i\in N} p_i\cdot\omega=1$ and $\pi\cdot\omega=\sum_{i\in N} p_i\cdot x_i$.

Suppose that $z\succsim_i x_i$. Since $\pi\geqslant p_i$, we have $\pi\cdot z\geqslant p_i\cdot z\geqslant p_i\cdot x_i$ because $p\in P(x)$. If $\pi\cdot x_i>p_i\cdot x_i$ for some i, then $\pi\cdot\omega=\sum_{i\in N}\pi\cdot x_i>\sum_{i\in N} p_i\cdot x_i$

which contradicts the fact that $p \in P(x)$. Hence $\pi \cdot x_i = p_i \cdot x_i$ and so $\pi \cdot z \geqslant \pi \cdot x_i$ whenever $z \succsim_i x_i$, as we wanted to prove. Q.E.D.

Denote by Δ the $n-1$ dimensional simplex. We now proceed to specify a correspondence,

$$\Phi: \Delta \to T = \left\{ t \in R^n: \sum_{i \in N} t_i = 0 \right\},$$

whose zeroes correspond to quasi-equilibria.

For every i, let u_i: $[0, \omega] \to R$ be a continuous function representing $\succsim_i$ on $[0, \omega]$. Such a function exists (see Richard and Zame [16]). We can scale u_i so that $u_i(0) = 0$ and $u_i(\omega) = 1$. The set $\hat{U} = \{(u_1, ..., u_N): 0 \leqslant u_i \leqslant u_i(x_i)$ for all i and some $x \in X\}$ is compact because X is $\sigma(L^n, L^{*n})$-compact and each $u_i(\cdot)$ is $\sigma(L, L^*)$-upper semicontinuous (the closed sets $\{z \in [0, \omega]: z \succsim_i x_i\}$ are convex, hence $\sigma(L, L^*)$-closed).

For every $s \in \Delta$ define $f(s) = \max\{\alpha \in R: \alpha s \in \hat{U}\}$ and $X(s) = \{x \in X: U(x) \geqslant f(s) s\}$. Of course, every member of the (nonempty) set $X(x)$ is a weak optimum. Also, monotonicity and τ-continuity of preferences imply that for every s there is $x \in X(s)$ with $u_i(x_i) = f(s) s_i$ for all i.

Finally, define $\Phi: \Delta \to T$ by

$$\Phi(s) = \left\{ (\pi \cdot (\omega_1 - x_1), ..., \pi \cdot (\omega_n - x_n)): x \in X(x) \right.$$
$$\left. \text{and } \pi = \bigvee_{i \in N} p_i \text{ for some } p \in P(x) \right\}.$$

Because of the proposition the zeroes of Φ yield quasi-equilibria.

Lemma 2. *The function $f: \Delta \to R$ is continuous and the correspondence $X(\cdot): \Delta \to X$ is convex valued and u.h.c. (for the $\sigma(L^n, L^{*n})$-topology on X).*

Proof. The continuity of f was proved in Mas-Colell [14, pp. 1045–1046]. The convexity of $X(s)$ is an obvious consequence of the definition. Because X is $\sigma(L^n, L^{*n})$-compact it suffices, in order to prove u.h.c., to show that for any nets $\{s^k\} \subset \Delta$ converging to $s \in \Delta$, and $\{x^k\} \subset X$ with $x^k \in X(s^k)$ $\sigma(L^n, L^{*n})$-converging to $x \in X$, we have $x \in X(s)$. Clearly $\sum_{i \in N} x_i \leqslant \omega$. Furthermore, each u_i is $\sigma(L, L^*)$-upper semicontinuous so that lim sup $u_i(x_i^k) \leqslant u_i(x_i)$, which implies that $f(s) s \leqslant U(x)$. Q.E.D.

Remark. If X and K are metrizable then nets can be replaced by sequences in the above and all subsequent proofs.

The next lemma will allow us to prove the u..h.c. and the convex valuedness of Φ.

LEMMA 3. *Consider nets* $\{x^k\} \subset X$, *and* $\{p^k\} \subset K$ *such that* x^k *is a weak optimum which satisfies* $\sum_{i \in N} x_i^k = \omega$ *and* $\sigma(L^n, L^{*n})$*-converges to* $x \in X$, *and such that* p^k *belongs to* $P(x^k)$ *and* $\sigma(L^{*n}, L^n)$*-converges to* $p \in K$. *Denote* $\pi^k = \bigvee_{i \in N} p_i^k$, $\pi = \bigvee_{i \in N} p_i$. *Then*:

(a) $\pi^k \cdot z \to \pi \cdot z$ *for all* $z \in [0, \omega]$,

(b) $\pi^k \cdot x_i^k \to \pi \cdot x_i$ *for all i, and*

(c) $p \in P(x)$.

Proof. Part (a). Take any $z \in [0, \omega]$. Since $[0, \omega]$ is compact, so is $[0, z]$. Let $\pi \cdot z = \sum_{i \in N} p_i \cdot z_i$, $z_i \geqslant 0$, $\sum_{i \in N} z_i = z$. Then $\pi^k \cdot z \geqslant \sum_{i \in N} p_i^k \cdot z_i$ for all k. Hence $\liminf \pi^k \cdot z \geqslant \sum_{i \in N} p_i \cdot z_i = \pi \cdot z$. Since the same inequality holds for $\omega - z$ it suffices to show that $\limsup \pi^k \cdot \omega \leqslant \pi \cdot \omega$. Since $\omega = \sum_{i \in N} x_i^k$ this is implied in turn by $\limsup \pi^k \cdot x_i^k \leqslant \pi \cdot x_i$. Fix $\varepsilon > 0$. Because every $\succsim_i$ is $\sigma(L, L^*)$-upper semicontinuous we can assume that $x_i + \varepsilon\omega \succ_i x_i^k$. Hence $p_i^k \cdot (x_i + \varepsilon\omega) \geqslant p_i^k \cdot x_i^k$. Taking limits and letting $\varepsilon \to 0$ we get $\limsup p_i^k \cdot x_i^k \leqslant p_i \cdot x_i$. But, by Lemma 1, $p_i^k \cdot x_i^k = \pi^k \cdot x_i^k$. Furthermore, $p_i \leqslant \pi$ implies $p_i \cdot x_i \leqslant \pi \cdot x_i$. Hence $\limsup \pi^k \cdot x_i^k \leqslant \pi \cdot x_i$, as we wanted.

Part (b). From the definition of π^k, $\pi^k \cdot \omega \geqslant \sum_{i \in N} p_i^k \cdot x_i'$ for all $x' \in X$. Hence $\liminf \pi^k \cdot \omega \geqslant \sum_{i \in N} p_i \cdot x_i'$ for all $x' \in X$. In particular, $\liminf \pi^k \cdot \omega \geqslant \pi \cdot \omega$. Therefore $\liminf \sum_{i \in N} \pi^k \cdot x_i^k \geqslant \sum_{i \in N} \pi \cdot x_i$. Since $\limsup \pi^k \cdot x_i^k \leqslant \pi \cdot x_i$ for all i, as proved in part (a), we conclude $\pi^k \cdot x_i^k \to \pi \cdot x_i$ for all i.

Part (c). For arbitrary $\varepsilon > 0$ we eventually have $x_i + \varepsilon\omega \succ_i x_i^k$. Therefore $p_i^k \cdot x_i + \varepsilon p_i^k \cdot \omega \geqslant p_i^k \cdot x_i^k$. Hence $p_i \cdot x_i \geqslant \limsup p_i^k \cdot x_i^k$. Suppose that $p_i \cdot x_i > \limsup p_i^k \cdot x_i^k$ for some i. Because $\sum_{i \in N} p_i^k \cdot x_i^k \geqslant \sum_{i \in N} p_i^k \cdot x_i$ we then have

$$\liminf \sum_i p_i^k \cdot x_i^k \geqslant \sum_i p_i \cdot x_i > \sum_i \limsup p_i^k \cdot x_i^k \geqslant \limsup p_i^k \cdot x_i^k$$

which is a contradiction. Therefore $p_i \cdot x_i = \limsup p_i^k \cdot x_i^k$ for all i. But $p_i^k \cdot x_i^k = \pi^k \cdot x_i^k$ and, from part (b), $\limsup \pi^k \cdot x_i^k = \pi \cdot x_i$. Hence $p_i \cdot x_i = \pi \cdot x_i$.

Obviously $p \in L_+^{*n}$ and $\sum_{i \in N} p_i \cdot \omega = 1$.

If $z \succsim_i x_i$ then for arbitrary $\varepsilon > 0$ we eventually have $z + \varepsilon\omega \succ_i x_i^k$. Hence $p_i^k \cdot z + \varepsilon p_i^k \cdot \omega \geqslant p_i^k \cdot x_i^k = \pi^k \cdot x_i^k$. Taking limits and letting $\varepsilon \to 0$ we get $p_i \cdot z \geqslant \pi \cdot x_i = p_i \cdot x_i$.

Now consider an arbitrary $x' \in X$. We have $\sum_{i \in N} p_i^k \cdot x_i' \leqslant \sum_{i \in N} p_i^k \cdot x_i^k = \sum_{i \in N} \pi^k \cdot x_i^k$. Taking limits, $\sum_{i \in N} p_i \cdot x_i' \leqslant \sum_{i \in N} \pi \cdot x_i = \sum_{i \in N} p_i \cdot x_i$.

Therefore, $p \in K$. Q.E.D.

A simple consequence of Lemma 3 is:

LEMMA 4. *$\Phi: \Delta \to T$ is upper hemicontinuous (u.h.c.).*

Proof. First note that $\Phi(\Delta)$ is contained in a compact subset of T since

$$|\pi \cdot (\omega_i \cdot x_i)| \leqslant \pi \cdot \omega = \sum_{i \in N} p_i \cdot x_i \leqslant \sum_{i \in N} p_i \cdot \omega = 1.$$

Let $s^k \to s$, $s^k \in \Delta$, and $t^k \to t$, $t^k \in \Phi(s^k)$. Choose $x^k \in X(s^k)$, $p^k \in P(x^k)$ with $t_i^k = \pi^k \cdot (\omega_i - x_i^k)$, $\pi^k = \bigvee_{i \in N} p_i^k$. By going to a subnet we can assume that $\{x^k\}$ $\sigma(L^n, L^{*n})$-converges to an $x \in X$, and $\{p^k\}$ $\sigma(L^{*n}, L^n)$-converges to a $p \in K$. Denote $\pi = \bigvee_{i \in N} p_i$. By Lemma 2, $x \in X(s)$. By Lemma 3, $p \in P(x)$ and $t_i = \lim \pi^k \cdot (\omega_i - x_i^k) = \pi \cdot (\omega - x_i)$. Therefore $t \in \Phi(s)$. Q.E.D.

It is not immediately obvious that Φ is convex-valued. Nonetheless,

LEMMA 5. *Φ is convex-valued.*

Proof. Fix an $s \in \Delta$ and choose $x \in X$ such that $u_i(x_i) = f(s) s_i$ for all i.

We show first that if $x' \in X(s)$ then $p_i \cdot x_i' = p_i \cdot x_i$ for all $i \in N$ and $p \in P(x)$. Indeed, $p_i \cdot z \geqslant p_i \cdot x_i$ whenever $z \succsim_i x_i$. Since $u_i(x_i') \geqslant u_i(x_i)$ we have $p_i \cdot x_i \leqslant p_i \cdot x_i'$ for all i. But $\sum_{i \in N} p_i \cdot x_i \geqslant \sum_{i \in N} p_i \cdot x_i'$ because $x' \in X$. Therefore $p_i \cdot x_i = p_i \cdot x_i'$ for all i. But $\sum_{i \in N} p_i \cdot x_i \geqslant \sum_{i \in N} p_i \cdot x_i'$ because $x' \in X$. Therefore $p_i \cdot x_i = p_i \cdot x_i'$ for $i \in N$. An immediate consequence of this is that $P(x) = P(x')$. From now on we put $P(s) = P(x)$, $x \in X(s)$. Clearly, the set $P(s)$ is convex.

Let $t^1, t^2 \in \Phi(s)$ and, correspondingly, take $x^1, x^2 \in X(s)$, $p^1, p^2 \in P(s)$. For any $\alpha \in [0, 1]$, denote $p^\alpha = \alpha p^1 + (1-\alpha) p^2 \in P(s)$, and $\pi^\alpha = \bigvee_{i \in N} p_i^\alpha$. It suffices to show that $\pi^\alpha \cdot (\omega_i - x_i^2) = (\alpha\pi^1 + (1-\alpha)\pi^2) \cdot (\omega_i - x_i^2) = \alpha t^1 + (1-\alpha) t^2$; of course, only the first equality needs proof.

We prove a stronger property, namely that $\pi^\alpha \cdot z = (\alpha\pi^1 + (1-\alpha)\pi^2) \cdot z$ for all $z \in [0, \omega]$. Because $\pi^1 \cdot \omega = \sum_{i \in N} p_i^1 \cdot x_i^1 = \sum_{i \in N} p_i^1 \cdot x_i^2$ and $\pi^2 \cdot \omega = \sum_{i \in N} p_i^2 \cdot x_i^2$ we have $(\alpha\pi^1 + (1-\alpha)\pi^2) \cdot \omega = \sum_{i \in N} p_i^\alpha \cdot x_i^2 = \pi^\alpha \cdot \omega$. The last equality follows from $p^\alpha \in P(s)$. For $z \in [0, \omega]$ and $\alpha \in [0, 1]$ let z_i^α, $i \in N$, be such that $z_i^\alpha \geqslant 0$, $\sum_{i \in N} z_i^\alpha = z$, and $\pi^\alpha \cdot z = \sum_{i \in N} p_i^\alpha \cdot z_i^\alpha$. (Note that the superscript in the definition of π^α is realized because $[0, z] \subset [0, \omega]$ is $\sigma(L, L^*)$-compact.) Then

$$\begin{aligned} \pi^\alpha \cdot z &= \alpha \sum_{i \in N} p_i^1 \cdot z_i^\alpha + (1-\alpha) \sum_{i \in N} p_i^2 \cdot z_i^\alpha \\ &\leqslant \alpha \sum_{i \in N} p_i^1 \cdot z_i^1 + (1-\alpha) \sum_{i \in N} p_i^2 \cdot z_i^0 = (\alpha\pi^1 + (1-\alpha)\pi^2) \cdot z. \end{aligned}$$

Similarly, because $(\omega - z) \in [0, \omega]$, $\pi^\alpha \cdot (\omega - z) \leqslant (\alpha\pi^1 + (1-\alpha)\pi^2) \cdot (\omega - z)$. Therefore, using $\pi^\alpha \cdot \omega = (\alpha\pi^1 + (1-\alpha)\pi^2) \cdot \omega$, we conclude $\pi^\alpha \cdot z = (\alpha\pi^1 + (1-\alpha)\pi^2) \cdot z$ for all $z \in [0, \omega]$.

Hence Φ is convex-valued. Q.E.D.

LEMMA 6. *If* $s_i = 0$ *and* $t \in \Phi(s)$ *then* $t_i \geqslant 0$.

Proof. If $s_i = 0$ then $f(s)s_i = 0$. Hence $\omega_i \succsim_i x_i$ for any $x \in X(s)$. Let $x \in X(s)$, $p \in P(x)$ and $\pi = \bigvee_{i \in N} p_i$ generate t. Because π supports x_i we have $t_i = \pi \cdot (\omega_i - x_i) \geqslant 0$. Q.E.D.

By Lemmas 4, 5, and 6 the "vector field" $s + \Phi(s)$ on Δ is u.h.c., convex-valued, and inward pointing at the boundary. Therefore, it has a fixed point; i.e., there is s^* such that $0 \in \Phi(s^*)$. In other words, there are $x \in X$ and $p \in K$ such that for $\pi = \bigvee_{i \in N} p_i$, (x, π) constitutes a quasi-equilibrium. Also, $\pi \cdot \omega \geqslant \sum_{i \in N} p_i \cdot ((1/n)\omega) = 1/n$.

This concludes the proof of the theorem.

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