Chapter 34

EQUILIBRIUM THEORY IN INFINITE DIMENSIONAL SPACES*

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*Support from the National Science Foundation and the Deutsche Forschungsgemeinschaft, Gottfried–Wilhelm–Leibniz–Förderpreis during BoWo'89 is gratefully acknowledged. Thanks are also due to R.A. Dana, P.K. Monteiro and N. Yannelis for careful reading of the manuscript.

Handbook of Mathematical Economics, Volume IV, Edited by W. Hildenbrand and H. Sonnenschein

1. Introduction

In this chapter, we attempt to give a summary account of the extension of the classical general equilibrium model to an infinite dimensional setting. Our account centers on the existence of competitive equilibrium.

The finite dimensional theory was surveyed by Debreu (1982) in Volume II of the *Handbook of Mathematical Economics*. Although some of the pioneering contributions to the infinite dimensional theory had already appeared at that point [Debreu (1954b), Gabszewicz (1968a,b), Bewley (1972, 1973), Prescott and Lucas (1972)], it has only been in the last ten years that the theory has undergone explosive growth.

In the classical finite dimensional theory, the commodity space is the canonical finite dimensional linear space $\mathbb{R}^n$. By contrast, there is no canonical infinite dimensional linear space. Different economic applications require models involving different (non-isomorphic) infinite dimensional linear spaces. Fortunately, the mathematical discipline of functional analysis has already been well developed as a tool for the abstract study of linear spaces. In this survey, we shall follow the methodology of functional analysis, and attack the existence problem from the abstract point of view. The advantage of this method is that it yields general results, capable of application in a wide variety of specific models. But the abstract approach also has a cost. Much interesting economics lies in the details of particular models. For example, in intertemporal models the functional analytic treatment typically abstracts away the inner recursiveness of the models, which are themselves at the heart of a rich body of economic theory [see Prescott and Mehra (1980)].

As an indication of the way in which different infinite dimensional spaces arise naturally in economics, we briefly describe three modeling problems which lead to quite different infinite dimensional commodity spaces.

(A) In intertemporal allocation problems, the natural commodity bundles are consumption streams. If we consider consumption of a single physical commodity, taking place at discrete intervals, over an infinite time horizon, the appropriate consumption streams are sequences of real numbers. Since the universe is finite, it is natural to consider only bounded sequences. We are led naturally, therefore, to consider the space l_∞ of bounded sequences (of real numbers). We interpret an element $x \in l_\infty$ as a discrete consumption stream, and $x(t)$ as consumption in the t-th period. Alternatively, we may consider consumption of a single physical commodity, taking place continuously through time, at a bounded rate, in which case we are led to consider the spaces $L_\infty([0, T])$ or $L_\infty([0, \infty])$ of bounded measurable functions. Again, we interpret an element $x \in L_\infty([0, T])$ (or $x \in L_\infty([0, \infty]))$ as a consumption stream, but now $x(t)$ is viewed as an instantaneous rate of consumption. For further discussion, see Bewley (1972, 1973). We might also consider consumption streams which are required to depend continuously on time, as in Gabszewicz (1968a,b) and Horsley and Wrobel (1988).

(B) In allocation problems under uncertainty, the natural commodity bundles are consumption patterns which depend on the state of the world. Such consumption patterns are most naturally modeled as random variables (i.e. measurable functions) on some probability space (S, Σ, μ). If, as is the case in many financial applications, we insist that consumption patterns have finite means and variances, we are led to consider the space $L_2(S, \Sigma, \mu)$ of square integrable functions on (S, Σ, μ). For an element $x \in L_2(S, \Sigma, \mu)$, we interpret $x(s)$ as consumption if state s occurs.

More generally, we can model intertemporal allocation problems under uncertainty by equipping the σ-algebra Σ, with an increasing time filtration $\{\Sigma_t\}$; Σ_t is the set of events that are known at time t. In this case, commodity bundles are naturally modeled as stochastic processes X_t adapted to the filtration $\{\Sigma_t\}$ (roughly speaking, this means that X_t depends only on information available up to time t). For further discussion, see, for instance, Duffie and Huang (1985) or Duffie (1988).

(C) In models of commodity differentiation, to allow for many different commodity characteristics, we are led to take as commodity space the space $M(K)$ of (signed) Borel measures on a compact metric space K. We interpret K as representing commodity characteristics, and a positive measure x on K as a commodity bundle comprising various characteristics in various quantities. That is, for each Borel set B of possible characteristics, $x(B)$ represents the number of units of those characteristics represented in the commodity bundle x [see Mas-Colell (1975), Jones (1983a, 1984), Podczeck (1985), Ostroy and Zame (1988)].

Following this Introduction, Section 2 summarizes the basic mathematical structures. This section is technical and may simply be used as reference. Next we concentrate on exchange economies and specify the basic assumptions which are maintained throughout the rest of the paper (Section 3), discuss the meaning of topological assumptions (Section 4), and introduce the fundamental concept of a price system (Section 5). We then isolate (Section 6) three difficulties which lie at the heart of the existence problem in infinite dimensions. Section 7 contains the basic fixed point argument, which is then applied in Sections 8 and 9 to a wide variety of infinite dimensional spaces. Sections 10, 11, 12 discuss important extensions. Section 13 summarizes a few alternative approaches to the existence problem. Section 14 incorporates production, and finally, Section 15 presents some concluding comments. Throughout we have tried to provide examples and counter-examples, as well as theorems.

As with any survey, this one reflects the points of view – and even prejudices – of the authors. We are well aware that this is not the only survey possible.

2. The essential mathematical structures

Throughout, we shall let L be a *topological vector space*; i.e. a (real) vector space, equipped with a topology τ having the property that the vector space operations (vector addition and scalar multiplication) are (jointly) continuous. We shall also assume that the topology τ is Hausdorff and *locally convex*; i.e. that τ has a neighborhood base at 0 consisting of convex sets. (Continuity of scalar multiplication implies, in addition, that there is always a neighborhood base at 0 consisting of convex and *symmetric* sets; i.e. convex sets W such that $W = -W$.)

By a *linear functional* on L, we mean a linear mapping from L to $\mathbb{R}$. We denote the value of the linear functional p at the vector x by $p \cdot x$.

The fundamental distinction between finite dimensional and infinite dimensional topological vector spaces is expressed in the existence and continuity of linear functionals. If L is finite dimensional, every linear functional is continuous, and every disjoint pair of convex sets can be separated by a linear functional; i.e. if A, B are disjoint convex sets then there is a (necessarily continuous) non-zero linear functional p on L such that $p \cdot x \leqslant p \cdot y$ for every $x \in A$, $y \in B$ (Minkowski's theorem). If L is infinite dimensional however, the existence of a continuous linear functional separating disjoint convex sets A, B is not guaranteed; indeed, there may even be no discontinuous linear functional separating A and B.

The most important facts about the existence of continuous linear functionals on locally convex spaces are the Hahn–Banach theorem and its corollaries, the extension theorem and the separation theorem. The Hahn–Banach theorem can be formulated in a number of ways. The following is the simplest.

Hahn–Banach Theorem. *Let L be a real vector space, $L_0 \subset L$ a subspace, $W \subset L$ a convex symmetric set containing 0 and $p : L_0 \to \mathbb{R}$ a linear functional such that $|p \cdot w| \leqslant 1$ for every $w \in W \cap L_0$. Then there is a linear functional $\tilde{p} : L \to \mathbb{R}$ which extends p and has the property that $|\tilde{p} \cdot w| \leqslant 1$ for every $w \in W$.*

In particular, if L is a locally convex topological vector space, L_0 is a subspace, and $p : L_0 \to \mathbb{R}$ is a continuous linear functional, then there is a continuous linear functional $\tilde{p} : L \to \mathbb{R}$ which extends p (this is the Hahn–Banach extension theorem).

In finite dimensional spaces, Minkowski's theorem guarantees that any two disjoint convex sets can be separated by a continuous linear functional. The corresponding result is false in infinite dimensional spaces, unless one of the convex sets has an interior point.

Separation Theorem. *Let L be a locally convex topological vector space and let A, B be disjoint convex sets, one of which has an interior point. Then there is a non-zero continuous linear functional $p : L \to \mathbb{R}$ such that $p \cdot x \leq p \cdot y$ for each $x \in A$, $y \in B$.*

In the particular case of most interest to us, the set A may consist of a single point x, and B may be the set of consumptions preferred to x. If B has an interior point, we can separate x from B, or equivalently, support B at x.

We denote the set of continuous linear functionals on L (the *dual space*) by L^*. The dual space is itself a vector space, and comes equipped with a number of natural topologies. To describe these topologies, it is convenient to abstract a bit and consider an arbitrary pair $\langle L, L' \rangle$ of vector spaces, together with a bilinear mapping $(x, p) \to p \cdot x : \langle L, L' \rangle \to \mathbb{R}$ which is non-singular in the sense that for each non-zero $x \in L$ there is a $p \in L'$ such that $p \cdot x \neq 0$ and for each non-zero $q \in L'$ there is a $y \in L$ such that $q \cdot y \neq 0$. Any such pairing gives rise to topologies on L and L', of which the most interesting are the weak topologies and the Mackey topology. The *weak topology* $\sigma(L, L')$ on L is the weakest topology for which the maps $x \to p \cdot x$ are continuous (for each $p \in L'$). Similarly, the *weak topology* $\sigma(L', L)$ on L' is the weakest topology for which the maps $p \to p \cdot x$ are continuous (for each $x \in L$). In terms of convergence of nets: $x_\alpha \to x$ in the topology $\sigma(L, L')$ exactly when $p \cdot x_\alpha \to p \cdot x$ for each $p \in L'$, and $p_\alpha \to p$ in the topology $\sigma(L', L)$ exactly when $p_\alpha \cdot y \to p \cdot y$ for each $y \in L$. The *Mackey topology* $\tau(L, L')$ is the topology for which convergence $x_\alpha \to x$ means that $p \cdot x_\alpha \to p \cdot x$ uniformly for p in any $\sigma(L', L)$-compact subset of L'. Similarly, the *Mackey topology* $\tau(L', L)$ is the topology for which convergence $p_\alpha \to p$ means that $p_\alpha \cdot y \to p \cdot y$ uniformly for y in any $\sigma(L, L')$-compact subset of L.

All these topologies are Hausdorff, locally convex vector space topologies. Moreover, when equipped with either the weak or the Mackey topology, the dual space of L is precisely L'. The fundamental fact about these topologies is that (among Hausdorff, locally convex topologies) the weak topology is the weakest with this property and the Mackey topology is the strongest (this is Mackey's theorem).

In particular, if we begin with the topological vector space L, equipped with the locally convex topology τ, then we obtain the dual pair $\langle L, L^* \rangle$. By Mackey's theorem, the topology τ lies between the weak topology $\sigma(L, L^*)$ and the Mackey topology $\tau(L, L^*)$. In particular, every $\sigma(L, L^*)$-closed set is τ-closed set, and every τ-closed set is $\tau(L, L^*)$-closed. For convex sets, we can say more: it follows from the Hahn–Banach theorem that the topologies $\sigma(L, L^*)$ and $\tau(L, L^*)$ (and hence the intermediate topology τ) have the same closed convex sets. For further information about topological vector spaces and duality, we refer to Schaefer (1971).

The crucial fact about the topology $\sigma(L^*, L)$, also called the weak star topology, is that many subsets of L^* are compact; this is Alaoglu's theorem.

Alaoglu's Theorem. *Let L be a locally convex topological vector space and let W be an open symmetric neighborhood of 0. Then the set $\{p \in L^*: |p \cdot w| \leq 1$ for every $w \in W\}$ is $\sigma(L^*, L)$-compact.*

In addition to the vector space structure, we shall wish to consider order structures. By an *ordered topological vector space* L we mean a topological vector space (assumed Hausdorff and locally convex) together with a reflexive, transitive, anti-symmetric relation $\leq$ on L. We assume that the order relation and the vector space structure are related in the following way: (a) if $x \leq y$ and $\alpha \in \mathbb{R}^+$ then $\alpha x \leq \alpha y$, (b) if $x \leq y$ and $0 \leq z$ then $x + z \leq y + z$. We define the *positive cone* $L^+ = \{x: x \geq 0\}$; note that L^+ is convex and is a proper cone, i.e. if $x \in L^+ \cap (-L^+)$ then $x = 0$. (Alternatively, given a proper convex cone $C \subset L$, we obtain an ordering of L by defining $x \leq y$ whenever $y - x \in C$.) We also assume that the ordering is continuous in the sense that the positive cone L^+ is closed. Note that if L is an ordered topological vector space, then the dual space L^* is also ordered, with positive cone $(L^*)^+ = \{p \in L^*: p \cdot x \geq 0$ for every $x \in L^+\}$. Moreover, the positive cone $(L^*)^+$ is evidently $\sigma(L^*, L)$-closed (and hence $\tau(L^*, L)$-closed).

For $x, y \in L$, define the order interval $[x, y] = \{z: y \leq z \leq y\}$. We say that a subset $A \subset L$ is *solid* if $[x, y] \subset A$ whenever $x, y \in A$.

The subset $A \subset L$ has a *supremum* (or *least upper bound*) if there is an element $\sup A \in L$ such that $x \leq \sup A$ for every $x \in A$ and $\sup A \leq y$ for every $y \in L$ which has the property that $x \leq y$ for every $x \in A$. Similarly, the subset $A \subset L$ has an *infimum* (or *greatest lower bound*) if there is an element $\inf A \in L$ such that $x \geq \inf A$ for every $x \in A$ and $\inf A \geq y$ for every $y \in L$ which has the property that $x \geq y$ for every $x \in A$. We usually write $x \wedge y$ rather than $\inf\{x, y\}$ and $x \vee y$ rather than $\sup\{x, y\}$. If every pair x, y of elements of L has a supremum $x \vee y$ and an infimum $x \wedge y$, we say that L is a *vector lattice* (or *Riesz space*). We write $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$ for the *positive* and *negative* parts of x; then $x = x^+ - x^-$. We write $|x| = x^+ + x^-$ for the *absolute value* of x. The notion of a vector lattice is much stronger than that of ordered vector space. For instance, if L is an n-dimensional ordered vector space, with positive cone L^+, then it is a vector lattice precisely when L^+ is generated (as a cone) by exactly n linearly independent vectors.

A linear functional $f: L \to \mathbb{R}$ is *order bounded* if it maps order intervals in L

to bounded subsets of $\mathbb{R}$. The collection of order bounded linear functionals on L is the *order dual* L^b. The order dual of a vector lattice is again a vector lattice. For $f, g \in L^b$, the supremum $f \vee g$ and infimum $f \wedge g$ are the (order bounded) linear functionals whose values at positive elements $x \in L^+$ are given by

$$f \vee g(x) = \sup\{f(y) + g(z): 0 \leqslant y, 0 \leqslant z, y + z = x\},$$

$$f \wedge g(x) = \inf\{f(y) + g(z): 0 \leqslant y, 0 \leqslant z, y + z = x\}.$$

A fundamental fact about vector lattices is the Riesz decomposition property.

Riesz Decomposition Property. Let L be a vector lattice and let $x_1, \ldots, x_n, z$ be positive elements of L such that $z \leqslant \Sigma x_i$. Then there are positive elements $z_1, \ldots, z_n$ of L such that $z = \Sigma z_i$ and $z_i \leqslant x_i$ for each i.

If the lattice operations $(x, y) \to x \wedge y$ and $(x, y) \to x \vee y$ are (uniformly) continuous, then L is a *topological vector lattice*. Continuity of the lattice operations is equivalent to the topology τ being *locally solid*; i.e. having a base of neighborhoods of 0 consisting of (symmetric, convex) solid sets. If L is a topological vector lattice, then every continuous linear functional on L is order bounded, so the dual space L^* is a subspace of the order dual L^b. Indeed, L^* is an *order ideal* in L^b; i.e. if $f \in L^*$, $g \in L^b$ and $0 \leqslant g \leqslant f$ then $g \in L^*$. In particular, L^* is itself a vector lattice.

The most important class of topological vector lattices are the *normed lattices*; i.e. topological vector lattices in which the topology is defined by a *norm* $\|\cdot\|: L \to \mathbb{R}^+$. We require that the norm satisfy: (a) $\|\alpha x\| = |\alpha|\,\|x\|$ for $\alpha \in \mathbb{R}$ and $x \in L$; (b) $\|x + y\| \leqslant \|x\| + \|y\|$; (c) $\|x\| = 0$ exactly if $x = 0$; (d) $\|x\| \leqslant \|y\|$ whenever $0 \leqslant x \leqslant y$. If in addition, L is complete in the metric induced by the norm, we say that L is a *Banach lattice*.

The most important examples of Banach lattices are the Lebesgue spaces. Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space. For $1 \leqslant p < \infty$, we write $L_p(\Omega, \mathscr{F}, \mu)$ for the space of (equivalence classes of) measurable functions $f: \Omega \to \mathbb{R}$ for which the norm

$$\|f\| = \left\{\int |f|^p \, \mathrm{d}\mu\right\}^{1/p}$$

is finite. We write $L_\infty(\Omega, \mathscr{F}, \mu)$ for the space of (equivalence classes of) bounded measurable functions $f: \Omega \to \mathbb{R}$, with the norm

$$\|f\|_\infty = \inf\{M: \mu\{x: f(\omega) < M\} = 0\}.$$

The ordering on $L_p(\Omega, \mathcal{F}, \mu)$ is defined pointwise; i.e. $f \geq g$ if $f(\omega) \geq g(\omega)$ almost everywhere. For $1 \leq p < \infty$, the dual of $L_p(\Omega, \mathcal{F}, \mu)$ is $L_q(\Omega, \mathcal{F}, \mu)$, where $(1/p) + (1/q) = 1$. The pairing is given by

$$f \cdot g = \int f(\omega) g(\omega) \, \mathrm{d}\mu .$$

The dual of $L_\infty(\Omega, \mathcal{F}, \mu)$ is the space $\mathrm{ba}(\Omega, \mathcal{F}, \mu)$ of bounded, finitely additive set functions on $\mathcal{F}$ which vanish on sets of μ-measure 0. This is much larger than $L_1(\Omega, \mathcal{F}, \mu)$, which may be identified as the subspace of $\mathrm{ba}(\Omega, \mathcal{F}, \mu)$ consisting of countably additive set functions.

If follows from Alaoglu's theorem that, for $1 < q \leq \infty$, subsets $A \subset L_q(\Omega, \mathcal{F}, \mu)$ that are norm bounded and closed with respect to the weak star topology $\sigma(L_q, L_p)$ are also compact with respect to this topology. In particular, order intervals are weak star compact. Since $L_1(\Omega, \mathcal{F}, \mu)$ is not the dual of $L_\infty(\Omega, \mathcal{F}, \mu)$, Alaoglu's theorem does not guarantee $\sigma(L_1, L_\infty)$ compactness (i.e. weak compactness) of order intervals. However, compactness of order intervals in $L_1(\Omega, \mathcal{F}, \mu)$ is a well-known fact [see Schaefer (1974)].

For more information about topological vector lattices and Banach lattices, we refer to Schaefer (1974), Aliprantis and Burkinshaw (1978, 1985), and Aliprantis, Brown and Burkinshaw (1989b).

Linear space structures were introduced in economics by Debreu (1954b); vector lattices were introduced by Aliprantis and Brown (1983).

3. Basic assumptions

From now until Section 14, when we introduce production, we shall restrict our attention to the pure exchange case. It is convenient to collect here the basic assumptions that will be maintained throughout. These assumptions are a minimal collection; we usually need to require more. In Section 15 we discuss briefly some of the ways in which the present assumptions may be relaxed.

The commodity space L is a (Hausdorff) locally convex, topological vector space. We denote the topology by τ. The commodity space is endowed with an order structure $\geq$ for which the positive orthant $L^+ = \{x: x \geq 0\}$ is a non-degenerate (i.e. $L^+ \neq \{0\}$), closed, convex cone.

There are N consumers. Each consumer i is described by a *consumption* set $X_i \subset L^+$, a *preference relation* $\succcurlyeq_i$ on X_i, and an *endowment vector* $\omega_i \in L^+$. It is assumed that, for each i:

(a) X_i is closed, convex, and satisfies the free-disposal property $X_i + L^+ \subset X_i$;

(b) the preference relation $\succcurlyeq_i$ is a complete pre-order which is *τ-continuous* (i.e. $\succcurlyeq_i$ is a closed subset of $X_i \times X_i$), *convex* (i.e. each of the sets $\{y\colon y \succcurlyeq_i x\}$ is convex), and monotone (i.e. $x + v \succcurlyeq_i x$ for every $x \in X_i$ and $v \in L^+$). We shall also assume *strict monotonicity* in the (relatively weak) sense that there is some $v_0 \in L^+$ such that $x + \alpha v_0 \succ_i x$ for every $x \in X_i$ and $\alpha > 0$.

(c) $\omega_i \in X_i$.

4. Preferences and continuity

If the commodity space L is finite dimensional, it admits a unique locally convex vector space topology. If L is infinite dimensional however, it will always admit many such topologies. Some of these topologies will be comparable (i.e. some will be finer and some will be coarser), and some may not be. It should be stressed that the choice of topology on L can only be dictated by economic, rather than mathematical, considerations.

In essence, a topology on L is a notion of "closeness" between vectors in L. For economic purposes, it seems natural to treat elements x, $y \in L$ as close if they are regarded as such by agents in the economy (consumers and producers). This suggests that the relevant restrictions on a topology on L are that the given consumption sets (and production sets) be closed and that the given preferences be continuous. However, we shall actually adopt a weaker restriction; we say that the topology σ is *compatible* if consumption sets are closed and preferences are upper semi-continuous (i.e. for each consumer i, the preferred sets $\{y\colon y \succcurlyeq_i x\}$ are σ-closed). When we consider production, we shall also require that production sets be closed.

It might appear that allowing for upper semi-continuity of preferences, rather than requiring full continuity, is nitpicking. In fact this is not so; indeed the distinction is quite important. The reason for this is that, in locally convex spaces, it is "easier" for a convex set to be closed than it is for a non-convex set. For instance, if $C \subset L$ is convex and closed in the locally convex topology τ, it is automatically closed in the weak topology σ associated with τ. (This is a consequence of the Separation Theorem; see Section 2.) Thus, if $\succcurlyeq$ is a τ-continuous, convex preference relation, then all preferred sets $\{y\colon y \succcurlyeq x\}$ are τ-closed and convex, hence σ-closed. In particular, the weak topology associated with any compatible topology is itself a compatible topology. As we shall see, this fact plays an important role in many places. In particular, it makes it possible to begin with a topology in which preferences are continuous, but carry out the technical work in a weaker – but still compatible – topology.

Because infinite dimensional spaces admit many topologies, many economic restrictions on preferences can be expressed in a natural way in terms of topological hypotheses. Some examples may serve to illustrate this; the first two are from Bewley (1972).

Example 4.1. Let $L = l_\infty$. As in the Introduction, we interpret sequences $x \in l_\infty$ as (discrete) consumption streams (of a single physical good) over an infinite time horizon. Consider a consumer whose consumption set is the positive cone and whose preference relation is $\succcurlyeq$. Norm continuity of $\succcurlyeq$ imposes no restrictions on the time preferences of this consumer. In particular, norm continuity is consistent with preferences which give the same utility to a single unit of consumption, independent of the date. On the other hand, upper semi-continuity of $\succcurlyeq$ with respect to the weak topology $\sigma(l_\infty, l_1)$ imposes a kind of upper impatience (upper myopia). To be specific, suppose that $x \succ y$; for each n, let $z^n \in l_\infty$ be the consumption stream which is 0 in the first n periods and 1 thereafter. Then $y + z^n \rightarrow y$ in the weak topology $\sigma(l_\infty, l_1)$, so upper semi-continuity implies that $x \succ y + z^n$ for sufficiently large n. Informally, gains in the distant future are negligible. Similarly, lower semi-continuity of $\succcurlyeq$ in the weak topology $\sigma(l_\infty, l_1)$ corresponds to lower impatience (lower myopia): losses in the distant future are negligible. The typical impatient (myopic) utility function is of course $U(x) = \Sigma\, \delta^n u(x(n))$, for some single period utility function u and discount factor $\delta < 1$. It is important to note, however, that impatience (myopia) is not tied to separability: there are many impatient (myopic) preferences that are not separable. For an extensive analysis of impatience (myopia) and its implications, see Brown and Lewis (1981), Araujo (1985), Raut (1986), and Sawyer (1987).

Example 4.2. Let $L = L_\infty(S, \Sigma, \mu)$, where (S, Σ, μ) is a probability space; to distinguish this case from the preceding, we assume that (S, Σ, μ) is not purely atomic. We interpret S as the set of states of the world, and elements $x \in L_\infty(S, \Sigma, \mu)$ as random, state dependent consumption patterns. As in the preceding example, norm continuity of a preference relation $\succcurlyeq$ on the positive cone $L_\infty(S, \Sigma, \mu)^+$ has no strong economic implications. However, upper semi-continuity in the weak topology $\sigma(L_\infty, L_1)$ (or, equivalently if $\succcurlyeq$ is convex, in the Mackey topology $\tau(L_\infty, L_1)$) has a natural and important interpretation. To be specific, suppose that $x \succ y$; let $\{E^n\}$ be a sequence of measurable sets such that $\mu(E^n) \rightarrow 0$, and let z^n be the characteristic function of E^n. Then $y + z^n \rightarrow y$ in the weak topology $\sigma(L_\infty, L_1)$ (and in the Mackey topology $\tau(L_\infty, L_1)$), so upper semi-continuity implies that $x \succ y + z^n$ for sufficiently large n. Informally, gains in events of low probability are negligible.

This example also provides a convenient place to illustrate our point about the distinction between continuity and upper semi-continuity of preferences.

The most important preferences are given by von Neumann–Morgenstern utility functions:

$$u(x) = \int v(x(t))\,\mathrm{d}\mu(t)$$

for $v : [0, \infty) \to (-\infty, \infty)$ a concave function. As Bewley (1972) shows, such utility functions are concave and continuous in the Mackey topology (and are therefore upper semi-continuous in the weak topology), but they are continuous in the weak topology only if the underlying felicity function v is linear. (Keep in mind that we have assumed the probability space (S, Σ, μ) is not purely atomic.) It is instructive to see why this is so for the typical case $S = [0, 1]$, Σ = Borel sets, μ = Lebesgue measure (the general case is quite similar). Since v is concave and not linear, we can find positive numbers $\alpha < \beta$ such that

$$v(\tfrac{1}{2}(\alpha + \beta)) > \tfrac{1}{2}v(\alpha) + \tfrac{1}{2}v(\beta)\,.$$

For each n, let $r^n : [0, 1] \to (-\infty, \infty)$ be the nth Rademacher function,

$$r^n(t) = \begin{cases} +1 & \text{if } m/2n \le t < (m+1)/2n,\ m \text{ even}\,, \\ -1 & \text{if } m/2n \le t < (m+1)/2n,\ m \text{ odd}\,. \end{cases}$$

This construction guarantees that, for each n, $\{t: r^n(t) = +1\}$ and $\{t: r^n(t) = -1\}$ have measure $\frac{1}{2}$, and that $q \cdot r^n \to 0$ for each $q \in L_1(S, \Sigma, \mu)$ (i.e. the sequence $\{r^n\}$ of Rademacher functions converges weakly to 0). Hence, if we set

$$x^n(t) = \tfrac{1}{2}(1 - r^n(t))\alpha + \tfrac{1}{2}(1 + r^n(t))\beta\,,$$

we obtain a sequence $\{x^n\}$ of positive functions in $L_\infty(S, \Sigma, \mu)$ such that, for each n, $\{t: x^n(t) = \alpha\}$ and $\{t: x^n(t) = \beta\}$ have measure $\frac{1}{2}$, and $\{x^n\}$ converges weakly to the constant function $\frac{1}{2}(\alpha + \beta)$. It follows that

$$\begin{aligned} u(x^n) &= \int v(x^n(t))\,\mathrm{d}\mu(t) \\ &= \tfrac{1}{2}v(\alpha) + \tfrac{1}{2}v(\beta) \\ &< v(\tfrac{1}{2}(\alpha + \beta)) \\ &= u(\tfrac{1}{2}(\alpha + \beta))\,; \end{aligned}$$

that is u is not weakly lower semi-continuous.

Example 4.3. Set $L = M(K)$, the space of (signed) Borel measures on a compact metric space K. As in the Introduction, we interpret K as representing commodity characteristics, and a positive measure $x \in M(K)^+$ as a commodity bundle comprising various characteristics in various quantities. Preferences that are continuous with respect to the weak topology $\sigma(M(K), C(K))$ find commodity bundles comprised of approximately equal quantities of nearby characteristics to be near perfect substitutes. In particular, given positive measures x, y we may choose sequences $x^n \to x$, $y^n \to y$ (weakly) such that each x^n, y^n has finite support. Then $x > y$ if and only if $x^n > y^n$ for all sufficiently large n [for applications to models of commodity differentiation, see Mas-Colell (1975), Jones (1983a, 1984), Podczeck (1987), Ostroy and Zame (1988)].

It should not be imagined, however, that all economic restrictions on preferences can be expressed as continuity requirements on preferences. The following example, which we shall encounter repeatedly in several guises, may serve to illustrate the point.

Example 4.4. Set $L = M([0, 1])$. Following Huang and Kreps (1987), we interpret the interval $[0, 1]$ as time, and a positive measure $x \in M([0, 1])^+$ as total consumption of a single physical commodity, so that $x([a, b])$ is consumption in the time interval $[a, b]$, etc. Motivated by the idea that consumptions at nearby times should be uniformly good substitutes, Huang and Kreps are led to focus on the weak topology $\sigma(M([0, 1]), \mathrm{Lip}([0, 1]))$ arising from the pairing of $M([0, 1])$ with the space $\mathrm{Lip}([0, 1])$ of Lipschitz functions [see also Jones (1983a, 1984)]. The weak topology $\sigma(M([0, 1]), \mathrm{Lip}([0, 1]))$ is weaker than the weak topology $\sigma(M([0, 1]), C([0, 1]))$, so in principle, continuity with respect to the former topology is a stronger requirement than continuity with respect to the latter. However, we are really interested only in preference relations defined on the positive cone $M([0, 1])^+$, and it may be shown that, on the positive cone, these two topologies coincide.

The point is simply that continuity properties of preferences may not allow us to discriminate finely enough between economic restrictions on preferences. As we shall see later, we may need to appeal to other considerations, such as restrictions on marginal rates of substitution.

As will become clear in subsequent sections, we place a great deal of emphasis on weak topologies, because the existence of a compatible weak topology is crucial in establishing the existence of an equilibrium.

A final observation: in what follows, we shall typically assume that preferences can be represented by utility functions. Given our monotonicity conditions, this involves no loss of generality [see Fishburn (1983), Mas-Colell (1986a), Shafer (1984), Monteiro (1987), Richard and Zame (1986)]. Similarly, there is no loss of generality in assuming that utility functions are also continuous.

5. Prices

Unless we specify to the contrary, by a *price* (or *price system*), we shall always mean a linear functional $p : L \to \mathbb{R}$ which is continuous with respect to the given topology τ on L. This definition demands some comment.

First, we require that p be linear. Since this is a familiar requirement, and its interpretation in the infinite dimensional setting is no different from its interpretation in the finite dimensional setting, we shall not elaborate on it.

Second, we require that p be defined and finite for each $x \in L$; i.e. that every (conceivable) commodity bundle be priced. This is certainly a desirable property, but it is also a strong one. This is especially true in the infinite dimensional setting because it is frequently the case that not all commodity bundles are "present in the market". A simple example may serve to illustrate the point. As we shall see in Section 10, this example is entirely representative (at least for exchange economies) of the situation in commodity spaces for which the positive cone has an empty interior.

Example 5.1. Take $L = L_2([0, 1])$, with consumption sets $X_i = L^+$. Let the aggregate endowment ω be the constant function with value 1. If ω has a finite price and prices are positive, then every commodity bundle $x \in L_2([0, 1])$ with $0 \leqslant x \leqslant \omega$ also has a finite price. Hence every commodity bundle y having the property that $y = \lambda x$ for some $x \in L$, $0 \leqslant x \leqslant \omega$ and $\lambda \in \mathbb{R}$, also has a finite price. However, since ω is identically 1, this set of commodities is precisely $L_\infty([0, 1])$, which is of course a proper subset of $L_2([0, 1])$. In particular, if $p \in L_1([0, 1])^+ \supset L_2([0, 1]) = L^*$, then p assigns a finite price to every element of $L_\infty([0, 1])$ (and hence to every commodity bundle "present in the market"), but if $p \not\in L_2([0, 1])$, then p does not assign a finite price to all elements of $L_2([0, 1])$ (so some conceivable commodity bundles are left unpriced).

Finally, we require that p be continuous. In part this is merely a mathematical and methodological desideratum. In some settings, continuity of prices will be a weak requirement, or will follow automatically. For instance, our monotonicity assumptions entail that equilibrium prices are positive, and in many commodity spaces (Banach lattices, in particular), positive linear functionals are automatically continuous.

In general however, continuity of prices reflects the choice of topology, and as we have already discussed, the choice of topology has economic meaning. To put it another way, continuity of equilibrium prices with respect to a weak topology yields more economic information than continuity of prices with respect to a strong topology. Ideally, we should ask that prices be continuous with respect to the weakest topology with respect to which preferences are continuous; call it σ. If consumption sets have non-empty interior with respect to the topology σ (in particular, since we require $X_i + L^+ \subset X_i$, if the positive cone L^+ has a non-empty interior with respect to σ), this is an unambiguous requirement and is automatically satisfied. (The half space defined by a supporting price must contain a preferred set, and hence a set which is open with respect to σ; this is enough to guarantee that p is continuous.) Unfortunately, consumption sets will generally have empty interior with respect to σ, and as the following example shows, it will not always be possible to find supporting prices that are continuous with respect to σ.

Example 5.2. As in Examples 4.3 and 4.4, we consider the commodity space $L = M([0, 1])$, paired with $L' = C([0, 1])$. Define the linear utility function $u : L^+ \to \mathbb{R}$ by

$$u(x) = \int t^{1/2}\, \mathrm{d}x(t)\,.$$

This utility function is continuous with respect to the weak topology $\sigma(L, L')$. If the endowment ω is the Lebesgue measure, then the unique supporting price at ω is the function $p \in C([0, 1]) = L'$ defined by $p(t) = t^{1/2}$, and p is continuous with respect to the topology $\sigma(L, L')$.

On the other hand, we may also consider the pairing of $L = M([0, 1])$ with $L'' = \mathrm{Lip}([0, 1])$. The utility function u is also continuous with respect to the even weaker topology $\sigma(L, L'')$, because the topologies $\sigma(L, L'')$ and $\sigma(L, L')$ coincide on L^+. However, the unique supporting price p is not continuous with respect to the topology $\sigma(L, L'')$, because it is not a Lipschitz function.

The end products of any equilibrium theory are equilibrium allocations and equilibrium prices. We require equilibrium prices to be continuous, but this requirement has a number of possible expressions:

(i) there is at least one equilibrium allocation supported by a continuous price (but there might also be equilibrium allocations supportable only by discontinuous prices);

(ii) every equilibrium allocation can be supported by a continuous price (but some equilibrium allocations might also be supportable by continuous prices);

(iii) every equilibrium price is continuous.

Of these, (i) seems a bit too weak, since there might be no natural way to decide which equilibrium allocation is the "correct" one. (This situation does not seem to have arisen in applications, but it has not been thoroughly studied.) On the other hand, (iii), while perhaps the most desirable, seems to be too much to ask for in general. In some settings, it will be possible to make "trivial" alterations in an equilibrium price which render it discontinuous and yet leave its equilibrium nature unchanged. For most purposes, (ii) is satisfactory, since it says that the set of equilibrium allocations is not affected by the methodological requirement of continuity. For related discussion, see Bewley (1972), Yannelis and Zame (1986), Podczeck (1987), Ostroy and Zame (1988) and Gilles and LeRoy (1987).

6. The main difficulties

In this section, we discuss some of the main difficulties that arise in infinite dimensional equilibrium theory. We do not suggest that they are the only difficulties, but they are central ones. Moreover, none of these difficulties are present in the finite dimensional setting, so they illuminate the differences between the finite dimensional and infinite dimensional theories. The three difficulties we isolate are:

(1) attainable sets may not be compact;
(2) preferred sets may not be supportable by prices;
(3) wealth may not be jointly continuous as a function of quantities and prices.

6.1. Compactness

The first difficulty is that some of the sets which are bounded in finite dimensions may not be bounded in the infinite dimensional setting. Indeed, this is typically the case for budget sets. For instance if the commodity space is $L = L_\infty([0, 1])$, the consumption set $X_i = L_\infty([0, 1])^+$, the endowment $\omega_i \in L_\infty([0, 1])^+$ is non-vanishing, and the price $p \in L_1([0, 1])^+$ is not 0, then the budget set $\{x \in L_\infty([0, 1])^+ : p \cdot x \leq p \cdot \omega_i\}$ is *never* bounded. It turns out, however, that this unboundedness of budget sets, while a serious obstacle for demand theory, is not a serious obstacle for the existence of equilibrium, and can be sidestepped by a suitable truncation argument.

Of more concern is the fact that the *attainable set*

$$Z = \left\{(x_1, \ldots, x_N) \in L^N : x_i \in X_i, \sum x_i \leq \omega\right\}$$

need not be bounded in the appropriate sense. (That it is always closed follows from the closedness of the consumption sets X_i and the positive cone L^+, and the continuity of addition.) An example will illustrate the point.

Example 6.1. Let $L = C^1([0, 1])$, the space of continuously differentiable functions on [0, 1], with the norm

$$\|x\|_1 = \sup|x(t)| + \sup|x'(t)|$$

and pointwise ordering. With two consumers, having consumption sets $X_1 = X_2 = L^+$ and endowments $\omega_1 = \omega_2 \equiv 1$, the attainable set is

$$Z = \{(x_1, x_2): x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 2\} .$$

Since arbitrarily small functions may have arbitrarily large derivatives, Z is evidently unbounded (in the norm $\|\cdot\|_1$).

A sufficient condition for norm boundedness of Z is that the norm and order structures of the commodity space make it into a Banach lattice. Then of course, Z will be norm bounded since, by construction, it is order bounded. However, even when the attainable set is bounded, it need not be compact. (Keep in mind that the Heine–Borel theorem is not generally valid in infinite dimensional spaces: closed and bounded sets need not be compact.) For instance, if $L = L_\infty([0, 1])$, $X_1 = X_2 = L^+$ and $\omega_1 = \omega_2 \equiv 1$, then of course the attainable set is

$$Z = \{(x_1, x_2): x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 2\}$$

which is norm bounded but not norm compact.

Roughly speaking, the attainable set Z will be compact in the topology of L only in two circumstances: if consumption sets are "thin enough" that order bounded sets are compact, or if order bounded sets themselves are "thin enough" to be compact. Two examples may serve to illustrate.

Example 6.2. (a) If $L = C([0, 1])$, then closed, norm bounded, equicontinuous subsets of L are norm compact (Ascoli's theorem). Hence, if norm bounded subsets of each consumption set X_i are equicontinuous, then the attainable set Z will be also norm compact [see Horsley and Wrobel (1988)]. Note, however, that norm bounded subsets of consumption sets cannot be equicontinuous if consumption sets contain the positive cone $C([0, 1])^+$, so this assumption is incompatible with our basic assumptions. However, if each consumption set is of the form $X_i = \hat{X}_i + L^+$, where $\hat{X}_i$ has the property that norm bounded subsets are equicontinuous, we shall still be able to push the analysis forward.

(b) If $L = l_p$, for $1 \leq p < \infty$, then order intervals $[0, \omega]$ are norm compact, whence the attainable set Z is also norm compact.

As we shall see, however, economic considerations lead us to consumption sets which may coincide with the positive cone, and to commodity spaces in which order intervals are not compact in the given topology of L. Hence we cannot expect the attainable set to be compact in the given topology of L. Fortunately, it is usually not necessary that the attainable set Z be compact in the given topology of L; all that is necessary is that the attainable set be compact in some (weaker) compatible topology. Some examples follow.

Example 6.3. (a) If L is a reflexive Banach lattice (e.g. $L = L_p(S, \Sigma, \mu)$ for $1 < p < \infty$), then all norm closed, bounded, convex sets are compact in the weak topology $\sigma(L, L^*)$ (Alaoglu's theorem; see Section 2). As noted in Section 4, preferences that are convex and norm continuous are automatically upper semi-continuous in the weak topology $\sigma(L, L^*)$, so we obtain compactness of the attainable set in a compatible topology for free.

(b) If $L = L_1(S, \Sigma, \mu)$, which is not a reflexive space, then norm closed, bounded, convex sets need not be compact in the weak topology $\sigma(L, L^*)$ (indeed the unit ball is not weakly compact). Nevertheless, order intervals are weakly compact (i.e. $L_1(S, \Sigma, \mu)$ has *order continuous norm*, see Section 2), so we again obtain compactness of the attainable set in a compatible topology for free.

(c) If L is the dual of a Banach lattice L_* (e.g. $L = L_\infty(S, \Sigma, \mu)$, which is the dual of the Banach lattice $L_* = L_1(S, \Sigma, \mu)$, or $L = M(K)$, which is the dual of the Banach lattice $L_* = C(K)$), then convex, norm bounded sets that are closed in the weak star topology $\sigma(L, L_*)$ are also weak star compact (Alaoglu's theorem again). Hence the attainable set will be weak star compact provided only that consumption sets are weak star closed; this will be so if consumption sets coincide with the positive cone. As discussed in Section 5, the weak star topology $\sigma(L, L_*)$ will be compatible whenever the Mackey topology $\tau(L, L_*)$ is compatible.

(d) More generally, let L be the dual of the Banach lattice L_*, and let L' be a separating subspace of L^* (e.g. $L = M([0,1])$, $L_* = C([0,1])$, $L' = \mathrm{Lip}([0,1])$). Then the topology $\sigma(L, L')$ is Hausdorff and is weaker than the weak star topology $\sigma(L, L_*)$, so these two topologies coincide on weak star compact sets. (The identity mapping of a weak star compact set K into itself is $\sigma(L, L_*)$ to $\sigma(L, L')$ continuous. Continuous mappings preserve compactness, and compact subsets of Hausdorff spaces are closed. Hence the identity mapping sends closed sets to closed sets, whence its inverse is continuous also, so the topologies on K coincide.) Hence the attainable set will be $\sigma(L, L')$ compact provided that consumption sets are $\sigma(L, L')$ closed.

Compactness of the attainable set with respect to a compatible topology has many useful consequences. The one which we use most often is closedness of the utility possibility set.

To be precise, choose utility functions $u_i : X_i \to \mathbb{R}$ representing the given preferences (as we remarked at the end of Section 4, our monotonicity assumptions guarantee that this is always possible). Write $u = (u_1, \ldots, u_N)$. The *utility possibility* set is

$$U = u(Z) - (\mathbb{R}^N)^+$$
$$= \{(u_1(x_1), \ldots, u_N(x_N)) \in \mathbb{R}^N : (x_1, \ldots, x_N) \in Z\} - (\mathbb{R}^N)^+ .$$

Note that monotonicity of preferences implies that U is bounded above by $(u_1(\omega), \ldots, u_N(\omega))$. It is easily seen that the compactness of Z and the upper semi-continuity of each u_i together imply that U is closed. (If $\{u(x^n)\}$ is a sequence in U converging to $v \in \mathbb{R}^N$, compactness of Z implies the existence of a subnet of the sequence $\{x^n\}$, convergent to some $x \in Z$. Upper semi-continuity of each u_i implies that $v \leq u(y)$, so $v \in u(X) - (\mathbb{R}^N)^+ = U$.) Without these two hypotheses, the utility possibility set U may not be closed; indeed, Pareto optima need not exist at all.

Example 6.4. Let $L = l_\infty$, $X_1 = X_2 = l_\infty^+$, $\omega_1 = \omega_2 = (1, 1, \ldots)$, and define utility functions by

$$u_1(x_1) = \sum 2^{-t} x_1(t) ,$$

$$u_2(x_2) = \liminf x_2(t) .$$

These utility functions are norm continuous, but u_2 is not $\sigma(l_\infty, l_1)$ upper semi-continuous; the set of allocations is $\sigma(l_\infty; l_1)$ compact but not norm compact. The utility possibility set is

$$U = \{(a_1, a_2) \in \mathbb{R}^2 : a_1 < 2 \text{ and } a_2 \leq 2, \text{ or } a_1 \leq 2 \text{ and } a_2 \leq 0\} ,$$

which is evidently not closed [see Araujo (1985)].

6.2. *Supportability*

If L is finite dimensional, $C \subset L$ is a convex subset, and $x \in L \backslash C$, then Minkowski's theorem guarantees that we can find a non-zero linear functional p separating x from C (i.e. $p \cdot x \leq p \cdot z$ for every $z \in C$). Taking C to be the set of consumption bundles strictly preferred to x, we conclude as usual that (convex) preferred sets can be price supported. But if L is infinite dimensional, the existence of separating functionals and supporting prices is not guaranteed.

Example 6.5. Let $L = l_2$, and define a utility function $u : l_2^+ \rightarrow \mathbb{R}$ by $u(x) = \Sigma\, v_t(x(t))$, where

$$v_t(x(t)) = \begin{cases} 2^t x(t) & \text{if } x(t) \leq 2^{-2t}, \\ 2^{-t}[x(t) + 1 - 2^{-2t}] & \text{if } x(t) > 2^{-2t}. \end{cases}$$

It is easily checked that u is norm continuous (indeed, even weakly continuous), concave and monotone. But if $\omega \in l_2^+$ is defined by $\omega(t) = 2^{-4t}$, then the preferred set to ω cannot be supported by a non-zero price. (The only candidates are multiples of the sequence $\{2^t\}$, which do not define linear functionals on l_2, even discontinuous ones.)

Note that the utility function u is defined on the positive cone l_2^+, but cannot be extended (as a continuous, concave function) to all of l_2.

In the infinite dimensional setting, the Separation Theorem guarantees that it will be possible to separate a convex set C from a point $x \not\in C$, provided that the interior of C is not empty (see Section 2). Hence, if consumption sets have non-empty interior, then the continuity and convexity of preferences will guarantee that preferred sets (which in this case will also have non-empty interior) can be price supported. From the point of view of supporting preferred sets, therefore, the best-behaved commodity spaces are those for which the positive cone has non-empty interior. Of the spaces discussed to this point, only $C(K)$ and L_∞ have this property (and in a certain sense, these are the "universal" spaces with this property; see Section 10). In other spaces, there is no alternative but to make assumptions on preferences that guarantee supportability of preferred sets.

6.3. *Joint continuity*

The wealth map $(x, p) \to p \cdot x$ arises in many arguments in equilibrium theory. In the finite dimensional setting, this map is jointly continuous, and this continuity plays an important role (in fixed point arguments for instance). In the infinite dimensional setting, there are many possible topologies on the commodity space L and its dual L^*, and hence many possible senses in which we could ask for the wealth map to be jointly continuous. In order that the set of allocations be compact, we are led to consider a weak topology on the commodity space L; in order that the set of supporting prices be compact, we will similarly be led to consider a weak topology on the price space L^*. Unfortunately, such a pair of choices usually leads to failure of joint continuity of the wealth map.

Example 6.6. Let $L = L_2([0, 1])$, so $L^* = L_2([0, 1])$. As in Example 4.2, let r^n be the nth Rademacher function,

$$r^n(t) = \begin{cases} +1 & \text{if } m/n \leq t < (m+1)/n,\ m \text{ even}, \\ -1 & \text{if } m/n \leq t < (m+1)/n,\ m \text{ odd}, \end{cases}$$

and set $x^n = p^n = 1 + r^n$. Then $x^n \to 1$ in the weak topology $\sigma(L, L^*)$ and $p^n \to 1$ in the weak star topology $\sigma(L^*, L)$, but $p^n \cdot x^n = 2$ for each n.

Roughly speaking, in order to be sure that the wealth map $(x, p) \to p \cdot x$ is jointly continuous with respect to topologies τ, τ^* on L, L^*, we need to know that τ is at least as strong as the Mackey topology $\tau(L, L^*)$ or that τ^* is at least as strong as the Mackey topology $\tau(L^*, L)$. Since neither the set of allocations nor the set of supporting prices will generally be Mackey compact, this presents a potentially serious problem. As we shall see in Section 7, however, we can usually circumvent these difficulties, because we need information about behavior of the wealth map only along very special sequences (or nets) of consumptions and prices.

7. The basic fixed point argument

We have identified three difficulties arising in infinite dimensional spaces: supportability, compactness and joint continuity. In succeeding sections, we shall have a great deal to say about the first two of these. In this section we show that joint continuity questions arise only at particular combinations of allocations and prices, and that, as a consequence, it turns out that the joint continuity difficulties can simply be finessed (given appropriate solutions to the supportability and compactness problems).

There are many possible approaches to the infinite dimensional existence proof; some of them are discussed in Section 13. The approach we take in this section, and that we use as our main organizational principle, is based on the Second Fundamental Theorem of welfare economics. The strategy is to look in the Pareto frontier of the set of attainable utilities; this is an approach pioneered by Negishi (1960) and Arrow and Hahn (1971) in the finite dimensional case, and used by Bewley (1969), Magill (1981) and Mas-Colell (1986a) in the infinite dimensional setting. We adopt this approach here simply because it most easily allows us to make our points about the main difficulties.

As in Section 6, we write $\omega = \Sigma\, \omega_i$ for the *aggregate endowment*, and denote by

$$Z = \left\{ x = (x_1, \ldots, x_N) \in X_1 \times \cdots \times X_N \colon \sum x_i \leq \omega \right\},$$

the *attainable set* (assuming free disposal) of the economy; elements of Z are *allocations*. The *utility possibility* set of the economy is

$$U = \{v \in \mathbb{R}^N : v \leq u(x) = (u_1(x_1), \ldots, u_N(x_N)), \text{ some } x \in Z\} ;$$
$$= u(Z) - (\mathbb{R}^N)^+ ;$$

elements of U are *utility vectors*. Without loss of generality, we normalize each u_i so that $u_i(\omega_i) = 0$.

The utility vector $u \in U$ is a *weak optimum* if there is no $u' \in U$ such that $u'_1 > u_i$ for each i; it is an *optimum* if there is no $u' \in U$ such that $u'_i \geq u_i$ for each i, with strict inequality for at least one i. An allocation x is a *weak optimum* (respectively, *optimum*) if the corresponding utility vector $u(x) = (u_1(x_1), \ldots, u_N(x_N))$ is a weak optimum (respectively, optimum).

A pair $(x, p) \in X \times L^*$ is a *quasi-equilibrium* if $p \cdot \omega \neq 0$, and for each i, $p \cdot x'_i \geq p \cdot \omega_i$ whenever $u_i(x'_i) > u_i(x_i)$. We focus throughout on quasi-equilibrium rather than on equilibrium only because the conditions which guarantee that the two notions coincide are entirely parallel to the well-understood, finite dimensional case [see McKenzie (1959), Arrow and Hahn (1974)]. We should also note that, under our maintained hypotheses on preferences, every equilibrium is a quasi-equilibrium.

The First Fundamental Theorem of welfare economics is valid in our setting; every equilibrium allocation is an optimum. Indeed, suppose that (x, p) is an equilibrium and that x' is an allocation with the property that $u_i(x'_i) \geq u_i(x_i)$ for each i, with strict inequality for at least one i. Then $p \cdot x'_i \geq p \cdot x_i$ for each i, with strict inequality for at least one i. Hence $p \cdot \Sigma\, x'_i > p \cdot \omega$. Monotonicity of preferences guarantees that p is positive, so this inequality contradicts feasibility of the allocation x'. It should be noted that this argument depends only on the equilibrium nature of the price p and on its linearity on the set of attainable consumption bundles; the argument does not depend on the continuity of p or its finiteness on all of L.

What about the Second Fundamental Theorem of welfare economics? Let us say that the price vector $p \in L^*$ *supports* the utility vector $u \in U$ if $p \cdot \omega \neq 0$ and $p \cdot (\Sigma\, x'_i - \omega) \geq 0$ whenever $u_i(x'_i) \geq u_i$ for all i. Similarly, p *supports* the allocation $x \in X$ if it supports the corresponding utility vector $u(x)$. Note that monotonicity of preferences guarantees that supporting prices are positive. If p supports x then $p \cdot (\Sigma\, x'_i - \Sigma\, x_i) \geq 0$ whenever $u_i(x'_i) \geq u_i(x_i)$ for each i, so $p \cdot x'_i \geq p \cdot x_i$ for each i. Let $P(u)$ be the set of prices supporting the utility vector u; note that $P(u)$ is a convex set.

The Second Fundamental Theorem asserts that every weak optimum can be supported by some price, or equivalently, that for every weak optimum u, the set $P(u)$ is not empty. As we have discussed in the previous section, this is in general not true in the infinite dimensional setting. We will have a great deal to say about this problem in succeeding sections, but for the moment our focus is elsewhere, so we shall simply assume that $P(u)$ is not empty for each weak optimum u. In fact we shall need to assume more, namely that the supporting prices can be chosen in some $\sigma(L^*, L)$-compact set.

The above takes care of the supportability problem. To deal with the compactness problem we shall simply assume that the utility possibility set U is closed. (Recall that monotonicity of preferences implies that U is always bounded above by $(u_1(\omega), \ldots, u_N(\omega))$.)

As we have discussed in the previous section, *U will be closed if the attainable set Z is compact in a compatible topology*. However, two points about compatible topologies should be kept in mind. First, the use of compatible topologies is purely a technical device to establish the compactness of U (in particular, we never alter our assumption that utility functions be continuous in the topology τ). Second, the requirement that U be closed is strictly weaker than the requirement that Z be compact in some compatible topology; this extra sharpness may be of value in some economic applications. For instance, U will be closed whenever there are subsets $\hat{X}_i \subset X_i$, compact in a compatible topology, with the property that $u(X) = u(\hat{X})$. This is exactly the circumstance alluded to in the final remark of Example 6.2(a). For another example where the utility possibility set is closed even though the set of allocations is not compact, see Cheng (1988).

With the supportability and compactness issues taken care of, the existence of a quasi-equilibrium is guaranteed.

Theorem 7.1. *Assume, in addition to the basic assumptions, that*:

(i) *U is closed*;

(ii) *there is a convex, $\sigma(L^*, L)$-compact set $K \subset L^*$ such that $p \cdot \omega \neq 0$ for all $p \in K$, and every weak optimum can be supported by some $p \in K$.*

Then the economy has a quasi-equilibrium.

Proof. Let Δ be the $N-1$ simplex. For any $s \in \Delta$, denote by $v(s)$ the point in $U \cap (\mathbb{R}^N)^+$ which is furthest from 0 on the ray from 0 through s. It is immediate that $s \to v(s)$ is an upper semi-continuous function (see figure 34.1).

For $s \in \Delta$, write $Q(s) = P(v(s)) \cap K$, and choose an allocation $x(s) \in X$ such that $u(x(s)) \geq v(s)$ and $\Sigma\, x_i(s) = \omega$. Our assumptions imply that $Q(s)$ is non-empty, convex and $\sigma(L^*, L)$ compact.

We define a correspondence $F: \Delta \to\!\!\to \mathbb{R}^N$ by

$$F(s) = \{(s_1 + q \cdot (x_1(s) - \omega_1), \ldots, s_N + q \cdot (x_N(s) - \omega_N)) : q \in Q(s)\}\,.$$

Since $Q(s)$ is non-empty, convex and compact, it follows that F has non-empty, convex, compact values. We claim that F is in fact an upper hemi-continuous correspondence.

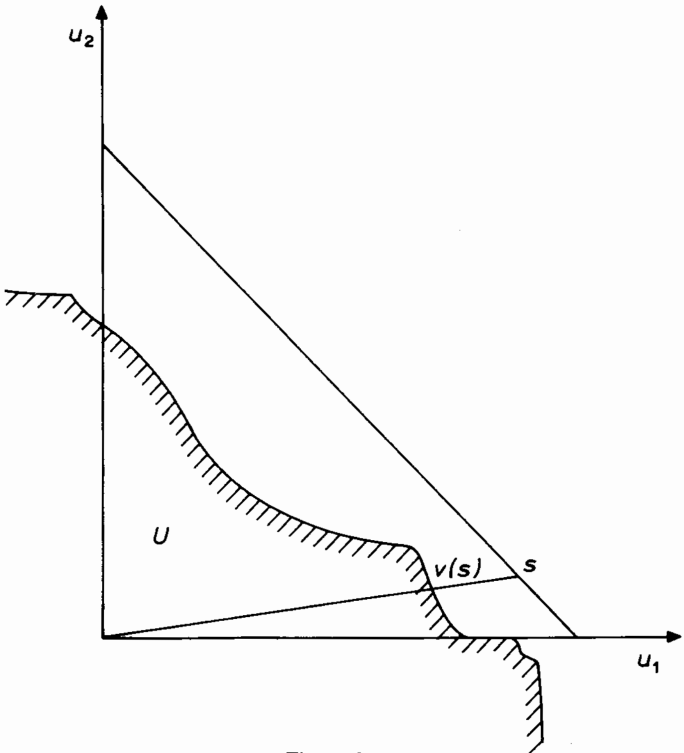

Figure 34.1

To see this, consider sequences $\{s^n\}$, $\{t^n\}$ where $s^n \to s$ in Δ and $t^n \in F(s^n)$ for each n. Choose $q^n \in Q(s^n)$, so that $q^n \cdot (x_i(s^n) - \omega_i) = t_i^n$ for each n, i. Passing to a subnet if necessary, we may assume that $q^n \to q$ for some $q \in K$. Set $t_i = q \cdot (x_i(s) - \omega_i)$; we will show that $t = (t_1, \ldots, t_N) \in F(s)$ and that $t^n \to t$; this will yield the upper semi-continuity of F. We assert first that $q \in Q(s)$. Indeed, suppose that $u_i(z_i) > v_i(s)$ for each i. Upper semi-continuity of v implies that $u_i(z_i) > v_i(s^n)$ for large n, so $q^n \cdot \Sigma(z_i - \omega_i) \geq 0$ and hence $q \cdot \Sigma(z_i - \omega_i) \geq 0$. Monotonicity of preferences now implies that $q \cdot \Sigma(z_i - \omega_i) \geq 0$ whenever $u_i(z_i) \geq v_i(s)$ for each i, which is to say $q \in Q(s)$, as asserted. Now suppose that $z_i \in X_i$ and $u_i(z_i) > v_i(s)$. Again, $u_i(z_i) > v_i(s^n)$ for large n, so we obtain

$$
\begin{aligned}
0 &\leq q^n \cdot (z_i - \omega_i) + \sum_{j \neq i} q^n \cdot (x_j(s^n) - \omega_j) \\
&= q^n \cdot (z_i - \omega_i) - q^n \cdot (x_i(s^n) - \omega_j) \\
&= q^n \cdot (z_i - x_i(s^n)) \,.
\end{aligned}
$$

Monotonicity of preferences implies that

$$0 \leq \liminf q^n \cdot (x_i(s) - x(s^n)) .$$

Since $q^n \to q$, we conclude that $q \cdot x_i(s) \leq \limsup q^n \cdot x_i(s^n)$. On the other hand, $\Sigma\, x_i(s^n) = \Sigma\, x_i(s) = \omega$, so that

$$q \cdot \omega \geq \limsup \sum q^n \cdot x_i(s^n) = \limsup q^n \cdot \omega = q \cdot \omega .$$

Hence, $q^n \cdot x_i(s^n) \to q \cdot x_i(s)$ for each i. We conclude that $t \in F(s)$ and $t^n \to t$, and hence that F is upper semi-continuous.

Finally, note that if $s_i = 0$ for some i, then $v_i(s) = 0$. Hence $q \cdot [\omega_i + \Sigma_{j \neq i} x_j(s) - \omega] \geq 0$, which yields $q \cdot [\omega_i - x_i(s)] \geq 0$. We conclude that $t_i \leq 0$, whence $s_i = 0$, $t_i \in F(s)$. Hence it follows from a standard application of Kakutani's fixed point theorem that F has a fixed point $\bar{s}$. Taking $p \in Q(\bar{s})$ and writing $x = x(\bar{s})$, we see that (x, p) is a quasi-equilibrium. ■

Note that the failure of joint continuity does not present a problem in the above proof because we need only consider very special sequences of consumptions and prices. To make the same point in a slightly different way, consider a sequence $\{p^n\}$ of price vectors and a sequence $\{x^n\}$ of consumption profiles (so that $x^n \in X_1 \times \cdots \times X_N$ for each n). Assume that $p^n \to p$ in the topology $\sigma(L^*, L)$ and that $x^n \to x$ in the topology $\sigma(L, L^*)$. In general, there is no reason to suppose that $p^n \cdot x^n \to p \cdot x$. However, the argument we have given [which goes back to Bewley (1968)] shows that this will be the case provided that: (1) $\Sigma\, x_i^n = \omega$, and (2) p^n supports x^n. To put it another way, restricted to the domain of price/consumption pairs satisfying (1) and (2), the map $(x, p) \to p \cdot x$ is jointly continuous. (The argument used in the proof above is actually a bit more subtle, since convergence of utilities substitutes for convergence of allocations, but the essence is the same.)

We conclude this section with a remark. The attentive reader will have noted that the above proof makes no use whatever of any continuity hypotheses on preferences, although upper semi-continuity is implicit in the assumption that the utility possibility set U is closed. At first sight this may seem surprising since it is well known – even in the finite dimensional setting – that upper semi-continuity of preferences does not suffice for the existence of equilibrium. Remember, however, that we have only established the existence of a quasi-equilibrium. It is in showing that a quasi-equilibrium is indeed an equilibrium that full continuity of utility functions will be required. Suppose for instance that (x, p) is a quasi-equilibrium and that for every i there is a $z_i \in X_i$ with $p \cdot z_i < p \cdot \omega_i$. If x_i is not preference maximizing in the budget set of consumer i, then there is a $y_i \in X_i$ such that $u_i(y_i) > u_i(x_i)$ and $p \cdot y_i = p \cdot \omega_i$. If u_i is continuous (indeed, if it is continuous on the segment $[z_i, y_i]$), then $u(y_i') > u_i(x_i)$ for some $y_i' \in X_i$ with $p \cdot y_i' < p \cdot \omega_i$. But this contradicts the quasi-equilibrium nature of (x, p). Therefore x_i is in fact preference maximizing on the budget set of consumer i, and so (x, p) is an equilibrium.

8. Interior consumption and L_∞

Of the three main difficulties we have identified in infinite dimensional equilibrium theory, the previous section has shown how to address one, the joint continuity difficulty, given solutions to the other two, and it is to these that our attention now turns.

As we have discussed earlier, supportability of optima is not a problem in commodity spaces for which the positive orthant has non-empty interior, and closedness of the utility possibility set is not a problem in commodity spaces for which order intervals are weakly compact. Unfortunately, there are *no* infinite dimensional spaces which enjoy both of these properties. In this section, we shall consider commodity spaces for which the positive orthant has non-empty interior. This makes the supportability problem easy to handle; to obtain closedness of the utility possibility set we shall have to impose additional assumptions. In the following section, we treat general commodity spaces, where, for the supportability of optima we will also need additional assumptions.

We therefore assume for the remainder of this section that the commodity space L is a topological vector space for which the interior $\operatorname{int} L^+$ of the positive cone L^+ is non-empty. Typical examples of such spaces are $C([0, 1])$ with the uniform norm and the positive cone $C([0, 1])^+ = \{x: x(t) \geq 0 \text{ all } t\}$, and $L_\infty(S, \Sigma, \mu)$ (for (S, Σ, μ) a σ-finite measure space), with the essential supremum norm and positive cone $L_\infty(S, \Sigma, \mu)^+ = \{x: x(t) \geq 0 \text{ almost all } t\}$.

The first thing to observe is that if $\omega \in \operatorname{int} L^+$, then $K = \{p \in L^*: p \geq 0 \text{ and } p \cdot \omega = 1\}$ is $\sigma(L^*, L)$-compact. Indeed, let W be an open, symmetric neighborhood of 0 such that $\omega + W \subset L^+$. If $p \in K$ then the restriction of p to $\omega + W$ is positive, so the restriction of p to W is bounded below by -1; since W is symmetric, it follows that the restriction of p to W lies between -1 and $+1$, and Alaoglu's theorem (see Section 2) then implies that K is compact.

If (adopting the terminology and notation of the previous section), $u \in \mathbb{R}^N$ is a weakly optimal utility vector, set

$$V = \left\{\sum z_i\colon u_i(z_i) \geq u_i \text{ for each } i\right\} - \{\omega\}\,.$$

It is evident that $0 \notin \operatorname{int} V$ (otherwise, u could not be weakly optimal) and

$\text{int}\, V \neq \emptyset$ ($\text{int}\, L^+ \neq \emptyset$ and preferences are monotone, so that $V \supset L^+$). Hence we may apply the Separation Theorem (see Section 2) to find a continuous linear functional $p \neq 0$ such that $p \cdot v \geq 0$ for each $v \in V$. We have $p \geq 0$ and $p \cdot \omega > 0$ (since $\omega \in \text{int}\, V$). We may therefore assume that $p \cdot \omega = 1$, and hence $p \in K$. Moreover, if $u = u(x)$ for the allocation x, then for each consumer i, p supports the preferred set $\{z_i: u_i(z_i) \geq u_i\}$ at x_i. In particular, we conclude that the Second Fundamental Theorem of welfare economics holds in this setting [a fact first established by Debreu (1954b) in his pioneering study of equilibrium in infinite dimensional spaces]. Combining all of this with Theorem 7.1, we obtain the following result.

Theorem 8.1. *Assume, in addition to the basic assumptions, that $\omega \in \text{int}\, L^+$. Then every weak optimum can be supported by a price vector. If, in addition, the utility possibility set U is closed, then a quasi-equilibrium exists.*

Versions of this result have been established by El-Barkuki (1977), Bojan (1974), Magill (1981), Yannelis and Prabhakar (1983), Horsley and Wrobel (1988); the result is already in Bewley (1972) for the case $L = L_\infty$.

As we have discussed, closedness of the utility possibility set U is not automatic, and will typically require strong, but economically meaningful, restrictions.

If $L = C(K)$ then its dual is $L^* = M(K)$, so that prices are countably additive measures on K; the value of the bundle x at prices p is $p \cdot x = \int x(t)\, \mathrm{d}p(t)$, which has a natural and obvious interpretation. Unfortunately, in this case it seems quite difficult to identify natural conditions guaranteeing that the utility possibility set U is closed. (If K is an infinite, compact metric space, for example, there will be no natural topology in which the set of allocations is compact.) Perhaps the most promising methodology is the one described in Section 6: search for norm compact sets $\hat{X}_i \subset X_i$ such that

$$u\left(\left\{x \in \prod \hat{X}_i : \sum x_i \leq \omega\right\}\right) - (\mathbb{R}^N)^+ = U .$$

If $L = L_\infty(S, \Sigma, \mu)$, then as discussed in Section 6, we can identify natural conditions which imply that the utility possibility set U is closed. For instance, this will be the case if consumption sets are closed and preferences are upper semi-continuous with respect to the Mackey topology $\tau(L_\infty, L_1)$ (equivalently, with respect to the weak star topology $\sigma(L_\infty, L_1)$). The first of these conditions will certainly be met if consumption sets are Mackey closed (hence weak star closed), and the second will be met if preferences are (upper) impatient. Under these conditions, Theorem 8.1 yields an equilibrium price in the dual space $L_\infty(S, \Sigma, \mu)^*$. But what, in concrete terms, is the dual space $L_\infty(S, \Sigma, \mu)^*$?

Unfortunately, the answer is that the dual space is unmanageably large. To be precise, $L_\infty(S, \Sigma, \mu)^*$ may be identified with the space $\mathrm{ba}(S, \Sigma, \mu)$ of bounded, finitely additive set functions on Σ which vanish on sets of μ measure 0. Among the finitely additive set functions in $\mathrm{ba}(S, \Sigma, \mu)$ are the countably additive ones; i.e. the countably additive measures on (S, Σ) that are absolutely continuous with respect to μ. In view of the Radon–Nikodym theorem, these countably additive set functions may be identified with functions in $L_1(S, \Sigma, \mu)$, with the pairing $p \cdot x = \int p(s)x(s)\,\mathrm{d}\mu(s)$ (see Section 2). Such prices have very natural economic interpretations. For instance, if we interpret elements of S as representing states of the world, so that a function in $L_\infty(S, \Sigma, \mu)$ represents a bundle of contingent commodities, then a function in $L_1(S, \Sigma, \mu)$ represents commodity/state prices. However, prices in $\mathrm{ba}(S, \Sigma, \mu)$ that do not belong to $L_1(S, \Sigma, \mu)$ seem to have no natural economic interpretation. (It seems that they have no concrete mathematical interpretation, either; indeed, their very existence depends on the Axiom of Choice.)

As the following examples shows, the possibility that equilibrium prices might not be in L_1 is quite real.

Example 8.1. Let $L = l_\infty$, the space of bounded sequences. (We identify l_∞ with the space of bounded measurable functions on the positive integers, with counting measure.) Consider a one consumer economy with $\omega = (1, 1, \ldots)$, $X = l_\infty^+$ and the utility function $u : l_\infty^+ \to \mathbb{R}$ defined by $u(x) = \liminf x(t)$. It is easily seen that u is concave and norm continuous, so there is a price $p \in l_\infty^*$ such that $p \cdot x \geqslant p \cdot \omega > 0$ whenever $u(x) \geqslant u(\omega) = 1$, but no such p can belong to l_1. (To see this, define, for each k, an element $x^k \in l_\infty$ by $x^k(t) = 0$ for $t < k$ and $x^k(t) = 2$ for $t \geqslant k$. Then $u(x^k) = 2 > u(\omega)$, but if $p \in l_1$ then $p \cdot x^k \to 0$, while $p \cdot \omega > 0$.)

The supporting price in Example 8.1 has the property that all its mass is "concentrated at infinity." (Economically, this is not surprising, since utility depends only on what happens at infinity.) Results of Yosida–Hewitt [see the discussion in Bewley (1972)] show that this is quite typical of finitely additive measures. To be more precise, let (S, Σ, μ) be a σ-finite measure space, and let $p \in \mathrm{ba}(S, \Sigma, \mu)$ be a positive finitely additive measure. Then p can be written uniquely as a sum $p = p_c + p_f$, where p_c is a positive, countably additive measure and p_f is a positive, finitely additive measure with the property that there is no positive, countably additive measure q such that $p_f \geqslant q \geqslant 0$; we refer to p_c as the *countably additive part* and to p_f as the *purely finitely additive part*. Purely finitely additive measures are supported on arbitrarily small sets, in the sense that, for every purely finitely additive measure $p_f \in \mathrm{ba}(S, \Sigma, \mu)$, there is a descending sequence $\{E^n\}$ of measurable subsets of Ω such that $\mu(E^n) \to 0$ and $p_f(\Omega \setminus E^n) = 0$ for each n.

Our discussion in Section 5 suggests that we should not expect that prices be more continuous than preferences. Therefore, we should not hope to find supporting prices in L_1 unless preferences are continuous in the stronger topology that forces continuous prices to be in L_1; i.e. the Mackey topology. The following example from Sawyer (1987) shows that Mackey upper semi-continuity will not suffice.

Example 8.2. Again, this is a one-consumer example. Let $L = l_\infty$, $X = l_\infty^+$. Define the endowment ω by $\omega(1) = 2$, $\omega(t) = 1 + 10^{-t}$ for $t > 1$; and define the utility function u by $u(x) = \inf x(t) + q \cdot x$, where $q \in l_1$ is given by $q(1) = 2$, $q(t) = 10^{-t}$ for $t > 1$. The utility function u is concave, strictly monotone, norm continuous and Mackey (hence weak star) upper semi-continuous. There is a price $p \in l_\infty^*$ that supports the preferred set at ω, but no such price can belong to l_1. (If $p \in l_1$, a simple argument shows that $p = \alpha q$ for some $\alpha > 0$. However, if we define $x \in l_\infty$ by $x(1) = 1.8$, $x(t) = 1.4$ for $t > 1$, we see that $u(x) > u(\omega)$ and $q \cdot (x - \omega) < 0$, a contradiction.)

However, even Mackey continuity of preferences will not suffice to yield prices in L_1 if consumption sets do not coincide with the positive orthant, as the following example of Back (1988) shows.

Example 8.3. Let $L = l_\infty$. The economy has two consumers, with consumption sets $X_1 = l_\infty^+$, $X_2 = \{x \in l_\infty^+ : x(0) + x(t) \geq 4 \text{ for } t > 0\}$. Utility functions u_1, u_2 are defined by $u_1(x) = \Sigma\, 3^{-t} x(t)$, $u_2(x) = x(0) + 2\, \Sigma_{t>0}\, 3^{-t} x(t)$. Finally, endowments ω_1, ω_2 are given by $\omega_1(t) = \omega_2(t) = 2$, for $t \geq 0$. Note that preferences are linear and weak star continuous, and that the endowments belong to the (norm) interior of l_∞^+. However, we claim that this economy has no quasi-equilibrium supported by a price $p \in l_1$.

Observe first that the initial endowment (ω_1, ω_2) is an optimum. (To see this, note that if X_2 were all of l_∞^+, (ω_1, ω_2) would not be an optimum, but any improvement would involve transferring some amount of commodity $t = 0$ from the second consumer to the first consumer. The actual definition of X_2 makes this impossible.) Hence if $p \in l_1^+$ is a quasi-equilibrium price, utility maximization by the first consumer would entail that $p(t) = \alpha 3^{-t}$ for some $\alpha > 0$. However, no such price system can support the preferred set of the second consumer at ω_2. Indeed, define $x_2 \in X_2$ by $x_2(0) = \omega_1(0) - \varepsilon$, $x_2(t) = \omega_2(t) + \varepsilon$ for $t > 0$. Then $u_2(x_2) = u_2(\omega_2)$ and $p \cdot x_2 = p \cdot \omega_2 - \frac{1}{2}\alpha\varepsilon < p \cdot \omega_2$. Hence, for $\delta > 0$ sufficiently small, $u_2(x_2 + \delta\omega) > u_2(\omega_2)$ and $p \cdot (x_2 + \delta\omega) < p \cdot \omega_2$, as desired. We conclude that there is no quasi-equilibrium price $p \in l_1$.

The budget set X_2, while "untraditional", is economically meaningful. If we interpret $t = 0$ as representing consumption today and $t > 0$ as representing consumption in various possible states of the world tomorrow, the constraints defining X_2 may be read as stating that subsistence requires a total of 4 units of consumption over the two dates.

As these three examples suggest, to obtain equilibrium prices in L_1 we shall have to require that preferences be Mackey continuous and that consumption sets coincide with the positive orthant. Bewley (1972) showed that these conditions are indeed sufficient.

Theorem 8.2. *Assume, in addition to the basic assumptions, that:*

(i) $X_i = L_\infty(S, \Sigma, \mu)^+$ *for each* i;

(ii) *each* $\succcurlyeq_i$ *is Mackey continuous;*

(iii) *each* $\succcurlyeq_i$ *is strictly monotone, in the sense that if* $x_i \in L_\infty(S, \Sigma, \mu)^+$ *and* $v \in \operatorname{int} L_\infty(S, \Sigma, \mu)^+$ *then* $x_i + v >_i x$;

(iv) $\omega \in \operatorname{int} L_\infty(S, \Sigma, \mu)^+$.

Then the economy has a quasi-equilibrium, and every quasi-equilibrium price belongs to $L_1(S, \Sigma, \mu)$.

Proof. Since consumption sets are closed and preferences are upper semi-continuous in the Mackey, and hence weak star, topology, our earlier discussion shows that the utility possibility set U is closed. The existence of a quasi-equilibrium now follows from Theorem 8.1.

Let (x, p) be a quasi-equilibrium. It follows easily from strict monotonicity that $\Sigma\, x_i = \omega$. (This is the only place where strict monotonicity is used.) By the Yosida–Hewitt theorem quoted earlier, we may decompose $p = p_c + p_f$ into a countably additive and a purely finitely additive part. We wish to show that $p_f = 0$. Because ω is strictly positive, it suffices to show that $p_f \cdot \omega = 0$. Suppose to the contrary that $p_f \cdot \omega > 0$. Then $\Sigma\, p_c \cdot x_i = p_c \cdot \omega < p \cdot \omega$, so there is a j such that $p_c \cdot x_j < p \cdot \omega_j$. Choose $\varepsilon > 0$ so that $p_c \cdot x_j + \varepsilon p \cdot \omega < p \cdot \omega_j$. As mentioned earlier, we can find a descending sequence $\{E^n\}$ of measurable sets such that $\mu(E^n) \to 0$ and $p_f(\Omega \setminus E^n) = 0$ for each n. Define y^n by $y^n(t) = 0$ for $t \in E^n$ and $y^n(t) = x_j(t)$ for $t \notin E^n$. For each n, $y^n \in X_j$ (recall that $X_j = L_\infty(S, \Sigma, \mu)^+$). Because $y^n \to x_j$ in measure, and hence in the Mackey topology (see Section 2), lower semi-continuity of preferences implies that $y^n + \varepsilon\omega >_j x_j$ for n sufficiently large. However, since $p_f(\Omega \setminus E^n) = 0$ for each n, we have $p_f \cdot y^n = 0$ for each n, so $p \cdot (y^n + \varepsilon\omega) < p \cdot \omega_j$, which is a contradiction. We conclude that $p_f = 0$, and hence that $p \in L_1(S, \Sigma, \mu)$, as desired. ■

Note that strict monotonicity is used in the above argument only to guarantee that $\Sigma\, x_i = \omega$; without strict monotonicity, we cannot rule out quasi-equilibria for which $\Sigma\, x_i < \omega$, and such quasi-equilibria may be supported by prices $p \notin L_1(S, \Sigma, \mu)$. In that case, however, it is possible to show that the countably additive part p_c also supports the same quasi-equilibrium allocation

[Bewley (1972)]. In our setting, we note that our proof technique always yields quasi-equilibrium allocations such that $\Sigma x_j = \omega$, so our quasi-equilibrium prices are necessarily in $L_1(S, \Sigma, \mu)$.

The proof of Theorem 8.2 also helps to understand the hypotheses that consumption sets be the positive orthant and that preferences be Mackey continuous. These hypotheses are used precisely to ensure that if $y, z \in X_i$ with $y >_i z$, $\{E^n\}$ is a descending sequence of measurable sets such that $\mu(E^n) \to 0$, and we define y^n by putting $y^n(t) = 0$ for $t \in E^n$ and $y^n(t) = y(t)$ for $t \not\in E^n$, then we obtain a sequence $\{y^n\}$ of vectors that, first, belong to the consumption set X_i, and, second, have the property that $y^n >_i z$ for sufficiently large n. Any hypotheses that yield this conclusion can fulfill the same function [see Prescott and Lucas (1972)].

9. Properness and general commodity spaces

We turn now to general commodity spaces, for which the positive cone L^+ may have empty interior. As we have indicated, the central problem in such spaces is supportability of optima, and this section will be devoted largely to this problem. It is important to keep in mind that the list of commodity spaces for which the positive orthant has empty interior includes many of the most important commodity spaces, including the L_p spaces (and more generally, the reflexive Banach lattices). Recall that those are well behaved from the point of view of compactness of the attainable set.

We treat first the one consumer case. Afterward, we address the general situation, where optimal allocations involve real trade between consumers.

9.1. One consumer

Supporting prices are differentials, or more generally, subdifferentials, of utility functions. They are measures of marginal rates of substitution. When consumption sets have non-empty interior (and preferences are continuous), such supporting prices are guaranteed to exist (and to be continuous). When consumption sets have empty interior, however, marginal rates of substitution may be unbounded in such a way as to preclude the existence of supporting prices (see Examples 5.1 and 6.5). It seems natural therefore to *require* of well-behaved preferences that they admit supporting prices. This leads to the notion of *properness*, which was introduced by Mas-Colell (1986a). Antecedents to this notion appear in the economics literature in the notes of Debreu and Hildenbrand (1970) [see Bewley (1972) for a discussion], and the papers of Chichilnisky and Kalman (1980), Jones (1984) and Ostroy (1984).

We say that the preference relation $\succcurlyeq$, defined on the consumption set X, is *proper* at x with respect to the vector v, if there is an open cone Γ_x at 0, containing v, such that $x - \Gamma_x$ does not intersect the preferred set $\{x' \in X: x' \succcurlyeq x\}$; i.e. if $x' \succcurlyeq x$ then $x - x' \not\in \Gamma_x$ (see Figure 34.2). The interpretation we have in mind is that the commodity bundle v is desirable, in the sense that loss of an amount αv (with $\alpha > 0$) cannot be compensated for by an additional amount αz of any commodity bundle z, if z is sufficiently small. We say that $\succcurlyeq$ is uniformly proper with respect to v on the subset $Y \subset X$ if it is proper at every $y \in Y$, and we can choose the properness cone independently of y.

When preferences are convex, properness of $\succcurlyeq$ at x with respect to v is equivalent to the existence of a price $p \in L^*$ which supports the preferred set $\{x' \in X: x' \succcurlyeq x\}$ at x and has the additional property that $p \cdot v > 0$. Indeed, if such a p exists, we can simply take $\Gamma_x = \{z: p \cdot z > 0\}$. Conversely, if $\succcurlyeq$ is proper at x with respect to v, then $\{x' \in X: x' \succcurlyeq x\}$ and $x - \Gamma_x$ are disjoint convex sets, and the latter has non-empty interior, so the Separation Theorem (see Section 2) provides a continuous linear functional $p \in L^*$ that separates them; i.e. $p \cdot z \leqslant p \cdot x'$ for each $z \in (x - \Gamma_x)$ and $x' \succcurlyeq x$. Because Γ_x is an open cone at 0, containing v, it follows that $p \cdot z < 0$ for each $z \in \Gamma_x$, and hence that $p \cdot v > 0$ and $p \cdot x' \geqslant p \cdot x$ for $x' \succcurlyeq x$, as asserted. (For non-convex preferences,

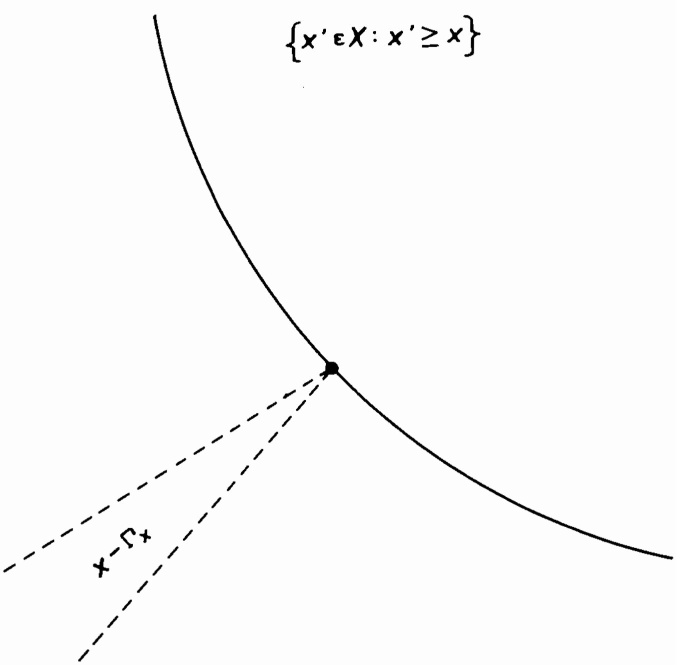

Figure 34.2

properness may still be interpreted in terms of marginal rates of substitution, but is more general than linear supportability.) Properness is thus a requirement which is no stronger than necessary for our purpose. Uniform properness, however, is a more serious restriction than properness, and will fail for some important preference relations; see Sections 10 and 11.

A related notion was introduced by Yannelis and Zame (1986) in the context of unordered preferences (see also Sections 13 and 15). We say $\succcurlyeq$ is *F-proper* (F for forward) at $x \in X$ with respect to v if there is an open cone Γ_x (at 0) such that $v \in \Gamma_x$ and $(x + \Gamma_x) \cap X \subset \{x' \in X: x' \succcurlyeq x\}$; i.e. if $z \in \Gamma_x$ and $x + z \in X$ then $x + z \succcurlyeq x$ (see Figure 34.3). We say that $\succcurlyeq$ is uniformly F-proper on $Y \subset X$ with respect to v if it is F-proper at each point $y \in Y$ and the properness cone may be chosen independently of y.

In general, properness and F-properness are incomparable conditions, but it is easy to see that uniform properness on X (with respect to v) is equivalent to uniform F-properness on X (with respect to v).

It seems natural to surmise that properness is related to extendibility of preferences, and Richard and Zame (1987) have shown that this is indeed the case. To be precise, take $X = L^+$. Uniform properness of $\succcurlyeq$ on X implies the existence of a convex cone $\tilde{L}$ containing L^+ and having non-empty interior, and a convex preference relation $\tilde{\succcurlyeq}$ on $\tilde{L}$ that extends $\succcurlyeq$. In general, the extended preference relation $\tilde{\succcurlyeq}$ may be chosen to be either upper or lower

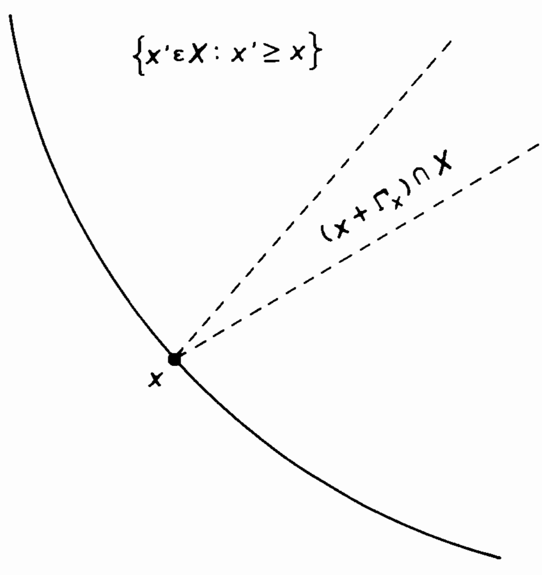

Figure 34.3

semi-continuous; if order intervals are weakly compact, it may be chosen to be both upper and lower semi-continuous (i.e. to be continuous). Conversely, the restriction to the positive cone of a continuous, convex preference relation $\tilde{\succcurlyeq}$ defined on a convex cone $\tilde{L}$ containing L^+ and having non-empty interior (for instance, the entire space L), is uniformly proper on order bounded sets. In particular, this provides a simple way to construct uniformly proper preferences.

It is easy to see that the preferences described in Example 6.5 are not proper (at ω). The preferences described in Example 5.2 are proper (at ω) in the topopology $\sigma(M([0, 1]), C([0, 1]))$, but not in the topology $\sigma(M([0, 1]), \mathrm{Lip}([0, 1]))$. This should serve as a reminder that the notion of properness depends on the topology of the space (through the requirement that the cone Γ_x be open).

9.2. *Several consumers*

It would be most convenient if properness, which is equivalent to supportability of individual preferred sets, were sufficient to guarantee supportability of weak optima. Unfortunately, this is not so, as the examples below demonstrate; this first is from Richard and Zame (1986) and the second from Jones (1987).

Example 9.1. Let $L = l_2$, $X_1 = X_2 = l_2^+$. As in Example 6.5, define $u : l_2^+ \to \mathbb{R}$ by $u(x) = \Sigma\, v_t(x(t))$, where

$$v_t(x(t)) = \begin{cases} 2^t x(t) & \text{if } x(t) \leq 2^{-2t}\,, \\ 2^{-t}[x(t) + 1 - 2^{-2t}] & \text{if } x(t) > 2^{-2t}\,. \end{cases}$$

This utility function has no supporting price at the vector $\omega \in l_2^+$ defined by $\omega(t) = 2^{-4t}$; the only candidate is the unbounded sequence $\{2^t\}$. Now let q_1, q_2 be non-collinear, strictly positive linear functionals on l_2, and define utility functions u_1, u_2 by

$$u_i(x_i) = \min\{u(x_i),\ q_i \cdot x_i + u_i(\omega) - q_i \cdot \omega\}\ .$$

These utility functions u_i are continuous, concave and strictly monotone. Let endowments be $\omega_1 = \omega_2 = \omega$. It may be seen that consumer i's preferred set to ω_i is

$$\{x_i \in l_2^+ : u(x_i) \geq u(\omega_i)\} \cap \{x_i \in l_2^+ : q_i \cdot x_i \geq u(\omega_i)\}$$

so that (ω_1, ω_2) is an optimal allocation. On the other hand, the only prices supporting these preferred sets are (up to positive multiples) on the line segment joining q_i to the unbounded sequence $\{2^t\}$; the only such price that belongs to l_2 is q_i itself. Since q_1, q_2 were chosen non-collinear, this means that no common supporting price exists.

These preferences are evidently proper at the endowments (since supporting prices exist), but they fail to be F-proper at the endowments, or to be uniformly proper on the attainable portion of the consumption sets.

Example 9.2. Set $L = L_\infty([0,1])$, equipped with the weak topology $\sigma(L_\infty([0,1]), C^1([0,1]))$ from the pairing of $L_\infty([0,1])$ with $C^1([0,1])$, so that the price space is $L^* = C^1([0,1])$. Set $X_1 = X_2 = L^+$, $\omega_1 = \omega_2 \equiv 1$. Define utility functions u_1, u_2 by

$$u_1(x_1) = \int t x_1(t) \, \mathrm{d}t ,$$

$$u_2(x_2) = \int (1-t) x_2(t) \, \mathrm{d}t .$$

These utility functions are continuous and uniformly proper (since they are linear). However, the optimum x_1 = characteristic function of $[0, 1/2]$, $x_2 = 1 - x_1$ is not supportable by any continuous price. The reason is not difficult to see: the only candidates for a supporting price are positive multiples of the function

$$p(t) = 1 - t \quad \text{for } 0 \leqslant t \leqslant 1/2 ,$$

$$p(t) = t \quad \text{for } 1/2 \leqslant t \leqslant 1 .$$

This price does not belong to $C^1([0,1])$, and so is not continuous in the weak topology $\sigma(L_\infty([0,1]), C^1([0,1]))$.

Note that the lattice operations are not continuous in the topology $\sigma(L_\infty([0,1]), C^1([0,1]))$; equivalently, this topology is not locally solid (use the Rademacher functions discussed in Example 4.2).

We also refer to Example 8.3, which may easily be modified to apply to any of the l_p spaces, $1 < p < \infty$, in order to show that even with linear preferences, when consumption sets differ from the positive orthant, optima may not be supportable by prices [see Back (1988)].

As the examples indicate, we need to assume uniform properness, not just properness; we need to assume that the commodity space is a topological vector lattice, not just a vector lattice; and we must assume that consumption sets coincide with the positive orthant. With all these restrictions, however, supportability of optima is guaranteed. The following results are due to Mas-Colell (1986a).

Theorem 9.1. *Assume, in addition to the basic assumptions, that:*
(i) *L is a topological vector lattice;*
(ii) *$X_i = L^+$ for each i;*
(iii) *preferences are uniformly proper on the order interval $[0, \omega]$ with respect to ω.*
Then there is a $\sigma(L^*, L)$ compact, convex set $K \subset (L^*)^+$ such that $p \cdot \omega = 1$ for every $p \in K$, and every weak optimum can be supported by some $p \in K$.

Proof. Let u be a weakly optimal utility vector. Monotonicity and the restriction $X_i = L^+$ guarantee that there is a weakly optimal allocation x such that $\Sigma\, x_i = \omega$ and $u_i = u_i(x_i)$ for each i. For each i, let Γ_i be a properness cone for $\succcurlyeq_i$ and $\Gamma = \bigcap \Gamma_i$. Set

$$V = \left\{ \sum (z_i - x_i) \colon u_i(z_i) \geqslant u_i(x_i) \text{ for each } i \right\}.$$

It is evident that V is convex; we assert that $V \cap (-\Gamma) = \emptyset$.

To see this, let $W \subset L$ be a neighborhood of 0 such that $\omega + W$ generates Γ. Because the topology is locally convex and locally solid, there is no loss of generality in assuming that W is convex, symmetric and solid. If $z - \omega \in V \cap (-\Gamma)$ then there are $z_i \geqslant 0$ such that $z = \Sigma\, z_i$, $u_i(z_i) \geqslant u_i(x_i)$ for each i and $z - (1-\alpha)\omega \in \alpha W$ for some $\alpha > 0$. Of course, $(1-\alpha)\omega - z \leqslant \omega$. Hence $[(1-\alpha)\omega - z]^+ \leqslant \omega$ and so

$$\begin{aligned} z &= (1-\alpha)\omega - \omega + \omega - [(1-\alpha)\omega - z]^+ + [(1-\alpha)\omega - z]^- \\ &\geqslant -\alpha\omega + [(1-\alpha)\omega - z]^- . \end{aligned}$$

It follows that $[(1-\alpha)\omega - z]^- \leqslant z + \alpha\omega$. The Riesz Decomposition Property of vector lattices (see Section 2) allows us to find vectors $s_i \in L$ such that $0 \leqslant s_i \leqslant z_i - \alpha\omega$ for each i and $[(1-\alpha)\omega - z]^- = \Sigma\, s_i$.

We now set $v_i = z_i + \alpha\omega - s_i \geqslant 0$. Note that $[(1-\alpha)\omega - z] \in \alpha W$ and

$$0 \leqslant s_i \leqslant [(1-\alpha)\omega - z]^- \leqslant |(1-\alpha)\omega - z|$$

so that $s_i \in \alpha W$. Properness at v_i therefore implies that $u_i(v_i) > u_i(z_i)$ for each i. On the other hand,

$$\begin{aligned}\sum v_i &= z + \alpha\omega - [(1-\alpha)\omega - z]^- \\ &\leqslant z + \alpha\omega - [(1-\alpha)\omega - z]^- + [(1-\alpha)\omega - z]^+ \\ &= z + \alpha\omega + [(1-\alpha)\omega - z] \\ &= \omega ,\end{aligned}$$

which contradicts the optimality of the allocation x. We conclude that $V \cap (-\Gamma) = \emptyset$, as asserted. We can now apply the Separation Theorem to find a continuous linear functional p separating V from $-\Gamma$; as in the proof of Theorem 8.1, p is the desired supporting price. ■

From the above and Theorem 7.1 we immediately obtain the following theorem.

Theorem 9.2. *Assume, in addition to the basic assumptions, that:*
(i) *L is a topological vector lattice;*
(ii) *$X_i = L^+$ for each i;*
(iii) *preferences are uniformly proper on the order interval $[0, \omega]$ with respect to ω;*
(iv) *the utility possibility set U is closed.*
Then a quasi-equilibrium exists.

As we have noted several times, if L is a reflexive Banach lattice (e.g. $L = L_p$, $1 < p < \infty$) or a Banach lattice with order continuous norm (e.g. $L = L_1$), then it is automatically the case that the weak topology is compatible and the set of allocations is weakly compact, so closedness of the utility possibility set is also automatic.

Since the Mackey topology on L_∞ is locally solid, Theorem 9.1 might be viewed as a generalization of Theorem 8.2, but it should be noted that Mackey lower semi-continuity is a weaker condition than Mackey uniform properness (for example, the former is compatible with infinite marginal utility at zero consumption, while the latter is not).

We should also point out that, even in the finite dimensional case, properness-like assumptions are not dispensable if we insist (as seems most reasonable) on finding a quasi-equilibrium price p with $p \cdot \omega > 0$ [Mas-Colell (1985), Yannelis and Zame (1986)].

While we have established the existence of quasi-equilibria supported by continuous prices, we have left open the possibility of quasi-equilibria supported by discontinuous prices; see the discussion in Section 10.

Are we at the end of the road? Not quite. Aside from the unfortunate restriction that consumption sets coincide with the positive orthant (see Section

15 for further discussion), two of our assumptions will not be satisfied in some economically interesting settings. The first is uniform properness, which rules out infinite marginal utility for zero consumption and is thus incompatible with some models used in finance. The other is local solidness of the topology, which rules out commodity spaces such as $L = M(K)$, with the weak topology $\sigma(M(K), C(K))$, and is thus incompatible with some models of commodity differentiation.

In the next three sections, we see that, in many cases of interest, these assumptions can be relaxed. In Section 10 we show that, even without uniform properness, it is still possible to find price systems which are not defined for all consumption bundles. (Properness will then suffice to guarantee that such price systems can be extended continuously to all commodity bundles.) In Section 11 we apply these ideas to a financial model. Finally, Section 12 shows how the assumption of a locally solid topology may be eliminated.

10. The order ideal $L(\omega)$

As we noted in Section 5, requiring that prices be defined and finite on all of L amounts to requiring that every conceivable commodity bundle be assigned a finite price. In this section, we explore the consequences of relaxing this requirement. As we shall see, this leads naturally to a weaker notion of equilibrium, whose existence can be established even if preferences are not proper. When preferences *are* proper, this weaker equilibrium notion yields an equilibrium in the usual sense.

Informally, we shall insist that endowments (and hence all feasible bundles) be assigned a finite price, but we allow for the possibility that commodity bundles "not present in the market" are left unpriced. Such a possibility was first considered by Peleg and Yaari (1970) in the context of intertemporal equilibrium theory.

In what follows, we assume that L is a topological vector lattice with topology τ, that consumption sets $X_i = L^+$ for each i, and that the attainable set Z is compact in some compatible topology σ, or simply that the utility possibility set U is closed.

The key notion is the *order ideal* generated by the aggregate endowment ω:

$$L(\omega) = \{x \in L: |x| \leq \lambda\omega \text{ for some } \lambda > 0\} .$$

If $L = L_\infty(S, \Sigma, \mu)$ and ω is bounded away from 0, then $L(\omega) = L$. In general however, $L(\omega)$ is much smaller than L. For instance, if $L = L_1(S, \Sigma, \mu)$, then $L(\omega)$ consists of functions x for which the ratio $|x(t)/\omega(t)|$ is bounded. If $L = M(K)$, then $L(\omega)$ consists of measures x that are absolutely continuous with respect to ω and have bounded Radon–Nikodym derivatives (see also Example 5.1).

Note that, in an exchange economy, $L(\omega)$ contains all the feasible consumption bundles so that [as pointed out by Brown (1983)], we can determine all Pareto optimal and core allocations by considering the restriction of the economy to $L(\omega)$. (That is, we consider the economy with consumption sets $X_i(\omega) = X_i \cap L(\omega)$, preferences obtained by restricting to $L(\omega)$, and the same endowments). This suggests that we look for quasi-equilibria of the restriction of the economy to $L(\omega)$, and then consider the relationship between such quasi-equilibria and quasi-equilibria of the original economy. This strategy has been employed by Zame (1987), Aliprantis, Brown and Burkinshaw (1987b), Araujo and Monteiro (1989a) and Duffie and Zame (1989).

The search for quasi-equilibria in $L(\omega)$ is much easier than in L because $L(\omega)$ carries a lattice norm (i.e. a norm with respect to which the lattice operations are uniformly continuous) with respect to which the positive cone has a non-empty interior. This norm is defined by setting for $x \in L(\omega)$

$$\|x\|_\omega = \inf\{\lambda > 0: |x| \leq \lambda\omega\} .$$

It is easy to check that $\|\cdot\|_\omega$ is a lattice norm on $L(\omega)$ and that the $\|\cdot\|_\omega$ topology is stronger than the topology τ (because τ is locally solid). Moreover, ω is in the $\|\cdot\|_\omega$ interior of the positive cone $L(\omega)^+ = L(\omega) \cap L^+$. Thus, $L(\omega)$ is much like L_∞. Indeed, in many cases of interest, $L(\omega)$ is actually isomorphic to $L_\infty(S, \Sigma, \mu)$ for some measure space (S, Σ, μ) [see Zame (1986)]. For our present purposes, we need only observe that the restriction of the economy to $L(\omega)$ enjoys all the properties required in Theorem 8.1. Hence, the restriction of the economy to $L(\omega)$ has a quasi-equilibrium (x, p), where p is a positive, $\|\cdot\|_\omega$ continuous linear functional on $L(\omega)$, $\Sigma\, x_i = \omega$ and $p \cdot \omega \neq 0$.

It should be emphasized that (x, p) is not a quasi-equilibrium in the usual sense, since we have not priced all commodity bundles in L. Moreover, at this point we can draw no conclusions about continuity of the price p (with respect to the topology τ) or its extendibility to all of L. On the other hand, to this point we have made no assumptions about preferences other than convexity and continuity with respect to τ.

To study the continuity of p (with respect to τ) and its extendibility to all of L, we make use of the notion of F-properness discussed in Section 9. Recall that F-properness of $\succcurlyeq_i$ at x_i with respect to the vector ω (and the topology τ) means that there is a τ-neighborhood W_i of 0 such that every point of the forward cone $\Gamma = \{x_i + \lambda\omega - \lambda z: \lambda > 0,\ z \in W_i\}$ which also belongs to L^+ is preferred to x_i. As we have already noted, properness and F-properness are closely related; in particular, uniform properness and uniform F-properness are equivalent. Moreover, it is easily seen that F-properness of $\succcurlyeq_i$ at x_i implies properness at x_i of the restriction of $\succcurlyeq_i$ to $L(\omega)$.

We assert that if each preference relation $\succcurlyeq_i$ is F-proper at x_i (with respect to ω), then the price p is continuous with respect to the original topology τ. (Note that our assumption is only on the behavior of the preference relations at a single point.) To see this, set $W = \cap W_i$; without loss of generality, we may assume that W is solid and symmetric (i.e. $W = -W$) and that $p \cdot \omega = 1$. To show that p is continuous, it suffices to show that it is bounded on some neighborhood of 0. Since W is symmetric, it suffices to establish this for $y \geqslant 0$. We claim that in fact $p \cdot y \leqslant N$ for each $y \in W \cap L(\omega)^+$, whence $|p \cdot y| \leqslant 2N$ for each $y \in W \cap L(\omega)$. Since $y \in L(\omega)$, there is a $\lambda > 0$ such that $0 \leqslant y \leqslant \lambda\omega$; set $z = (1/\lambda)y \leqslant \omega$. Because $\Sigma\, x_i = \omega$, we conclude that $\Sigma\,(x_i + (1/\lambda)\omega) - z \geqslant 0$. We now apply the Riesz Decomposition Property of vector lattices (see Section 2) to find $z_1, \ldots, z_N \in L(\omega)^+$ such that $\Sigma\, z_i = z$ and $x_i + (1/\lambda)\omega - z_i \geqslant 0$ for each i. Since $z_i \leqslant z$ and $z \in (1/\lambda)W$, solidity of W implies that $z_i \in (1/\lambda)W$ for each i. F-properness implies that $x_i + (1/\lambda)\omega - z_i \succcurlyeq_i x_i$, and the quasi-equilibrium conditions then imply that $p \cdot (x_i + (1/\lambda)\omega - z_i) \geqslant p \cdot x_i$ for each i. Summing over all consumers, rearranging terms, and keeping in mind that $p \geqslant 0$, we conclude that

$$N/\lambda = p \cdot (N/\lambda)\omega \geqslant \sum p \cdot (1/\lambda)\omega \geqslant p \cdot (1/\lambda)y$$

so that $p \cdot y \leqslant N$, as asserted. We conclude that p is continuous. [This argument is from Yannelis and Zame (1986).]

If $L(\omega)$ is dense in L (i.e. if ω is in the *quasi-interior* of L^+), then the price p has a unique continuous (with respect to τ) extension $\hat{p}$ to all of L. It is easily checked that $(x, \hat{p})$ is a quasi-equilibrium for the original economy. (If $L(\omega)$ is not dense in L, the price p may have many continuous extensions to L, and it might happen that none of them is a quasi-equilibrium price. However, if each of the preference relations $\succcurlyeq_i$ is uniformly proper, it may be shown that there is some continuous extension $\tilde{p}$ of p to L such that $(x, \tilde{p})$ is a quasi-equilibrium.)

Summarizing, we obtain the following result [which is a variant of results obtained by Zame (1987), Aliprantis, Brown and Burkinshaw (1987b), Araujo and Monteiro (1989a) and Duffie and Zame (1989)].

Theorem 10.1. *Assume, in addition to the basic assumptions, that*:

(i) $X_i = L^+$ *for each* i;

(ii) *the attainable set* Z *is compact in some compatible topology* (*or simply that the utility possibility set* U *is closed*).

Then:

(a) *the restriction of the economy to the order ideal* $L(\omega)$ *has a quasi-equilibrium* (x, p), *such that the price* p *is continuous in the* $\|\cdot\|_\omega$ *norm on* $L(\omega)$;

(b) *if each preference relation* $\succcurlyeq_i$ *is F-proper at* x_i *then* p *is continuous in the topology of* L;

(c) *if, in addition, either* $L(\omega)$ *is dense in* L *(i.e.* ω *is in the quasi-interior of* L^+*), or every* $\succcurlyeq_i$ *is uniformly proper, then* p *extends to a continuous price* $\hat{p}$ *on all of* L, *and* $(x, \hat{p})$ *is a quasi-equilibrium for the original economy.*

Three final comments are in order here. First, note that the result above yields a quasi-equilibrium price $\hat{p}$, provided only that preferences are F-proper at a single particular allocation – a quasi-equilibrium allocation for the restriction of the economy to the order ideal $L(\omega)$. This will certainly be the case if preferences are F-proper at every individually rational, Pareto optimal allocation. Second, the hypothesis that ω be in the quasi-interior of L^+ is quite weak in many circumstances. For example, if $L = L_p$ $(1 \leq p < \infty)$ with the norm topology, or L_∞ with the Mackey topology, this restriction means only that ω is non-vanishing except on a set of measure 0. However, if $L = M(K)$ with the norm topology, this restriction is unpleasantly strong, since $M(K)^+$ has no quasi-interior points unless K is countable. Third, we should not forget that we are restricting ourselves to exchange economies. The discussion we have given here depends crucially on the fact that the feasible set is a subset of an appropriate order interval. This is in the nature of things for an exchange economy, but quite problematical in the more general production context. Nevertheless, by appealing to truncation arguments, the order ideal approach remains a powerful technique even in the production context [see Zame (1987)].

11. Separable utilities and the finance model

As we have discussed in the Introduction, in finance models it is common to take the commodity space to be $L_2(S, \Sigma, \mu)$ (for (S, Σ, μ) a probability space) and consumption sets to be the positive cone $L_2(S, \Sigma, \mu)^+$. Since the positive cone has an empty interior, Theorem 8.1 does not apply. Moreover, much of finance theory assumes instantaneous utility functions with infinite marginal utility for consumption at zero, a requirement incompatible with uniform properness, so Theorem 9.1 also does not apply. In what follows, we show how the special structure of the finance model may be combined with Theorem 10.1 to sidestep these difficulties. As we shall see, the idea is to exploit separability of utility functions and the nature of optimum allocations. Our discussion follows Araujo and Monteiro (1989a) and Duffie and Zame (1989).

Let (S, Σ, μ) be a probability space. We take as commodity space $L = L_p(S, \Sigma, \mu)$ (with the norm topology and pointwise ordering) and as price space the dual $L^* = L_q(S, \Sigma, \mu)$, where $1 \leq p < \infty$ and $(1/p) + (1/q) = 1$ (in the finance setting, $p = q = 2$). We assume that individual utility functions $u_i : L^+ \to \mathbb{R}^+$ are norm continuous, strictly monotone, concave and additively separable. That is, there are concave, continuous, strictly monotone functions $v_i : [0, \infty) \times S \to \mathbb{R}^+$ such that

$$u_i(x) = \int v_i(x(s), s)\, \mathrm{d}\mu(s)$$

for each $x \in L^+$. We shall also assume that each $v_i(\cdot, s)$ is continuously differentiable on $(0, \infty)$ for each s; we write $v_1'(\cdot, s)$ for its derivative (and $v_1'(0, s)$ for the right-hand derivative at 0).

For each i, define $Q^i : L_p^+ \times S \to \mathbb{R}^+$ by $Q^i(z, s) = v_i'(z(s), s)$. Araujo and Monteiro (1989a) show that properness of u_i at z is equivalent to F-properness of u_i at z, which in turn is equivalent to the assertion that the function $Q^i(z, \cdot)$ belongs to L_q. In this case, $Q^i(z, \cdot)$ is a supporting linear functional at z. If z is strictly positive then $Q^i(z, \cdot)$ is (up to scalar multiples) the *unique* supporting linear functional at z. From this it follows, incidentally, that u_i cannot be uniformly proper if $v_i'(0, s) = \infty$ for each s.

On the other hand, Theorem 10.1 assures us that an equilibrium will exist provided that ω is strictly positive (and hence belongs to the quasi-interior of L_p^+) and that for every weak optimum x which is individually rational (i.e. $u_i(x_i) \geq u_i(\omega_i)$ for each i), each u_i is F-proper (equivalently in this setting, proper) at x_i. In fact, it suffices to have properness for a single allocation z. Araujo and Monteiro (1989a) establish the following theorem for the case $z_i = \omega_i$; Duffie and Zame (1989) use (in essence) the case $z_i = \omega/N$.

Theorem 11.1. *If $\omega(s) > 0$ for almost all $s \in S$, and there is any allocation $z \geq 0$ with $\Sigma\, z_i = \omega$, and such that u_i is proper at z_i for each i, then the economy has a quasi-equilibrium.*

Proof. Let x be any individually rational weak optimum; we wish to show that each v_i is proper at x_i. To this end, we use optimality to choose weights α_i, $0 < \alpha_i < 1$, such that the weighted sum $\Sigma\, \alpha_i v_i(y_i(s), s)$ is maximized (over all allocations y) by taking $y = x$. It follows that for almost all $s \in S$, if $x_i(s) > 0$ then

$$\alpha_i Q^i(x_i, s) \geq \alpha_j Q^j(x_j, s) \quad \text{for every } j\,.$$

For each k, set $S_k = \{s \colon x_k(s) > z_k(s)\}$. If $s \in S_k$, then for every i we have

$$\alpha_i Q^i(x_i, s) \leq \alpha_k Q^k(x_k, s) \leq \alpha_k Q^k(z_k, s)$$

and if $x \in S \setminus \bigcup S_k$, then for every i we have

$$\alpha_i Q^i(x_i, s) = \alpha_i Q^i(z_i, s) .$$

Since $\alpha_i < 1$ for each i, we conclude that

$$\alpha_i Q^i(x_i, s) \leq \max_k Q^k(z_k, s) .$$

Because $Q^k(z_k, \cdot)$ belongs to L_q for each k and $\alpha_i > 0$ for each i, we conclude that $Q^i(x_i, \cdot)$ belongs to L_q for each i. Hence u_i is proper at x_i for each i, as desired.

Therefore, for every individually rational, weak optimum x, each u_i is proper, and hence F-proper, at x_i. The existence of a quasi-equilibrium now follows from Theorem 10.1. ∎

Although we have derived Theorem 11.1 via Theorem 10.1, it could also be derived directly from Theorem 7.1. To accomplish this, note that, in the argument we have given, strict monotonicity of preferences guarantees that the weights α_i are uniformly bounded away from 0. Therefore, we can support individually rational weakly optimal allocations by prices lying in a compact set.

For more on existence issues in the finance model, see Karatzas, Lakner, Lehoczky and Shreve (1990), Dana and Pontier (1989) and Dana (1990).

12. The lattice structure of the price space

In the preceding sections, we have seen that the order structure of the commodity space plays a key role when consumption sets have empty interior. Indeed, many of the arguments we have given to this point depend heavily on the assumptions that the commodity space is a lattice and that the lattice operations are (uniformly) continuous, or equivalently, that the topology is locally solid. As Example 9.2 shows, these assumptions are not entirely dispensable. Unfortunately, they rule out some economically important examples, including the commodity space $M(K)$ equipped with the weak topology $\sigma(M(K), C(K))$, and the commodity space $M([0,1])$ equipped with the weak topology $\sigma(M([0,1]), \mathrm{Lip}([0,1]))$. As we have discussed (see the Introduction and Examples 4.3 and 4.4), these commodity and price spaces have been used in models of product differentiation and intertemporal consumption. In this section we show how the assumptions on the commodity space can be weakened to incorporate examples such as these. Our discussion follows Mas-Colell and Richard (1991).

In what follows, we consider a commodity space L which is (Hausdorff) locally convex topological vector space with topology τ, and which is ordered by a closed, convex, non-degenerate, positive cone L^+. We also assume that L is a vector lattice with respect to this order, but we do *not* assume that the lattice operations are continuous (in particular, we do not assume that τ is locally solid). Instead, we assume only that the dual space L^* is a sublattice of the order dual (i.e. that for p, q in L^*, the supremum $p \vee q$ and infimum $p \wedge q$ are also in L^*). If L is a topological vector lattice, the lattice structure of L^* obtains automatically, but, as may be seen from the examples cited above, the lattice assumption on L^* is strictly weaker than the assumption that L is a topological vector lattice.

Note that the commodity/price duality of Example 9.2 ($L = M([0, 1])$ with the topology $\sigma(M([0,1]), C^1([0,1]))$, $L^* = C^1([0, 1])$) does not satisfy these assumptions; with the natural order, the commodity space is a vector lattice and the positive cone is closed, but the dual space is not a lattice. As the reader may see, it is precisely this failure of the dual to be a lattice (i.e. the failure of the supremum of two differentiable functions to be differentiable) that lies at the heart of Example 9.2. And it is precisely this failure to be a lattice that distinguishes between price spaces such as $C(K)$ and Lip([0, 1]) on the one hand and $C^1([0, 1])$ on the other.

The argument for the existence of equilibrium in this setting, like the argument given in Section 9, breaks into two parts. The first part establishes the existence of a compact set of supporting prices.

Theorem 12.1. *Assume, in addition to the basic assumptions, that:*
(i) *L is a vector lattice and L^* is a sublattice of the order dual;*
(ii) *for each i, $X_i = L^+$;*
(iii) *for each i, $\succcurlyeq_i$ is uniformly proper on the order interval $[0, \omega]$.*
Then there is a weak star compact, convex set $K \subset (L^{*+})^N$ such that $\Sigma\, p_i \cdot \omega = 1$ for every $(p_1, \ldots, p_N) \in K$, and every weak optimum is supported by a price of the form $p_1 \vee \cdots \vee p_N$ for some $(p_1, \ldots, p_N) \in K$.

The crucial difference between Theorems 12.1 and 9.1 is that here the supporting price is constructed in an explicit way (which makes quite clear the role played by the lattice structure of the price space). The explicit construction of the supporting price makes it possible to treat the set of supporting prices in a "disaggregated" fashion, and it is this avoidance of aggregation which allows us to dispense with local solidness of the topology τ in L.

Proof. Let $x = (x_1, \ldots, x_N)$ be a weak optimum (we assume $\Sigma\, x_i = \omega$), and for each i, set $W_i = \{z \in L^+ : z \succ_i x_i\}$, and $V_i = W_i + \Gamma$, where Γ is the properness cone. Write $V = \{(v_1, \ldots, v_N) \in L^N : v_i \in V_i\}$. Uniform properness implies that $V \cap Z = \emptyset$. Since V contains an open set, the separation theorem provides a linear functional $(p_1, \ldots, p_N) \in L^{*N}$ separating V from Z. There is no loss of generality in normalizing so that $\Sigma\, p_i \cdot \omega = 1$. To see that $p_1 \vee \cdots \vee p_N$ is a supporting price, we show first that $(p_1 \vee \cdots \vee p_N)\cdot x_i = p_i \cdot x_i$ for each i. Note that the definition of supremum for linear functionals (see Section 2) yields

$$\begin{aligned}\sum (p_1 \vee \cdots \vee p_N)\cdot x_i &= (p_1 \vee \cdots \vee p_N)\cdot \sum x_i \\ &= (p_1 \vee \cdots \vee p_N)\cdot \omega \\ &= \sup\left\{\sum p_i \cdot z_i \colon z_i \geqslant 0, \sum z_i \leqslant \omega\right\} \\ &= \sum p_i \cdot x_i\end{aligned}$$

(the last equality following because $(p_1, \ldots, p_N)$ separates V from Z). On the other hand, $(p_1 \vee \cdots \vee p_N)\cdot x_i \geqslant p_i \cdot x_i$ for each i. Combining these gives $(p_1 \vee \cdots \vee p_N) \vee x_i = p_i \cdot x_i$ for each i, as desired. Observe now that if $z_i \succ_i x_i$ then $p_i \cdot z_i \geqslant p_i \cdot x_i$ (this again follows from the separating property). Therefore

$$(p_1 \vee \cdots \vee p_N)\cdot z_i \geqslant p_i \cdot z_i \geqslant p_i \cdot x_i \geqslant (p_1 \vee \cdots \vee p_N)\cdot x_i\,.$$

Finally, we may take

$$K = \left\{(p_1, \ldots, p_N)\colon \sum p_i \cdot \omega = 1 \text{ and } p_i \cdot y \geqslant 0 \text{ for each } i \right.$$
$$\left. \text{and each } y \in \Gamma\right\},$$

so the proof is complete. ∎

As in Section 9, we obtain the existence of quasi-equilibrium.

Theorem 12.2. *Assume, in addition to the basic assumptions, that:*
(i) *L is a vector lattice and L^* is a sublattice of the order dual;*
(ii) *for each i, $X_i = L^+$;*
(iii) *for each i, $\succcurlyeq_i$ is uniformly proper on the order interval $[0, \omega]$;*
(iv) *the utility possibility set U is compact.*
Then the economy has a quasi-equilibrium.

The proof of Theorem 12.2 follows the same outline as the proof of Theorem 9.2, but it is more subtle, because the price set here is disaggregated. (Since the lattice operations in L are not assumed to be continuous, we cannot conclude that the aggregated price set $\{p_1 \vee \cdots \vee p_N \colon (p_1, \ldots, p_N) \in K\}$ is compact.) In essence, what is required is to reprove Theorem 7.1 with a disaggregated price set. Establishing the required upper hemi-continuity and convexity properties is delicate, and we refer to Mas-Colell and Richard (1991) for details.

13. Other approaches

To this point, we have focused on the approach to the existence of equilibrium via the Negishi method. There are at least three other approaches to the existence of competitive equilibrium that have been used in the infinite dimensional setting: finite approximations, core equivalence and excess demand. We cannot do justice here to the virtues of each of these methods; instead, we give a detailed sketch of the method of finite approximations, and content ourselves with merely indicating the way in which the other methods proceed. As we have noted previously, our main purpose for following the Negishi approach is that we are able to exhibit the main difficulties in a clear way. Of course, since these difficulties are central to the existence problem, they arise, in one way or another, in all proofs. We shall try to make this apparent in our discussions.

The idea underlying the method of finite approximations is to approximate the given economy (with an infinite dimensional commodity space) by a family of economies with finite dimensional commodity spaces. Familiar results then guarantee that each of these economies has an equilibrium. One then proves that an equilibrium for the original economy can be obtained as a limit of equilibria for the finite dimensional economies. To illustrate the details, we sketch a proof of Theorem 8.1 via finite approximations. With only one fairly small variation, the argument is Bewley's (1972) [see also Mertens (1970)].

Theorem 13.1. *Assume, in addition to the basic assumptions, that $\omega \in \operatorname{int} L^+$, and that the utility possibility set U is closed. Then a quasi-equilibrium exists.*

Proof. Let $\mathcal{F}$ be the family of finite dimensional subspaces of L which contain the initial endowments ω_i. Note that $\mathcal{F}$ is directed by set inclusion. For each $F \in \mathcal{F}$, let $\mathcal{E}^F$ be the economy obtained by restricting all the data to the subspace F; i.e. consumption sets in $\mathcal{E}^F$ are $X_i \cap F$, etc. The usual finite dimensional existence results guarantee that the economy $\mathcal{E}^F$ has a quasi-equilibrium (x^F, p^F).

There is no loss in normalizing so that $p^F \cdot \omega = 1$. Since $\omega \in \operatorname{int} L^+$, there is a symmetric neighborhood W of 0 such that $\omega + W \subset L^+$, whence $\omega + (W \cap F) \subset F \cap L^+$. Monotonicity implies that p^F is positive on $F \cap L^+$, so $p^F \cdot z \geq -1$ for $z \in W \cap F$. Symmetry of W now yields $-1 \leq p^F \cdot z \leq +1$ for $z \in W \cap F$. We can then apply the Hahn–Banach extension theorem (see Section 2) to find an extension $\hat{p}^F$ of p^F to all of L such that $\hat{p}^F \cdot \omega = 1$ and $-1 \leq \hat{p}^F \cdot z \leq +1$ for all $z \in W$.

Alaoglu's theorem (see Section 2) guarantees that the set of linear functionals

$$\Delta = \{q \in L^*: q \cdot \omega = 1 \text{ and } -1 \leq q \cdot z \leq +1 \text{ for all } z \in W\}$$

is $\sigma(L^*, L)$-compact, so, passing to a subnet if necessary, we may find a linear functional $p \in L^*$ such that $p \cdot \omega = 1$, $-1 \leq p \cdot z \leq +1$ for all $z \in W$ and $\hat{p}^F \cdot y \to p \cdot y$ for all $y \in L$. The assumption that the utility possibility set is closed implies that there is an allocation x such that $u_i(x_i) \geq \limsup u_i(x_i^F)$ for each i. We shall show that (x, p) is a quasi-equilibrium.

We first establish the following claim: If $u_i(y_i) > u_i(x_i)$ then $p \cdot y_i \geq p \cdot \omega_i$. If not, then there is a finite dimensional subspace $F_0 \in \mathcal{F}$ such that, whenever $F \supset F_0$, we have $u_i(y_i) > u_i(x_i^F)$ and $p \cdot y_i < p \cdot \omega_i$; there is no loss in assuming that F_0 contains y_i. Since $\hat{p}^F \to p$ and $\hat{p}^F$ is an extension of p^F, we may also choose F_0 so that $p^F \cdot y_i < p^F \cdot \omega_i$ whenever $F \supset F_0$. However, this contradicts the fact that (x^F, p^F) is a quasi-equilibrium, and this contradiction establishes the claim.

Finally, to show that (x, p) is a quasi-equilibrium, we must only verify that $p \cdot x_i \leq p \cdot \omega_i$ for each i. If, to the contrary, $p \cdot x_i > p \cdot \omega_i$ for some i, then, since $\Sigma x_i = \Sigma \omega_i$, it follows that $p \cdot x_j < p \cdot \omega_j$ for some j. Monotonicity then yields a contradiction to the above claim, so the proof is complete. ■

As we have noted many times, closedness of the utility possibility set follows from the existence of a compatible topology in which the set of allocations is compact. For instance, if the commodity space is $L = L_\infty(S, \Sigma, \mu)$, closedness of the utility possibility set follows if preferences are Mackey (and hence weak star) upper semi-continuous and the consumption sets are Mackey (hence weak star) closed. These are precisely the assumptions of Bewley's (1972, Theorem 1).

The arguments used above are readily adapted to the case of unordered preferences [see Khan (1984)]. We should also note that, although monotonicity of preferences, the assumption that $X_i + L^+ \subset X_i$, and the order structure of the commodity space L, all play a role in this argument, they are in fact superfluous; Zame (1987) shows how to eliminate them entirely.

It is instructive to compare the way in which the main difficulties we have isolated (supportability, compactness, joint continuity) are addressed in the argument sketched above and in the Negishi approach. As in the Negishi approach, compactness is assumed in the form of the assumption that the utility possibility set is closed. (This substitutes for the assumption of a compatible topology in which the set of allocations is compact.) As in the

Negishi approach, joint continuity is finessed by arguments that amount to proving the joint continuity of the wealth map $(x, p) \to p \cdot x$ only along special nets of consumptions and prices. Finally, supportability is guaranteed by the assumption that the positive cone L^+ has a non-empty interior, but here this assumption is used indirectly. On the one hand, it guarantees that the quasi-equilibrium prices p^F for the finite dimensional approximating economies lie in a $\sigma(L^*, L)$-compact subset; on the other hand, it guarantees that the limit of a subnet is not the zero price.

If L^+ has an empty interior, neither of these conclusions is necessarily valid. In general, it may not be possible to choose the finite dimensional equilibrium prices to lie in a $\sigma(L^*, L)$-compact set, or to be sure that the limit price is not identically zero. Here properness comes to the rescue. Yannelis and Zame (1986) show how to use uniform F-properness (which is equivalent to uniform properness) to obtain bounds on the finite dimensional equilibrium prices. These bounds guarantee that the finite dimensional equilibrium prices (suitably normalized) lie in a $\sigma(L^*, L)$-compact subset, and hence have a convergent subnet, and that the limit of this subnet is not the zero price. The argument is rather complicated, however, because the use of F-properness to obtain bounds on the finite dimensional equilibrium prices depends on being able to choose the finite dimensional subspaces $F \subset L$ to actually be *sublattices*. A more efficient route to the result is to use the arguments of the proof of Theorem 13.1 to establish the existence of equilibrium in the order ideal $L(\omega)$, and then use F-properness to conclude that the equilibrium price on $L(\omega)$ extends to an equilibrium price on all of L (see Section 10).

The method of core equivalence is based on the Debreu–Scarf theorem, which, in the finite dimensional setting, asserts the coincidence of the set of equilibrium allocations of an economy with the intersection of the cores of all replications. Since an algorithm of Scarf provides a direct proof of the non-emptiness of the core of a finite dimensional economy, the Debreu–Scarf theorem also provides a proof of the existence of competitive equilibrium in the finite dimensional setting. Aliprantis, Brown, and Burkinshaw (1987b, 1989b), following a precedent of Peleg and Yaari (1970), have used this method to establish the existence of equilibria in the infinite dimensional setting. First of all, they show that if there is a compatible topology in which the set of allocations is compact, then the economy has a non-empty and compact core. (This can be obtained via finite approximations or directly through Scarf's theorem.) It follows that the intersection of the cores of all replications (which Aliprantis, Brown and Burkinshaw call the set of *Edgeworth equilibria*) is non-empty. That every equilibrium allocation is an Edgeworth equilibrium is a simple extension of the first welfare theorem. The converse is true whenever the positive cone of the commodity space has a non-empty interior. In particular, an Edgeworth equilibrium can always be supported as a price equilibrium on the order ideal $L(\omega)$. An appeal to uniform properness then guarantees that the equilibrium price on $L(\omega)$ can be extended to all of L so as to be an equilibrium price for the original economy.

The excess demand approach has been used in the infinite dimensional context by a number of authors, including Aliprantis and Brown (1983), Bojan (1974), El-Barkuki (1977), Yannelis (1985), Florenzano (1983) and van Zandt (1989). As in the finite dimensional setting, it depends on the use of some form of the Kakutani fixed point theorem or its variant, the Gale–Debreu–Nikaido lemma. Assuming compactness of the set of allocations (or closedness of the set of utility possibilities), the crucial issue is obtaining a compact price simplex. If the positive cone of the commodity space has non-empty interior (and preferences are monotone), the set Δ identified in the proof of Theorem 13.1 is a suitable compact price simplex. If the positive cone has empty interior, however, it will in general not be possible to find a compact price simplex which does not include the zero price. Again, uniform properness comes to the rescue, since it guarantees that the only prices we need consider are those in the simplex

$$\{q \in L^*: q \cdot v = 1 \text{ and } -1 \leq q \cdot z \leq +1 \text{ for all } z \in V\}$$

where v is a properness vector (common to all consumers) and V is the neighborhood of 0 whose existence is assumed in the definition of uniform properness. The joint continuity problem can be finessed as in the proof of Theorem 13.1. (A minor complication is that, for a given price p, endowment ω_i and utility function u_i, an optimal consumption choice need not exist. Hence one cannot work directly with the excess demand mapping; a truncated version must be used.)

Finally, we should refer to Ionescu-Tulcea (1986, 1988b) for the infinite dimensional version of the approach to the existence of equilibrium via generalized games. (The technical issues are similar to those arising in the excess demand approach, although the disaggregated nature of the generalized games approach may be a potential advantage.)

14. Production

In this section, we review the extension of the previous results to a production context. This extension is less straightforward than in the finite dimensional setting. Over and above the familiar difficulties (compactness of the set of feasible allocations, supportability of optima, . . .) there are new ones specific to production. In retrospect this should not be surprising, since many of the previous results depended on order restrictions on consumption sets (for instance $X_i = L^+$), which have no obvious analog for production sets. We will concentrate on these additional difficulties, but we omit the proofs, referring the reader to the original papers – or challenging him/her to adapt the proofs for the exchange case!

Before proceeding further, let us agree on the data of a production economy. On the consumption side, we simply adopt the notation and assumptions introduced in Section 3 and maintained throughout. We describe the production side by a finite number M of *firms*, each of which is characterized by a production set Y_j. In the recursive treatment of production theory, it is customary to assume a countable number of firms, one for each date [see Malinvaud (1953). This difference is important for the study of production efficiency in the intertemporal context, but not for the study of equilibrium. We shall always assume that production sets are closed and convex, contain 0, and have the property that $y - L^+ \subset Y_j$ whenever $y \in Y_j$ (that is, we assume free disposal in production). Profits of the firms are distributed to consumers according to *firm shares* (θ_{ij}), where $\Sigma\, \theta_{ij} = 1$ for each j.

By an *allocation* we mean an $(N + M)$-tuple (x, y) where $x_i \in X_i$ for each i, $y_j \in Y_j$ for each j and

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} \omega_i + \sum_{j=1}^{M} y_j \, .$$

As usual, a *quasi-equilibrium* is an $(N + M + 1)$-tuple (x, y, p) where (x, y) is an allocation and p is a continuous linear functional on L such that $p \cdot \omega > 0$ and:

(1) $p \cdot y_j = \max\{p \cdot v \colon v \in Y_j\}$ for each j;

(2) $p \cdot x_i \leqslant p \cdot \omega_i + \Sigma\, \theta_{ij}(p \cdot y_j)$ for each i;

(3) if $u_i(v) > u_i(x_i)$ then $p \cdot v \geqslant p \cdot \omega_i + \Sigma\, \theta_{ij}(p \cdot y_j)$.

The boundedness assumptions that are typically used in the finite dimensional setting to obtain compactness of the set of attainable allocations are far from sufficient in the general infinite dimensional setting. On the other hand, the order boundedness properties that are so useful in the exchange case are far from automatically satisfied in the general production context. Thus, we shall need to make compactness assumptions on the attainable set directly. The hypotheses should not be difficult to verify in each particular application.

As might be expected from our discussions of the exchange case, the supportability problem disappears when the positive cone L^+, and hence the production sets Y_j, have a non-empty interior. With the appropriate compactness assumptions, this leads quickly to the following theorem, due essentially to Bewley (1972).

Theorem 14.1. *Assume that the maintained hypotheses on the consumption side and the above assumptions on the production side are valid. If*

(i) $\omega \in \operatorname{int} L^{+}$,

(ii) *there is a compatible topology in which the set of attainable allocations is compact,*

then the economy has a quasi-equilibrium.

As in the exchange case, this result is not completely satisfactory. If the commodity space is L_∞, we only obtain a quasi-equilibrium price in the dual space $L_\infty^* = \mathrm{ba}$, the space of finitely additive measures. We would like instead to obtain a quasi-equilibrium price which is a countably additive measure, i.e. an element of L_1. In the exchange case, we can do so (Theorem 8.2) if $X_i = L_\infty^+$ for each i, $\omega \in \operatorname{int} L_\infty^+$, and preferences are strictly monotone and continuous in the Mackey topology $\tau(L_\infty, L_1)$. In the production case, it is natural to require in addition that production sets be Mackey closed and that the set of attainable allocations be weak star compact. But, as the following example shows, more will be required.

Example 14.1. Take $L = l_\infty$. There is one consumer, with consumption set $X = l_\infty^+$, endowment $\omega = (1, 1, \ldots)$, and utility function $u(x) = \Sigma\, 4^{-n} x(n)$. There is one firm, whose production set is

$$Y = \{y \colon y^+(1) \leq \liminf y^-(n)\}\ .$$

Let (x, y, p) be a quasi-equilibrium, and suppose that $p \in l_1$. For each $n \geq 2$, write z^n for the sequence whose first n terms are 0, and whose remaining terms are 1. Profit maximization guarantees that $p(1) \leq p \cdot z^n$ for all $n \geq 2$. On the other hand, utility maximization guarantees that $p(1) > 0$. Hence $p \cdot z^n \not\to 0$. But this contradicts the supposition that $p \in l_1$.

With the interpretation of elements of l_∞ as commodity streams over an infinite time horizon, the above example is familiar from growth theory. In economic terms, the difficulty is that outputs come before inputs. To treat such difficulties, Prescott and Lucas (1972) suggested the following assumption.

Possibility of Truncation. If $(y(1), \ldots, y(n), \ldots) \in Y$, then $(y(1), \ldots, y(n), 0, 0, \ldots) \in Y$ for each n.

Using this assumption (for the commodity space $L = l_\infty$), Prescott and Lucas (1972) obtain the existence of supporting prices in l_1. In order to obtain prices in L_1 (for the commodity space $L = L_\infty$), Bewley (1972) uses the Yosida–Hewitt decomposition of linear functionals $p \in \mathrm{ba}^+$ into a countably additive part p_c and a finitely additive part p_f (see Section 8) to formulate the following assumption, which is a generalization of the Possibility of Truncation.

Exclusion Assumption. For each production set $Y \subset L_\infty = L_\infty(\Omega, \mathcal{F}, \mu)$ and each $p \in \text{ba}^+$, there is a sequence $\{F_n\}$ of measurable sets such that $p_c(F_n) \to 0$, $p_f(\Omega \backslash F_n) = 0$ for all n and $(y|\Omega \backslash F_n) \in Y$ for each $y \in Y$ and all n.

Together with the assumptions of Theorem 8.2 and the assumption that production admits constant returns to scale, the exclusion assumption is just what is required to yield quasi-equilibria with prices in L_1. In fact, it guarantees that all quasi-equilibria can be supported by prices in L_1.

Theorem 14.2. *Let $L = L_\infty$. Assume the maintained hypotheses and*:

(i) *for each i, $X_i = L_\infty^+$*;

(ii) *preferences are strictly monotone and Mackey continuous*;

(iii) *for each j, Y_j is a Mackey closed, convex cone at 0 and satisfies the exclusion assumption*;

(iv) $\omega \in \text{int}\, L_\infty^+$;

(v) *the set of attainable allocations is $\sigma(L_\infty, L_1)$-compact.*

Then the economy has a quasi-equilibrium. Moreover, if (x, y, p) is any quasi-equilibrium with $p \in \text{ba}^+$, then (x, y, p_c) is also a quasi-equilibrium (where p_c is the countably additive part of p).

For commodity spaces in which the positive cone has empty interior, failure of supportability may entail that quasi-equilibria need not exist, as the following example shows.

Example 14.2. The commodity space is $L = l_1$. There is a single consumer, with consumption set $X = l_1^+$, endowment $\omega_1 = (4^{-n})$ and utility function $u(x) = \Sigma\, x(n)$. There is one firm, whose production set Y is the closed convex cone (at the origin) generated by the negative cone $(-l_1^+)$ and the set of all vectors of the form $-\delta_k + 2\delta_{k+1}$, for every k which is not a power of 2. Interpreting a sequence in l_1 as a commodity stream over an infinite time horizon, this means that the production technology can produce, from one unit of input in a given period, two units of output in the next period (except for initial periods which are powers of two.) It is easily checked that the consumption side of this economy satisfies our maintained hypotheses and that the set of allocations is norm compact. However, this economy has no quasi-equilibrium.

To see that this is so, suppose to the contrary that (x, y, p) were a quasi-equilibrium, where $p \in l_\infty = l_1^*$ is a bounded sequence. Profit maximization by the firm implies that the functional p is positive, and that $p(k) \geqslant 2p(k+1)$ for each k which is not a power of 2. On the other hand, because there is no production in any period which is a power of 2, the consumer's final allocation x is certainly strictly positive in such periods. Utility maximization by the consumer therefore implies that, in particular $p(2^m) = p(2^n) > 0$ for each m

and n. Since the sequence p is unbounded, these conditions can only be compatible if $p(k)=0$ for each k, a contradiction.

A little reflection reveals the problem here: one unit of input, if used late enough, may be used to produce an arbitrarily large quantity of output (many periods later). In particular, the rates of technological transformation are unbounded. This difficulty can be treated by making assumptions which, directly or indirectly, bound (marginal) rates of technological transformation. Two assumptions of this kind have been used in the literature. Mas-Colell (1986b) and Richard (1989), assume that production sets satisfy a condition which is the analog, on the production side of the economy, of uniform properness on the consumption side of the economy. This condition indirectly bounds marginal rates of technological transformation, in much the same way that properness in consumption bounds marginal rates of substitution. Zame (1986) gives a condition which explicitly bounds marginal rates of technological transformation. Although they are different, both assumptions make essential use of the lattice structure of the commodity space. Since properness in production is easier to describe, we begin there (although Zame's approach was historically first and served as inspiration). We follow Richard (1989), which generalizes and simplifies Mas-Colell (1986b).

Let L be a topological vector lattice, Y a production set (in particular, Y is a closed, convex set containing the negative cone), and ω a positive element of L. (In practice, we shall want to take for ω the aggregate endowment). We say that Y is *ω-uniformly proper* if there is a neighborhood W of 0 in L such that, for each $y \in Y$, $(y-V) \cap \{x \in L\colon x^+ \leqslant y^+\} \subset Y$, where V is the cone $V=\{\lambda\omega+\lambda w\colon w \in W, \lambda>0\}$. Note that if $\omega \in \operatorname{int} L^+$ (which of course requires that $\operatorname{int} L^+ \neq \emptyset$), this condition is automatically satisfied, since we may take for W the translate to the origin of any open neighborhood of ω contained in L^+. (In particular, this covers the case where L is finite dimensional and all goods are represented initially.) Informally, ω-uniform properness is the assumption that ω can substitute for any other input in the production of any given output, and that the rate of substitution is uniformly bounded.

To describe the approach in Zame (1986), we assume that the commodity space L is a normed lattice. For the production set Y, we say that the *marginal rate of technological transformation is bounded* if there is a constant C such that, if $y=y^+-y^-$ is in Y and $0 \leqslant z^- \leqslant y^-$, then there is a z^+ such that $0 \leqslant z^+ \leqslant y^+$, z^+-z^- is in Y and $\|y^+-z^+\| \leqslant C\|y^- - z^-\|$. Informally, this is a condition on the marginal rates of transformation of inputs to outputs.

In finite dimensional spaces, ω-uniform properness is always satisfied if $\omega \gg 0$, but marginal rates of technological transformation may be unbounded near zero production.

Perhaps these conditions may be most easily understood in the context of a technology which produces a single output good according to some (smooth) production function $f : (-L^+) \to \mathbb{R}$. In that case, the marginal rate of technological transformation is bounded precisely when the directional derivatives $D_z f(y)$ of the production function f are (uniformly) bounded (for all inputs y and positive directions $z \leq y$). By contrast, production is ω-uniformly proper exactly if the ratios $D_z f(y)/D_\omega f(y)$ are uniformly bounded (for all inputs y and positive directions $z \leq y$). Thus, if $D_\omega f(y)$ is uniformly bounded away from 0, then ω-uniform properness implies a bounded marginal rate of technological transformation. However, if $D_\omega f(y)$ is not uniformly bounded away from 0, the two conditions are incomparable. The following example makes the same point.

Example 14.3. Let $L = l_1$; consider two production sets

$$Y_1 = \left\{ y : \sum_{n=2}^{\infty} y^+(n) \leq y(1) \right\},$$

$$Y_2 = \left\{ y : \sum_{n=k}^{\infty} y^-(n) \leq \sum_{n=k}^{\infty} y^+(n) \text{ for each } k \right\}.$$

As before, we interpret an element of l_1 as a stream of a single commodity over an infinite time horizon. The production set Y_1 corresponds to a storage technology in which any quantity of the commodity may be stored at date 1, for release at any future time(s); no new input to storage is possible. The production set Y_2 corresponds to a storage technology in which any quantity of the commodity may be stored at any date, for release at any future date(s), and new inputs to storage are possible at any time. (But release before storage is impossible.) If we take any $\omega \in l_1^+$ with $\omega(1) > 0$, then Y_1 is ω-uniformly proper, but Y_2 is not. On the other hand, for both Y_1 and Y_2, the marginal rates of technological transformation are bounded (by 1).

Either ω-uniform properness of production sets or bounded marginal rates of technological transformation are sufficient to guarantee that quasi-equilibria exist. Theorem 14.3 is from Richard (1989) and Theorem 14.4 is from Zame (1986). Zame (1986) also gives a result which is valid without the assumption of constant returns to scale in production. However, as McKenzie (1959) has shown, the assumption of constant returns to scale involves essentially no loss of generality.

Theorem 14.3. *Let L be a topological vector lattice. Assume the maintained hypotheses on the consumption side and:*

(i) *for each i, $X_i = L^+$;*
(ii) *preferences are ω-uniformly proper;*
(iii) *each production set Y_j is ω-uniformly proper;*
(iv) *the set of attainable allocations is compact in some compatible topology.*

Then the economy has a quasi-equilibrium.

Theorem 14.4. *Let L be a normed lattice. Assume the maintained hypotheses on the consumption side and:*

(i) *for each i,* $X_i = L^+$;

(ii) *preferences are* ω*-uniformly proper*;

(iii) *each production set* Y_j *is a closed convex cone at* 0 *and its marginal rate of technological transformation is bounded*;

(iv) *the set of attainable allocations is compact in some compatible topology. Then the economy has a quasi-equilibrium.*

15. Final comments

Lack of space has prevented us from discussing many other topics. Here we mention a few that seem important and promising for further research.

(A) With the exception of Theorems 7.1 and 8.1, we have not considered general consumption sets with empty interior. Although little has been done in this area, a tentative conclusion is that general consumption sets are similar to general production sets, and that methods analogous to those used in the production case may be relevant. Some special results and a striking counter-example have been given by Back (1988); see Example 8.3 and the remarks following Example 9.2. The free disposal assumption on consumption sets (i.e. $X_i + L^+ \subset X_i$) is also restrictive in some contexts (such as finance models with incomplete markets). See Boyd and McKenzie (1990) for more on consumption sets.

(B) An important line of research in classical general equilibrium theory has been the relationship of the core to the set of competitive allocations. In the infinite dimensional setting, Aliprantis, Brown and Burkinshaw have developed an extensive body of work centered around the infinite-dimensional version of the Debreu–Scarf core convergence theorem. We have briefly touched on this work in Section 13; for further details, we refer the reader to the original papers and especially to a recent monograph [Aliprantis, Brown and Burkinshaw (1989b)]. Nothing seems to have been done to date on more general core convergence results (i.e. without the assumption of replication). There is also an extensive literature on infinite dimensional versions of Aumann's core equivalence theorem for non-atomic economies, including Gabszewicz (1968a,b), Mertens (1970), Bewley (1973), Mas-Colell (1975), Jones (1984), Ostroy (1984), Gretsky and Ostroy (1986b), Zame (1986), Podczeck (1985), Rustichini and Yannelis (1987) and Ostroy and Zame (1988). The existence of equilibrium is a particularly thorny issue; see in particular the counter-examples in Zame (1986).

(C) Determinacy (local uniqueness) of equilibrium is largely unexplored in the infinite dimensional setting. Some early work was carried out by Chichilnisky and Kalman (1980) in the context of resource allocation problems and by

Araujo and Scheinkman (1977) in the context of capital theory. Kehoe, Levine, Mas-Colell and Zame (1989) have followed an approach that takes excess demand functions as primitives; their work uses the theory of Fredholm operators and Smale's infinite dimensional version of Sard's theorem. Approaches that take preferences and endowments as primitives seem to encounter many difficulties (in addition to the usual difficulties of doing calculus in infinite dimensional spaces). The natural domain of prices is the positive orthant $(L^*)^+$ of the dual space, but this set usually has empty interior, which is very inconvenient for doing calculus. Moreover, excess demand functions are typically not defined [Araujo (1987), Hildenbrand (1989)] and are not generally smooth even when they are defined. Indeed, Araujo (1987) argues that excess demand functions can be smooth only if the commodity space is a Hilbert space.

It might appear that the Negishi approach would avoid most of these difficulties by allowing us to work with the utility map on a finite dimensional space. This approach has indeed been applied in a special case by Kehoe, Levine and Romer (1989a,c), but carrying it through in reasonable generality has met with a serious technical difficulty: establishing the smoothness of the utility mapping.

(D) One limitation of the Negishi approach that we have adopted is that it is very dependent on utility functions, and therefore on the completeness and transitivity of preferences. To treat unordered preferences, the approach via finite dimensional approximations is superior (see Section 13). Existence results with unordered preferences have been obtained by Khan (1984), Toussaint (1985) and Yannelis and Zame (1986).

The Negishi approach also depends on the Pareto optimality of equilibria, and hence is not applicable to distorted or incomplete markets (where equilibria need not be Pareto optimal). Unfortunately, the approach via finite dimensional approximations also does not appear to work when markets are incomplete (even if the number of securities is finite). The difficulty (as a careful reading of our discussion in Sections 8 and 13 will show) lies in finessing the joint continuity of the wealth mapping.

Existence of equilibrium with incomplete markets and a countable number of states (or commodities) has been obtained by Zame (1988), Green and Spear (1988), Zevine (1989) and Hernandez (1988). The case of a continuum of states (or commodities) is difficult and remains largely unresolved. For related work, see Duffie, Geanakoplos, MacLennan and Mas-Colell (1988). For tax-distorted markets, some results have been obtained by Kehoe, Levine and Romer (1988) and Jones and Manuelli (1989).

(E) As our examples show, when the positive orthant has empty interior and preferences are not proper, weak optima may not be supportable by prices. A number of authors have studied the approximate supportability of weak optima. The sharpest results are due to Aliprantis and Burkinshaw (1988) and Becker, Bercovici and Foias (1990); see also the survey by Becker (1991). For the existence of approximate equilibria we refer to Khan and Vohra (1984) and Aliprantis and Burkinshaw (1988).

(F) Throughout, we have assumed that there are only a finite number of types of consumers. Allowing for the possibility of infinitely many types raises many new issues and goes well beyond the scope of this survey. For work on overlapping generations models, see Chapter 6.

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