Journal of Mathematical Economics 20 (1991) 397–409. North-Holland

On the finiteness of the number of critical equilibria, with an application to random selections

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Submitted September 1989, accepted May 1990

Abstract: This paper establishes the following result for exchange economies with l commodities and n agents: for a generic set of utility functions of the first consumer, the number of critical equilibria is at most ln at every endowment. This result has application to, for example, the theory of general equilibrium Cournot competition.

1. Introduction

In this paper we establish, by comparatively elementary mathematical means, the following theorem on the structure of the Walras equilibrium manifold of exchange economies with l commodities and n agents: for a *generic* (more precisely, for a residual) *set of utility functions of the first consumer the number of critical equilibria is at most* ln *for any collection of endowment vectors* (hence the total number of equilibria is at most countable). An analogous result for arbitrary paramerizations of exchange economies also holds.

Our motivation comes from the research of Allen (1984, 1985a, b). Working in a universe of parameterized excess demand functions she showed (1984) that for a residual set of parameterizations the number of equilibria is finite at every parameter value. Our result differs from Allen's in two respects. On the one hand, we are more demanding and take as primitive preferences and endowments rather than excess demand functions. But, on

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the other, we obtain only countability, rather than finiteness, of the equilibrium set.

We show in section 3 that, despite obtaining only countability, our result suffices for the intended application of Allen's [see Allen (1985a)]: if the equilibrium manifold is countable valued then it possessess a continuous random selection. This in turn has application to questions such as the existence of equilibrium in Cournot economies with general demand sectors [see Allen (1985b) and, for background Gabszewicz and Vial (1972), Roberts (1980), or Mas-Colell (1982)]. Specifically, if the price selection from the exchange economy price equilibrium correspondence, to be thought of as the 'demand function' facing the production sector, is required to be deterministic, the full economy integrating production with consumption may not have even a mixed-strategy equilibrium [Dierker and Grodal (1986)]. However, at parameter (netput) values for which there are multiple equilibria, it may be argued that a random, rather than a deterministic, selection makes sense. The randomization may be thought of as representing inherent ex-ante uncertainty in the minds of producers as to which price will actually arise in equilibrium. This uncertainty is not qualitatively different from uncertainty as to what netputs a firm's rivals will choose, uncertainty which provides the motivation for mixed-strategy equilibrium as a solution concept: see Aumann (1987), and Brandenburger and Dekel (1987). The reader will note that if prices or netputs, or both, are randomized, the resulting equilibrium may well be inefficient because the randomization introduces uncertainty against which consumers cannot insure.

2. The model and main result

The following definitions are standard. See Mas-Colell (1985) for additional detail and motivation.

Denote by $\mathscr{U}$ the space of C^∞, strictly increasing utility functions $u:\mathbb{R}^l_{++}\to\mathbb{R}$ satisfying: (i) the bordered Hessian of u is non-zero at every x, and (ii) the set $\{x':u(x')\geqq u(x)\}$ is closed in $\mathbb{R}^l$ for every $x\gg 0$. As is well known this implies that the demand function generated from u is C^∞ and proper (on a normalized set of prices).

Endow the set $\mathscr{U}$ with the topology of C^∞ uniform convergence on compacta. It is then topologically complete and, therefore, residual subsets (i.e. sets containing the intersection of a countable collection of open, dense sets) are dense.

We will consider n consumer economies. Each consumer i is specified by a pair $(u_i,\omega_i)\in\mathscr{U}\times\mathbb{R}^l$. Given an economy $\{(u_i,\omega_i)\}_{i=1}^n$, let $S'=\{p\in S:p\cdot\omega_i>0$ for all $i\}$, where $S=\{p\in\mathbb{R}^l_{++}:\|p\|=1\}$. Denote by $f:S'\to\mathbb{R}^l$ the excess demand function of the economy. A $p\in S'$ is a regular (resp. critical) equilibrium if $f(p)=0$ and $\operatorname{rank} Df(p)=l-1$ (resp. $<l-1$).

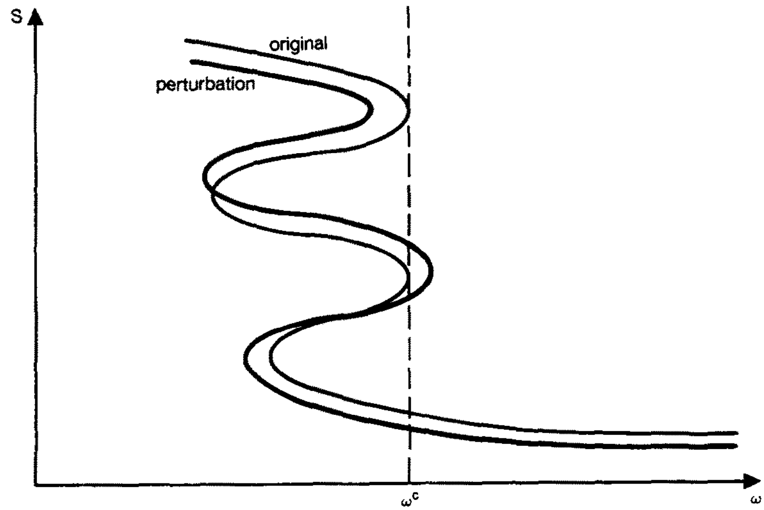

Fig. 1. Equilibrium manifold, before and after perturbation.

Fix once and for all the utility functions $u_2, \dots, u_n$ of consumers $i=2, \dots, n$. Our main result is:

Proposition 1. There is a set $\mathscr{U}^*$ *residual in* $\mathscr{U}$ *such that for every* $u_1 \in \mathscr{U}^*$ *and every* $\omega \in \mathbb{R}^{ln}$*, the number of critical equilibria of the economy* $\{(u_i, \omega_i)\}_{i=1}^n$ *is at most ln.*

The logic of Proposition 1 is symbolically illustrated in figs. 1 and 2. Fig. 1 displays the equilibrium correspondence for a one-dimensional parameter space. It is intuitively clear that the existence of two distinct critical equilibria for the same economy is coincidental. Fig. 2 illustrates a case with a two-dimensional parameter space. In the figure the lines represent the projection on the ω-space of fold lines (hence of lines of critical equilibria) in the equilibrium manifold. Again, the intersection of more than two of these lines appears intuitively coincidental.

Proof of Proposition 1. Let $B = \{B^1, \dots, B^{ln-1}\}$ be a collection of disjoint closed balls in S, each B^i having rational radius and a center with rational first $l-1$ coordinates. There is a countable number of such B. For every B and positive integer m put

$$K_{B,m} = \{\omega : (1/m) \leqq p^j \cdot \omega_i \leqq m, \text{ all } j, \text{ all } p^j \in B^j, \text{ all } i\}.$$

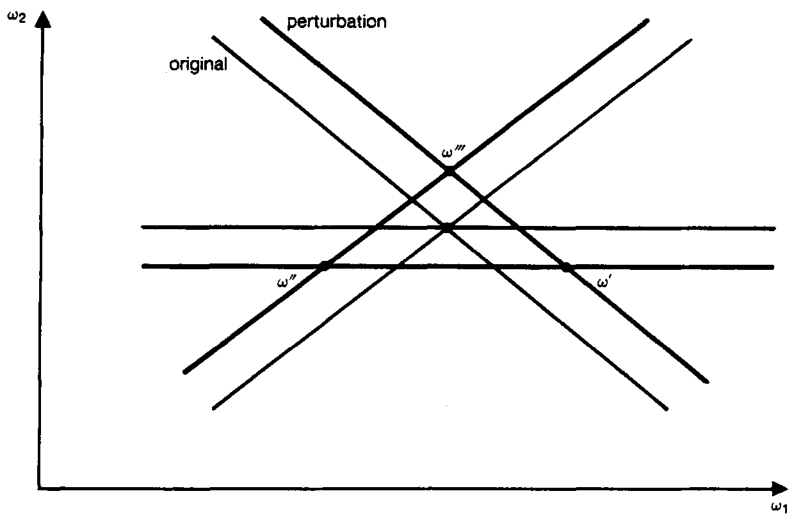

Fig. 2. Projection of fold lines into the parameter space, before and after perturbation.

For every $u_1 \in \mathscr{U}$, define $F_{u_1}: K_{B,m} \times B \to \mathbb{R}^{(l-1)(ln+1)}$ by letting $F^j_{u_1}(\omega, p^1, \ldots, p^{ln+1})$, $j=1,\ldots,ln+1$, be the excess demand function for commodities 1 to $l-1$ generated by the economy $\{(u_i, \omega_i)\}_{i=1}^n$ at the price vector p^j.

Lemma 2 There is an open and dense subset $\mathscr{U}' \subset \mathscr{U}$ such that for every element of $\mathscr{U}'$, 0 is a regular value of F_{u_1}.

Lemma 2, which will be proved below, established Proposition 1 as follows. Consider the intersection $\mathscr{U}^*$ of the sets $\mathscr{U}'$ obtained as we run over all the different B and m. This set is trivially residual. Suppose now that for some $u_1 \in \mathscr{U}^*$ and $\bar{\omega} \in \mathbb{R}^{ln}$ we have $ln+1$ critical equilibria $\bar{p}^1, \ldots, \bar{p}^{ln+1}$. Take a B and an m such that each B^j contains $\bar{p}^j$ in its interior and $\bar{\omega} \in K_{B,m}$. Define F_{u_1} with respect to this B and m. For notational simplicity we drop the subscript u_1. Now, the rank of $D_\omega F$ can be at most ln and by assumption for every j we have rank $D_{p^j}F^j(\bar{\omega}, \bar{p}^1, \ldots, \bar{p}^{ln+1}) \leq l-2$. It follows, taking into account the diagonal character of $D_{p^1,\ldots,p^{ln+1}}F$, that the rank of DF can at most be $(ln+1)(l-2)+ln=(ln+1)(l-1)-1$. On the other hand, full rank requires that rank $DF=(ln+1)(l-1)$. Thus if there are at least $ln+1$ critical equilibria, then 0 is not regular value of F (remember that $F(\bar{\omega}, \bar{p}^1, \ldots, \bar{p}^{ln+1})=0$), contradicting the lemma.

Proof of Lemma 2. Openness follows from continuity of the determinant and the fact that $K_{B,m}$ and the B^j are compact. For denseness we fix a $u_1 \in \mathscr{U}$ and show that we can perturb u_1 to a u_1' arbitrarily nearby and such that 0 is a regular value of $F_{u_1'}$. This we shall do by means of the Transversality Theorem.

Let B' be a collection (in the countable family) with $B^j \subset \operatorname{Int} B'^j$ for every j. Let also $\alpha_j: S \times \mathbb{R}_+ \to [0,1]$ be a C^∞ function satisfying

$$\alpha_j(B^j \times [1/m, m]) = 1$$

and

$$\alpha(S \times \mathbb{R}_+ \backslash B'^j \times [1/2m, 2m]) = 0.$$

Fix $\varepsilon > 0$ and let $v: S \times R_+ \to \mathbb{R}$ be the indirect utility function associated with u_1. For every j denote by $Y^j \subset \mathbb{R}^l$ the unit ball. Then for every $y \in Y^1 \times \cdots \times Y^{ln+1}$ we define

$$v_y(p,w) = v(p,w) + \varepsilon \sum_{j=1}^{ln+1} \alpha_j(p,w) y^j \cdot p.$$

It is a simple exercise in duality theory [see, e.g., Diewert (1982)] to verify that if ε is sufficiently small then, for all y, $v_y: S \times \mathbb{R}_+ \to \mathbb{R}$ satisfies all the properties of an indirect utility function. If u_1^y is the direct utility function corresponding to v_y, then, by taking ε arbitrarily close to 0, we have u_1^y arbitrarily close to u_1. Therefore the proof will proceed by showing that for any ε we can find a y such that 0 is a regular value of $F_{u_1^y}$. Henceforth, fix $\varepsilon > 0$.

Define $F: K_{B,m} \times B^1 \times \cdots \times B^{ln+1} \times Y^1 \times \cdots \times Y^{ln+1} \to \mathbb{R}^{(l-1)(ln+1)}$ by, as usual, letting $F^j(\omega, p^1, \ldots, p^{ln+1}, y)$, $1 \leqq j \leqq ln+1$, be the excess demand function for the first $(l-1)$ commodities generated by the economy $\{(u_1^y, \omega_1), (u_2, \omega_2), \ldots, (u_n, \omega_n)\}$ at the price vector p^j. Suppose that we can show that whenever $F(\bar{\omega}, \bar{p}^1, \ldots, \bar{p}^{ln+1}, \bar{y}) = 0$ we have rank $D_y F(\bar{\omega}, \bar{p}^1, \ldots, \bar{p}^{ln+1}, \bar{y}) = (l-1)(ln+1)$. Then we are done, since by the Transversality Theorem [see Hirsch (1976) or Mas-Colell (1985)], we have that for all but a measure zero subset of $Y^1 \times \cdots \times Y^{ln+1}$, and hence (trivially) for at least one y, we have that 0 is a regular value of $F_{u_1^y}$. Note that by construction $D_y F(\bar{\omega}, \bar{p}^1, \ldots, \bar{p}^{ln+1}, \bar{y})$ is diagonal, therefore it suffices to show that rank $D_{y^j} F^j(\bar{\omega}, \bar{p}^1, \ldots, \bar{p}^{ln+1}, \bar{y}) = l-1$.

By construction, y^j enters into F^j only via the excess demand for the first consumer. By Roy's formula this excess demand is

$$-[1/D_w v_y(\bar{p}^j \bar{p}^j \cdot \bar{\omega}^1)] D_p v_y(\bar{p}^j, \bar{p}^j \cdot \bar{\omega}^1) - \bar{\omega}^1.$$

But $D_p v_y(\bar{p}^j, \bar{p}^j \cdot \bar{\omega}^1) = D_p v(\bar{p}^j, \bar{p}^j \cdot \bar{\omega}^1) + \varepsilon y^j$, a linear map on y^j. Note also that $D_w v_y(\bar{p}^j, \bar{p}^j \cdot \bar{\omega}^1)$ does not depend on y^j. Summing up, the linear map $D_{y^j} F^j(\bar{\omega}, \bar{p}^1, \dots, \bar{p}^{ln+1}, \bar{y})$ is a multiple of the identity. Thus it has rank $l-1$. □

By the Inverse Function Theorem, regular equilibrium are locally isolated. Therefore we obtain:

Corollary 3. There is a set $\mathscr{U}^*$ residual in $\mathscr{U}$ such that for every $u_1 \in \mathscr{U}^*$ and every $\omega \in \mathbb{R}^{ln}$, the number of equilibria of the economy $\{(u_i, \omega_i)\}_{i=1}^n$ is at most countable.

Proposition 1 is only an example of an entire class of results. As an illustration we state another variant applicable to general parameterizations of economies.

Let K be a q-dimensional, σ-compact, smooth manifold. A function $\eta: K \to (\mathscr{U} \times \mathbb{R}^l_{++})^n$ is a C^∞ parameterization if, denoting by $f_\eta(p,t)$ the excess demand at p of the economy $\{\eta_i(t)\}_{i=1}^n$, the function $f_\eta: S \times K \to \mathbb{R}^l$ is C^∞. Let the space of C^∞ parameterizations on K be denoted $\mathscr{T}$. We give to $\mathscr{T}$ a complete topology by saying that $\eta_s \to \eta$ if $f_{\eta_s} \to f_\eta$ C^∞-uniformly on compacta.

Proposition 4. There is a set $\mathscr{T}^*$ residual in $\mathscr{T}$ such that for every $t \in K$, $f_\eta(\cdot, t)$ has at most q critical equilibria.

Although the setting of Proposition 4 is more general than that of Proposition 1, neither result implies the other. Nonetheless, the proof of Proposition 4 is so similar to that of Proposition 1 that we omit it.

A natural conjecture is that the above results can be strengthened in two directions. First, one would suppose that finiteness of the number of critical equilibria should hold on a set which is open-dense rather than just residual (of course, for this we must take the parameter set to be compact). Second, and most importantly, one would conjecture that the total number of equilibria at every parameter can be asserted to be finite, rather than just countable. In Mas-Colell (1985) it is in fact proved that if we restrict attention to a compact one-dimensional parameter space, then, for an open and dense set of parameterizations, *all* equilibria are locally isolated, hence finite in number, at every parameter value. This result has a dual, stated and proved in the appendix, which says that if the number of commodities is 2, so that the price space is one-dimensional, then for an arbitrary compact finite-dimensional parameter space there is an open and dense set of parameterizations, all equilibria of which are locally isolated. Both of these results involve looking at the higher derivatives of the excess demand function. The underlying idea is that failure of local isolation at a critical economy means a failure of the excess demand function to be sufficiently 'curved' (i.e. to have a non-zero derivative of some order) at a critical equilibrium. The proofs then proceed by showing that a 'flat' excess demand function is not generic in the parameterization of the first consumer.

Taking excess demand as primitive, the general result concerning generic (in the sense of residual) finiteness for all parameters was established by Allen (1984). For arguments based on preferences and endowments, unfortunately, the generalization has thus far resisted our attempts at proof. We are informed that D. Saari and C. Simon have been investigating this question and have reached some results. Mathematically, the basic problem is that for functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, the zero set in the neighborhood of a critical zero can be extremely complex if $n>1$. The one-dimensional intuition that the function must be 'flat' in the neighborhood of a non-isolated critical zero may still in some sense be valid, but it appears that its expression will be considerably more subtle. For a general smooth function $f:\mathbb{R}^n\to R^n$, $f(0)=0$, a known sufficient condition for local isolation at 0, one which perhaps may be the most useful achieveable, is hat the local ring of f be finite dimensional at 0 [see for example Golubitsky and Guillemin (1973)]. This is essentially a statement about the ideal generated by the power series expansions at 0 of the coordinate functions of f; finiteness dictates that the 'span' of the ideal be sufficiently large. If $n=1$, finiteness means precisely that at 0 a derivative of some order must be non-zero, the condition for local isolation used in the appendix and easily derived by more elementary means. For $n>1$, finiteness is far more difficult to check. It is known, however, that generically all smooth functions are finite at every point. This result, a theorem of Tougeron [Proposition 6.2 in Tougeron (1972)] was in fact the key technical fact employed by Allen. Unfortunately, imbedding the techniques that underly Tougeron's theorem into a framework of preferences and endowments is no easy task.

We want to emphasize, however, that even if our result is not the most general possible, it is of interest in its own right for at least two reasons: (i) comparatively speaking the proof is simple and elementary (no more than the Transversality Theorem is involved and higher derivatives of excess demand play no role), and (ii) as we will see in the next section, our result quite suffices for some of the intended applications of the theory.

3. An application

In this section we show that the results of section 2 admit the same application as that of Allen (1985a). Namely, we show that in the generic set of utility functions the equilibrium correspondence can support a continuous random selection. As noted in the Introduction, this result has applications

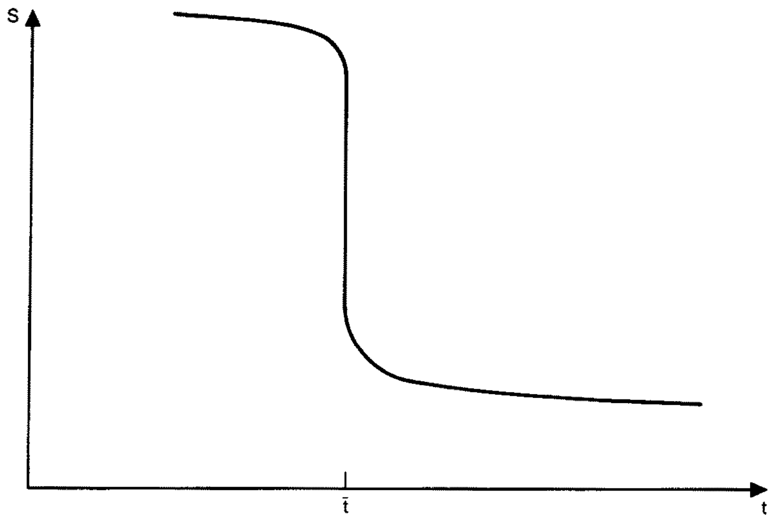

Fig. 3. Equilibrium manifold. No continuous random selection is possible in a neighborhood of the critical economy $\bar{t}$.

in the theory of general equilibrium Cournot competition. In particular, it can be used to prove the existence of (mixed-strategy) equilibria.[1]

The discussion here follows Allen (1985a) closely and it is included for completeness, not originality. The setup is as in section 2. The equilibrium correspondence is the set-valued map $E: K \to S$ where $W(t) = \{p \in S: f_t(p) = 0\}$, K is a parameter space and f_t is the aggregate excess demand corresponding to $t \in K$. Following Allen (1985a):

Definition. Let $M(S)$ be the set of probability measures on S. It is given the weak convergence topology. A *continuous random selection* of W is a continuous function $\rho: K \to M(S)$ such that $\rho(t)(W(t)) = 1$.

Proposition 5. If for every $t \in K$, $W(t)$ is at most countable, then W admits a continuous random selection.

Figure 3 illustrates the fact that without some restriction on the nature of the sets $W(t)$ the existence of a continuous random selection is not guaranteed.

[1]In a recent paper, Simon and Zame (1990) prove an equilibrium existence theorem of, roughly, the following form: for every game with an upper hemicontinuous payoff correspondence that takes non-empty, compact and convex values, there is a measurable selection of the payoff correspondence such that the derived game admits a Nash equilibrium, possibly in mixed strategies. As here, convex-valuedness may require randomization over payoffs. The results of this section can be interpreted as showing that in the case of general equilibrium Cournot games, we can, generically, take the selection to be continuous, rather than merely measurable.

To prove Proposition 5 we follow the approach of Allen. Suppose that we have a correspondence from a paracompact space to a complete, metrizable, locally convex, linear space. Then Michael's selection theorem [see Michael (1956)] tells us that every correspondence which is lower hemicontinuous (LHC) and takes (non-empty) closed and convex values admits a continuous selection. Convexity is achieved by considering probability mixtures over the set $W(t)$. The hurdle to applying Michael's theorem is the condition that the correspsondence be LHC. This will not in general be the case for the probabilistic convexification of W (refer again to fig. 3). Nonetheless, if W admits a LHC selection W', then the probabilistic convexification of this W' will be LHC (this we will check later on). Thus our problem is reduced to the search for a LHC selection W'. As in Allen (1985a), we shall let W' be the subcorrespondence that picks the 'nice' equilibria, where the definition of nice will be based on considerations of degree. Because we allow for a countable, rather than finite, number of equilibria at each t, our definitions and arguments will be slightly more delicate than in Allen.

Definition. A zero, $\bar{p}$, *of* f_t is *quasi-regular* if for any $\varepsilon > 0$ there exists some closed ball $B_\varepsilon \subset S$, centered at $\bar{p}$ and of radius less than ε, such that the degree of f_t restricted to ∂B_ε, the boundary of B_ε, is well-defined [i.e., $0 \notin f_t(\partial B_\varepsilon)$] and non-zero. [By degree of f_t on ∂B_ε we mean the degree of the map $p \to [1/\|f_t(p)\|] f_t(p)$ viewed as a map of the $(l-2)$ sphere into itself.]

For the definition of degree see Hirsch (1976), Mas-Colell (1985) or McLennan (1988). The discussion to follow will be phrased in terms of oriented degree theory but mod-2 degree theory works just as well. Of course, any regular zero is quasi-regular (and in fact yields degree either $+1$ or -1 for ε small enough) but the converse is not true; a critical zero may be quasi-regular.

Definition. W' is the subcorrespondence of W mapping from K to the quasi-regular zeros.

We need to demonstrate that this W' is LHC and non-empty valued. To do so a technical fact is needed.

Definition. A compact set A in a metric space M satisfies property $\mathscr{C}$ ($\mathscr{C}$ for 'covering') if for any ε the elements of A can be covered by a finite set of open balls $O_1, \dots, O_s$ of radius no more than ε such that the closures $B_1, \dots, B_s$ are disjoint.

Lemma 6. Let A be a compact subset of a metric space. Then A satisfies property $\mathscr{C}$ if it is countable.

Proof of Lemma 6. Since A is countable we may write $A=\{a_1,a_2,\ldots\}$. Fix an $\varepsilon>0$ no matter how small. Then we may find an open ball O_1 around a_1 of radius no more than ε such that its closure B_1 contains no point of A on its boundary. (If this were not possible, then A would be uncountable.) Consider the first point $\tilde{a}\notin O_1$. If there is no such point we are done. Otherwise, without loss of generality we may suppose $\tilde{a}=a_2$. Form an open ball O_2 of radius no more than ε around a_2 such that its closure B_2 is both disjoint from B_1 and contains no point of A on its boundary. Continuing in this way, either the procedure terminates with a finite number of balls, and we are done, or it does not terminate. If the latter, consider for each positive integer h the set $C_h=A\backslash\bigcup_1^h O_i$. If $c\in\bigcap_1^\infty C_h$, then c would not have been covered which is impossible because for some r, $c=a_r$, which is covered at the rth stage at the latest. Hence $\bigcap_1^\infty=C_h=\emptyset$. Since every C_h is compact, and $C_1\supset\cdots\supset C_h\supset\ldots$, we must have that $C_s=\emptyset$ for some s. Hence $\{B_1,\ldots,B_s\}$ is as desired. □

Because of the strict monotonicity of utility functions every $W(t)$ is compact. Therefore, in view of Corollary 3, we conclude that for every t, $W(t)$ satisfies property $\mathscr{C}$.

Lemma 7. Let B be a closed ball in S. If the (non-empty) zero set of f_t satisfies property $\mathscr{C}$ and if the degree of f_t is well-defined and non-zero on ∂B, then B contains a quasi-regular zero in its interior.

Proof. It is a well-known fact [see Hirsch (1976) or Mas-Colell (1985)] that B contains a zero in its interior. For specificity, suppose B has radius ε. By property $\mathscr{C}$ we know that we may cover the zeroes in the interior of B with finitely many disjoint closed balls B_i, each of radius no more than $\varepsilon/2$ and each with degree f_t well-defined on its boundary. Because the degree of f_t on ∂B is non-zero, and $f_t(\cdot)$ is non-zero outside each of the B_i, it follows that the degree of f_t on some ∂B_i must be non-zero as well. We have thus established that for some closed ball $B^1=B_i$, f_t has non-zero degree on ∂B^1. Continuing in this way, we generate a sequence of balls B_n with radius strictly decreasing and converging to zero and such that f_t is well-defined on ∂B_n and has non-zero degree. Thus $\cap B_n$ consists of a single point which by construction is a quasi-regular zero. □

We establish lower hemicontinuity as follows.

Lemma 8. If for each t the (non-empty) zero set of f_t satisfies property $\mathscr{C}$, then W' is LHC.

Proof. Fix some arbitrary $t\in K$ and consider any sequence $t_s\to t$. If p is a quasi-regular zero of f_t, then for any ε no matter how small there is a ball B_ε centered at p and of radius less than ε such that f_t restricted to ∂B_ε is of non-zero degree. By continuity, for s large enough, f_{t_s} is homotopic to f_t on ∂B_ε, hence f_{t_s} has non-zero degree on ∂B_ε. By Lemma 7, then, f_{t_s} has a quasi-regular zero p_{t_s} in the interior of B_ε. Of course, $\|p-p_{t_s}\|<\varepsilon$. Since ε was arbitrary, we are done. □

Establishing that W' is non-empty valued also follows from Lemma 7 since the boundary condition on excess demand implies that f_t has degree different from zero on a set diffeomorphic to the $l-2$ ball and close to the boundary of S.

Finally,

Proof of Proposition 5. It remains to be shown that the correspondence $t \to M(W'(t))$ is LHC. Let $t_s \to t$ and let $\mu \in M(W'(t))$. Fix an arbitrary $\varepsilon>0$. Choose a finite set $\{p_1,\dots,p_m\} \subset W'(t)$ with $\mu(\{p_1,\dots,p_m\})>1-\varepsilon$. For each $1 \leqq h \leqq m$ let $p_{hs} \to p_h$, $p_{hs} \in W'(t_s)$. Finally, for each s, take for μ_s a probability measure supported by $\{p_{1s},\dots,p_{ms}\}$ and such that $\mu_s(p_{hs}) \geqq \mu(p_h)$ for every $1 \leqq h \leqq m$. Because ε is arbitrarily small, this yields lower hemicontinuity. □

Appendix

The setup is as in Proposition 4. The reader is directed there for definitions. The statement below is for compact K.

Proposition A.1 If $l=2$, then there is a set $\tilde{\mathscr{T}}$ open and dense in $\mathscr{T}$ such that for every $\eta \in \tilde{\mathscr{T}}$ and every $t \in K$, the zeros of $f_\eta(\cdot,t)$ are locally isolated.

The corresponding result for the setting of Proposition 1 is true and is proved similarly.

Proof. Normalizing prices so that $p_2=1$ and appealing to Walras's Law, it is sufficient to consider f^1, the aggregate demand for the first good. For fixed $t \in K$, $f^1_\eta(\cdot,t)\colon \mathbb{R}_+ \to \mathbb{R}$. For such functions, a sufficient condition for local isolation of p_1 is that for some positive integer r:

$$D^r_{p_1} f^1(p_1,t) \neq 0. \tag{A.1}$$

Consider then the map $G_\eta = (f^1_\eta, D^1_{p_1} f^1_\eta, \dots, D^q_{p_1} f^1_\eta)\colon (0,\infty)\times K \to \mathbb{R}^{q+1}$. Remember that q is the dimension of K. Suppose that 0 is a regular value of G_η. Then $G_\eta(p_1,t)=0$ implies that $D^{q+1}_{p_1} f^1_\eta(p_1,t) \neq 0$. Therefore, p_1 is an isolated zero. The condition that 0 be a regular value of G_η is obviously open in $\mathscr{T}$. It remains to show that the condition is dense: for any η there is an η' arbitrarily nearby for which 0 is regular value of the associated $G_{\eta'}$.

By the Transversality Theorem (the style of the argument should by now be very familiar), it suffices to find a set of perturbation parameters y such that (1) $D_y G_\eta(p,t,y)$ has full rank, namely $q+1$, whenever $G_\eta(p,t,y)=0$, and (2) $G_\eta(p_1,t,0)=G_\eta(p_1,t)$.

Fix $\bar{t}$ and denote by $v(p_1,w)$ the indirect utility at $\bar{t}$ of the first consumer. We will consider additive perturbations of v, with $\mu_j(p_1,y)$ denoting the jth perturbation. There will be $q+1$ perturbations in all. For each j, μ_j is a symmetric j-form: $\mu_1(p_1,y)=y_1^1 p_1+y_1^2$ where $y_1\in\mathbb{R}^2$, $\mu_2(p_1,y)=y_2^1 p_1^2+y_2^2 p_1+y_2^3$, $y_2\in\mathbb{R}^3$, and so on. In general, $y_j\in\mathbb{R}^{j+1}$. Let Y denote the total space of parameters, namely $Y=\mathbb{R}^\gamma$ with

$$\gamma=\sum_{j=1}^{q+1}(j+1)=(q+1)(q+4)/2.$$

The perturbed indirect utility is then given by:

$$v_y=v+\sum_{j=1}^{q+1}\mu_j.$$

The perturbation is linear in y. It is also admissible: μ_j is smooth and if we restrict attention to $y\in Y$ sufficiently close to 0, then v_y is, locally, monotone and quasiconvex (with a bit of extra care the perturbation can be made globally admissible; see the proof of Proposition 1). Note also that this construction is independent of the particular $\bar{t}$. Using Roy's formula we see easily that we do indeed have $G_\eta(p_1,t,0)=G_\eta(p_1,t)$ for all p_1,t.

Let then $G_\eta(\bar{p},\bar{t},\bar{y})=0$. To simplify notation, and without loss of generality, we suppose that $\bar{y}=0$. To show that rank $D_y G(\bar{p},\bar{t},0)=q+1$, we define $q+1$ directions y^κ by $y^\kappa=(0,\dots,y_k,\dots,0)$ with y_k defined by

$$\mu_k(p_1,y_k)=(p_1-\bar{p}_1)^k.$$

Note that since μ_k is linear in y_k we have $\mu_k(p_1,sy_k)=s(p_1-\bar{p}_1)^k$ for any s.

Clearly it suffices to examine the demand of the first consumer. Let his perturbed demand be $\phi(p,w,y^\kappa)$. Put $\bar{w}=\bar{p}_1\omega_1^1(\bar{t})+\omega_2^1(\bar{t})$. Direct calculation (using Roy's formula) reveals that for any $0\leq r\leq k-2$,

$$D^r_{p_1}\phi(\bar{p}_1,\bar{w},sy^\kappa)=D^r_{p_1}\phi(\bar{p}_1,\bar{w},0),$$

while

$$D_{p_1}^{k-1}\phi(\bar{p}_1,\bar{w},sy^{\kappa}) = D_{p_1}^{k-1}\phi(\bar{p}_1,\bar{w},0) - \alpha D_{p_1}^{k}\mu_k(\bar{p}_1,sy^k)$$

$$= D_{p_1}^{k-1}\phi(\bar{p}_1,\bar{w},0) - s\alpha k!,$$

where $\alpha = 1/D_w v(\bar{p}_1,\bar{w}) \neq 0$. Thus $D_y G_\eta(\bar{p}_1,\bar{t},0)(y^{\kappa}) = (0,\ldots,0,\xi,*,\ldots,*)$ where $\xi = -\alpha k! \neq 0$ appears in the kth place. Therefore rank $D_y G_\eta(\bar{p}_1,\bar{t},0) \geqq q+1$ and we are done. □

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