**Indeterminacy in incomplete market economies**

Andreu Mas-Colell

Department of Economics, Harvard University, Cambridge, MA 02138, USA

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**Summary.** It is shown that in a two-period economy with a continuum of states and real assets, the following holds: (1) if the asset structure is complete, then generically the number of equilibria is finite; (2) if there are a finite number of real assets (this can approximate completeness arbitrarily close) then, for a non-empty open set of economies, there are a continuum of distinct equilibria. Asymptotic versions (on the number of states and on the number of assets) of the result are also given. It is argued, therefore, that incompleteness, by itself, may be a leading source of indeterminacy.

I. Introduction

It was established by Debreu in 1970 (see Debreu, 1970), that the Walrasian model of general equilibrium is, under classical hypothesis (say those in Debreu 1959; or Arrow-Hahn 1974), determined. Here determinacy means that except for very particular combinations of parameters (i.e., generically) the number of distinct equilibria is finite. As a matter of terminology, indeterminacy of a model means that there are robust examples with a continuum of equilibria. The issue of determinacy is relevant to comparative statics analysis. However, a careful study of the latter must deal not only with the cardinality but also the topological properties (e.g., local uniqueness, discreteness) of the equilibrium set. This we shall not do in this paper.

How far can Debreu’s result be generalized as we remove the classical hypothesis? Clearly, somewhere along the line it must break down because, very informally, as soon as we enter a strategic, dynamic setting we are in the domain of the theory of games and there indeterminacy is the rule. The question is how tight the classical hypotheses are? How singular is the classical model with respect

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to the determinacy property? This paper is an inquiry into an aspect of this question.

It is becoming increasingly clear that the room to maneuver is narrower than perhaps one would have thought at first and that Debreu’s theorem does not generalize easily in some important dimensions. The first evidence came, possibly, from the analysis of the Overlapping Generations Model (see Samuelson 1958; Gale 1973; Calvo 1978; Kehoe-Levine 1985; Woodford 1984; Grandmont 1985) where classes of robust indeterminacy soon emerged (even under the constraint of Pareto Optimality and nice asymptotic behavior). The culprit in the model was identified to be the combination of an infinite horizon with infinitely many significantly distinct agents (and indeed robust indeterminacy was also discovered in other models of this nature; see Kehoe-Levine-Mas-Colell-Zame 1986).

In this paper we examine another important departure from the classical hypothesis, namely, the general equilibrium model with incomplete markets (Radtner 1972; see Magill-Shafer 1989; and the special 1990 issue of Journal of Mathematical Economics, edited by J. Geanakoplos for overviews). To focus on the essential we emphasize that we study a finite horizon model (in fact, just two periods) with finitely many agents and assets with real returns (for the case with financial returns see Balasko-Cass 1989; and Geanakoplos-Mas-Colell 1989). Geanakoplos-Polemarchakis 1986, have shown that in a setting as in the previous paragraph and with a finite number of states, Debreu’s theorem generalizes: generically there are only finitely many equilibria. This result is, of course, entirely correct but it deserves qualification. Indeed, the aim of this paper is to argue that the parallel with the classical situation is not exact. The reason can be understood by means of a trivial example. Suppose there are no assets at all. Then the equilibria of the incomplete market model are the selections from the equilibrium correspondence from states to spot prices. Suppose that this correspondence is nontrivial (i.e., multiplicity of equilibrium in the spot markets). Then, in agreement with the Geanakoplos-Polemarchakis theorem, the number of selections is finite but this number will typically grow to infinity with the number of states. This (robust) dependence of the number of equilibria on the number of states does not arise in the complete market model (there is no reason to think that it does, but at any rate this will be proved). The basic point of this paper is that in what concerns the indeterminacy question the trivial no assets case is not trivial at all, but it represents perfectly well what happens in the general incomplete situation.

It is convenient to concentrate on the limit case where the number of states is infinite. Then we have a clear-cut result. If the market structure is complete then generically the number of equilibria is finite (this is shown in Sect. III). If the asset structure is incomplete (there are a few technical qualifications on the number of assets and its return vectors) then we can always find a two traders economy displaying robustly a continuum of equilibria (this is shown in Sect. IV to VI). In a slightly more restricted set-up we spell out in Sect. VIII the corresponding result for a sequence of finite state economies converging to an infinite state limit. There are robust examples where the cardinality of the equilibrium set sequence is bounded in the complete case and unbounded in the incomplete one.

Our conclusion is that the sharp contrast one gets in the infinite number of states case is a good interpretative guide for the situation where the number of states is not small. If we view the latter as determinate then the lenses of Debreu’s theorem for the complete model show us an equilibrium set of relatively low
cardinality while in fact its cardinality may (robustly) be of the same order than
the number of states.

Our analysis throws some light on another class of models where indeter-
mminacy has been found, namely, sunspot models (see Class-Shell 1983; Azaria-
dis-Guesnerie 1980; and the recent survey of Chiappori-Guesnerie 1989). Usually
this indeterminacy (in the sense of a continuum of equilibria) has been found in
dynamic models but it is clear that the infinite horizon is not essential and that
indeterminacy can be obtained with a finite horizon. What is crucial to the
indeterminacy results with sunspots is that a continuous range of possible prob-
bility distributions for the sunspots variables be allowed. This is because different
equilibria will require different sunspot probabilities. But this implies that the
underlying probability space cannot be purely atomic, i.e. in terms of our ap-
proach, sunspot models where indeterminacy is possible are particular cases of
incomplete market economies with infinitely many states. Thus one lesson of our
analysis is that the weight of responsibility for indeterminacy in sunspots models
falls on the inability to insure rather on the sunspot character (payoff irrelevancy)
of the signal. Another conclusion is, of course, that the lessons learned from
sunspot research hold much more generally.

There are many directions in which the research can be pursued. A particularly
intriguing problem is to obtain a full understanding of the role of Pareto Opti-
mality in getting determinacy. The exploitation of Pareto Optimality is essential
for the mathematical treatment of Sect.III. Per force the lack of it is key to the
indeterminacy results of Sect.IV–VI but this does not come out of the analysis
in a very direct way. In dynamic models indeterminacy has also been associated
to the lack of Pareto Optimality (see Obstfeld, 1984, and Matsuyama, 1989, in
the context of Brock’s, 1975, monetary model; Howitt-McAfee, 1988, for dynamic
models with externalities, or Kehoe-Levine 1989, for dynamic models with taxes)
but not always. As previously indicated in the Overlapping Generations Model
indeterminacy obtains with Pareto Optimality guaranteed.

Finally, a word on financial (i.e., with nominal returns) assets. With non-
classical hypothesis (dynamic models, incomplete markets) they constitute a source
of indeterminacy in a stronger sense than in this paper. The rule of thumb is that
they generate a continuum of equilibria for (almost) every economy.

II. The basic model

II.1 Dates, events and spot commodities

There are two dates \( t = 0,1 \). At \( t = 1 \) a state of the world \( s \) occurs. The set of the
states of the world is a probability space \((\mathcal{S}, \mathcal{B}, \mu)\). The only condition we impose
is that \( \mu \) not be the union of finitely many atoms.

Consumption takes place only a date 1. There are \( G \) physical commodities
tradable at every \( s \). We assume \( 1 \leq G < \infty \) and denote a generic commodity
by \( g \).

II.2 Trader's characteristics

Denote by \( U \) the space of \( C^2 \) utility functions on \( R^G_+ \) which are differentiably
strictly concave, strictly monotone and proper (these are standard definitions,
see Mas-Colell 1985, 2.4, 2.6; the space \( U \) is endowed with the topology of \( C^2 \) uniform convergence on compacta).

We shall assume that traders obey the von-Neumann-Morgenstern axioms and, to make things simple, that their subjective probabilities equal the objective probabilities \( \mu \). Thus, an individual trader can be described by two (measurable) functions \( \omega: S \to R_{++}^G, u: S \to U \) giving, respectively, the endowments \( \omega(s) \) and the utility function \( u(s, \cdot) \) at every state \( s \). We assume that all the \( u(s, \cdot) \) belongs to a fixed compact subset of \( U \) and that, similarly all the \( \omega(s) \) are bounded above and away from zero in any component by a priori fixed bounds.

Denote by \( \mathcal{X} \) the set of characteristics \( (u, \omega) \) so defined. We make \( \mathcal{X} \) into a metrizable space by means of the uniform convergence. More precisely, we say that \( (u_n, \omega_n) \to (u, \omega) \) if for every compact set \( K \subset R_{++}^G \), the essential suprema over \( s \) of
\[
\max \left( |u_n(s, z) - u(s, z)| + |\partial u_n(s, z) - \partial u(s, z)| \right) + \| \omega_n(s) - \omega(s) \|
\]
goes to zero with \( n \).

II.3 Economies

There is a finite set of traders \([1, \ldots, I]\) (or, more precisely, a finite set of types of traders). By a slight abuse of notation we also denote the set by \( I \). An economy \( \mathcal{E} \) is simply a list \((u_i, \omega_i) \in \mathcal{X}, i \in I \). Denote the space of economies by \( \mathcal{M} \); since \( \mathcal{M} = \prod_{i \in I} \mathcal{X} \) we give to \( \mathcal{M} \) the obvious product topology. Sometimes it is also convenient to view an economy as a function \( \mathcal{E}: S \to \prod_{i \in I} (U \times R_{++}^G) \).

More generally, instead of a finite number of types one could have a distribution of traders characteristics (as in Hildenbrand 1974). It can be conjectured that all the results of this paper would extend to the more general setting but the extension presents some technical difficulties and we shall not attempt it.

II.4 Allocations

Given an economy \( \mathcal{E} \) an allocation of commodities is a list \( x = (x_1, \ldots, x_I) \) where \( x_i: S \to R_{++}^G \) is measurable and \( \sum_i x_i(s) \leq \sum_i \omega_i(s) \) for a.e. \( s \).

II.5 Complete economies

To fully describe an economy we need to specify the trading possibilities across states. If there are no restrictions on them we say that the economy is complete.

The following definitions are standard.

A commodity price system is a (measurable) function \( p: S \to R_{++}^G \). For any \( z: S \to R_{++}^G \) we denote \( p \cdot z = \int p(s)z(s)d\mu(s) \).

The pair \((p, x)\) is a complete equilibrium for the economy \( \mathcal{E} \) if:
(a) \( x \) is an allocation,
(b) \( p \cdot x_i \leq p \cdot \omega_i < \infty \), for every \( i \),
(c) if \( p \cdot z \leq p \cdot \omega_i, z: S \to R_{++}^G \), then
\[
\int u_i(s, x_i(s))d\mu(s) \geq \int u_i(s, z(s))d\mu(s), \quad \text{for every } i.
\]
In the next two subsections we consider the situation where interstate trading possibilities are limited to what can be attained by trading a fixed number of assets.

II.6 Asset structures

An asset \( j \) is specified by a return function \( r_j : S \rightarrow R^G_j \). Note that the returns are in real terms.

We call a list \( r = (r_1, \ldots, r_J) \) of \( J \) assets an asset structure. We take \( J < \infty \). It is possible to consider asset structure with infinitely many assets but this adds many technical complications. We remark that the fact of having finitely many assets does not mean that the economy is far from complete. In our setting any economy can be made to be arbitrarily close to completeness by using only finitely many assets (see Sect. IX).

As a convenience hypothesis we assume that asset returns \( r_j \) are uniformly bounded above and, in every component, uniformly bounded away from zero. Also we let the asset structure to be linearly independent in the following sense: for any \( v \in R^G, v \neq 0 \), the functions \( \{v \cdot r_1(\cdot), \ldots, v \cdot r_J(\cdot)\} \) are linearly independent (this condition is generic in asset returns; recall we have many states). In fact our proofs only require that this be true for a \( v > 0 \).

II.7 Incomplete economies

An incomplete economy is specified by a pair \( (r, \mathbb{E}) \) where \( r \) is an asset structure and \( \mathbb{E} \) an economy.

The following definitions are standard (see, for example, Magill-Shafer 1989, or the special issue of the Journal of Mathematical Economics 1990).

Trade in assets takes place at \( t = 0 \). An asset trade is a list \( v = (v_1, \ldots, v_J) \) with \( v_i \in R^J \). The asset trade is an allocation if \( \sum v_i \leq 0 \). An asset price system is a vector \( q \in R^J \).

The quadruple \((q, v, p, x)\) is an equilibrium for the incomplete economy \((r, \mathbb{E})\) if:
(a) \( x \) and \( v \) are allocations,
(b) \( q \cdot y_i \leq 0 \) and \( p(s) \cdot (x_i(s) - \omega_i(s)) \leq \sum_j p(s) \cdot r_j(s) y_{ij} \), for every \( i \) and a.e. \( s \in S \).
(c) if \( q \cdot y_i' \leq 0 \), \( z : S \rightarrow R^G_+ \) and \( p(s) \cdot (z(s) - \omega_i(s)) \leq \sum_j p(s) \cdot r_j(s) y_{ij}' \) for a.e. \( s \in S \), then
\[ \int u_i(s, x_i(s)) d\mu(s) \geq \int u_i(s, z(s)) d\mu(s) , \]
for every \( i \).

III. Determinacy of complete economies

In this section we shall establish the following result:

Proposition 1. There is an open and dense set of economies \( \mathbb{M}^* \subset \mathbb{M} \) such that every \( \mathbb{E} \in \mathbb{M}^* \) admits only a finite number of complete equilibria.

The proposition is not covered by the standard theory of regular economies (see, for example, Mas-Colell 1985) because the number of states is infinite. The
theory of regular economies with infinitely many commodities is not yet well
developed (but see Chichilnisky-Kalman 1976; Kehoe-Levine-Mas-Colell-Zame
1986; Kehoe-Levine-Romer 1989; Dana 1990). Fortunately, our current set-up is
simple enough (no production, separable utility,...) that a proof is easily obtained.
Our treatment is similar to Kehoe-Levine 1985.

The key observation is that complete equilibria yield Pareto optimal allocations.
Therefore, the search for an equilibrium can be viewed as the search of a
Pareto optimal allocation which, when evaluated at the induced shadow prices,
does not require transfers across traders. (This is sometimes called the Negishi
approach to equilibrium.) Because the set of Pareto optimal allocations is a finite-
dimensional object (precisely, the number of dimensions is I-1) we conclude that
the First Fundamental Theorem allows us, in a natural way, to express the
complete equilibria as the zeros of a finite system of equations. This fact has also
proved useful for the study of existence (see Mas-Colell-Zame 1989, for a survey).
In our case it is immediate that the equations are C'. (Warning: proving this may
be very hard in more general settings with production or other nonseparabilities.
This is one of the main stumbling blocks for a general theory of regular economies
with infinitely many commodities.) Once we have this establishing the generic
finiteness of equilibria is a routine matter. The details follow.

Proof of Proposition 1. Let \( \mathcal{E} \in \mathcal{M} \) be a given economy. Denote by \( \Delta = \{ \lambda \in \mathbb{R}^I_+ : \sum_i \lambda_i = 1 \} \) the open \( I-1 \) simplex. For every \( s \in S \) and \( \lambda \in \Delta \) there are
\( x_i(s, \lambda) > 0, \ i \in I \), which constitute solutions to the problem:
Max \( \sum_i \lambda_i u_i(s, x_i) \) s.t. \( \sum x_i = \sum \omega_i(s) \). Denote also by \( p(s, \lambda) \) be corresponding price
variables, i.e., \( \lambda_i \partial u_i(s, x_i(s, \lambda)) = p(s, \lambda) \) for all \( i \).

Because of the hypothesis made on preferences and economies it is possible
to verify that the functions \( p(s, \lambda), x_i(s, \lambda) \) are \( C^1 \) with respect to \( \lambda \) and that both
the values of the functions and their partial derivatives are uniformly bounded
on \( (s) \) (see Mas-Colell 1985, 4.6 and 5.2 for more details).

Finally, define \( F_{\mathcal{E}} : \Delta \rightarrow \mathbb{R}^{1} = \{ v \in \mathbb{R}^I : \sum v_i = 1 \} \) by
\[
F_{\mathcal{E}}(\lambda) = (\int p(s, \lambda) \cdot (x_i(s, \lambda) - \omega_i(s)) \, du(s), \ldots, \int p(s, \lambda) \cdot (x_i(s, \lambda) - \omega_i(s)) \, du(s))
\]
This function is \( C^1 \) and \( x \) is a complete equilibrium allocation if and only if
\( x_i(s) = x_i(s, \lambda) \) for a.e. \( s \), all \( i \) and some \( \lambda \) with \( F_{\mathcal{E}}(\lambda) = 0 \). It is easy to see that the set \( \{ \lambda \in \Delta : F_{\mathcal{E}}(\lambda) = 0 \} \) is compact. Therefore a sufficient condition for the
finiteness of the set is that 0 be a regular value of \( F_{\mathcal{E}} \), i.e., rank \( \partial F_{\mathcal{E}}(\lambda) = \) I-1
whenever \( F_{\mathcal{E}}(\lambda) = 0 \).

Let \( \mathcal{M}^* = \{ \mathcal{E} \in \mathcal{M} : 0 \) is a regular value of \( F_{\mathcal{E}}(\lambda) = 0 \} \). We want to show
that \( \mathcal{M}^* \) is open and dense.

The openness of \( \mathcal{M}^* \) follows from the following two observations: (a) both
the values and the partial derivatives of \( F_{\mathcal{E}}(\lambda) \) are jointly continuous on \( \mathcal{E} \) and
\( \lambda \), (b) for any \( \mathcal{E} \) there is a compact set \( K \subset \Delta \) and a neighborhood of \( \mathcal{E} \) such
that \( K \) contains all the zeroes of \( F_{\mathcal{E}} \) for any \( \mathcal{E}' \) in this neighborhood.

We now prove density by a standard application of the Transversality The-
orem (see Mas-Colell 1985, 1.2 for details). Let \( \mathcal{E} \) be a fixed economy and \( V \subset \mathbb{R}^{I-1} \)
a sufficiently small open neighborhood of zero. For any \( v = (v_1, \ldots, v_{I-1}) \in V^{I-1} \)
to be thought as state-independent redistribution parameters for initial endow-
ments) define \( F_{\mathcal{E}}(\lambda, v) \) by:
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\[ F_{\mathcal{E}}(\lambda, v) = \left( \int p(s, \lambda) \cdot (x_1(s, \lambda) - \omega_1(s) - v_1) d\mu(s), \ldots, \right. \]
\[ \left. \int p(s, \lambda) \cdot (x_{I-1}(s, \lambda) - \omega_{I-1}(s) - v_{I-1}) \right), \]
\[ \int p(s, \lambda) \cdot \left( x_I(s, \lambda) - \omega_I(s) + \sum_{i=1}^{I-1} v_i \right) \]

It is obvious that rank \( \partial_v F_{\mathcal{E}}(\lambda, v) = I - 1 \). Therefore, by the Transversality Theorem, there is a \( v \) arbitrarily close to 0 such that 0 is a regular value of \( F_{\mathcal{E}}(\cdot, v) \) or, in other words, 0 is a regular value of \( F_{\mathcal{E}'}(\cdot) \) where \( \mathcal{E}' \) is the economy obtained from \( \mathcal{E} \) by replacing, for a.e. \( s, \omega_i(s) \) by \( \omega_i(s) + v_i \) for all \( i = 1, \ldots, I - 1 \) and \( \omega_I(s) \) by \( \omega_I(s) - \sum v_i \). Since \( \mathcal{E}' \) is arbitrarily close to \( \mathcal{E} \) this proves the denseness of \( \mathcal{M}^* \). (Note: we can without loss of generality assume that the \( \omega_i(s) \) satisfy with some slack the a priori given bounds and therefore guarantee that \( \mathcal{E}' \in \mathcal{M} \)). \( \square \)

For a more detailed proof of a similar result see Dana (1990).

IV. Lack of generic determinacy for incomplete economies

The generic finiteness of equilibrium does not generalize to incomplete economies. The following is the main result of this paper.

**Proposition 2.** Suppose that \( I > 1 \) (i.e., there is more than one physical good). Then for any asset structure \( r \) there is a nonempty open set \( \mathcal{M}, \subset \mathcal{M} \) of two types economies (i.e., \( I = 2 \)) such that for every \( \mathcal{E} \in \mathcal{M} \), the incomplete economy \((r, \mathcal{E})\) has a continuum of distinct equilibria. Moreover, \( \mathcal{M} \), can be chosen to contain an economy where preferences and endowments are state independent.

By a continuum of equilibria we simply mean that the set of equilibria has at least the cardinality of the continuum. As it will be clear from the proof much more could be asserted. Something we do not claim is that the equilibrium set is not discrete. Obviously the local uniqueness properties of equilibrium depend on the metric on price and allocations functions. For a separable metric (such as \( L_1 \)) discreteness implies countability but for a nonseparable one (such as \( L_\infty \)) discreteness is compatible with a continuum. We prefer not to introduce any metric on the equilibrium set and thus we restrict ourselves to analyzing its cardinality properties. Nonetheless, it is certainly true that for comparative statics purposes local uniqueness is the key property. Thus it could be asked if local uniqueness (relative to the \( L_\infty \) topology) may still in some sense be a generic property. This is a difficult question and we have no answer to offer.

Take the trivial case where \( r = \emptyset \), i.e., there are no assets. Then the conclusion of Proposition 2 is obviously true: choose state-independent utilities and endowments so that in every state the spot exchange economy has three equilibria \([\alpha, \beta, \gamma]\). Clearly, the equilibria of the incomplete economy are in one-to-one correspondence with the (measurable) functions from \( S \) to \([\alpha, \beta, \gamma]\) and, sure enough, there is a continuum of such selections. (But note that it is a \( L_\infty \) discrete continuum).
The next two sections (which contain the proof of Proposition 2) will attempt to convince the reader (and if some readers do not need convincing so much the better!) that for the purposes at hand the above extreme case is in no way special but it reflects perfectly well the general situation.

It is instructive to consider why the mathematical techniques used in the previous section fail to be applicable. The problem is that there is no natural way to express the equilibria as the zeroes of a system with the same finite number of equations and unknowns. It is entirely possible that the equilibria be the zeroes of an infinite system of equations, but we still would need that the system be differentiable and that the space used to model the variables be separable (otherwise local uniqueness is compatible with a continuum of equilibria). An examination (left to the reader) of the trivial case with \( r = 0 \), state independent utilities and endowments and nonunique spot prices, reveals the difficulty of getting both things. If we view excess demand as a map from \( L_\infty \) to \( L_\infty \) then the map is differentiable and quite well behaved. As already noted its zeroes are locally isolated but since \( L_\infty \) is not separable this does not contradict the existence of a continuum of equilibria. If we attempt viewing excess demand as a map from \( L_1 \) to \( L_1 \) then the map fails to be differentiable.

In Proposition 2, can we take \( \mathcal{M} \) to be open in some weaker topology (in particular, for the separable metric that looks at average rather than the supremum distance across \( s \))? We have not investigated this in detail but it would appear that with some strengthening on the basic hypothesis (in particular those concerning \( \mu \) and allowable set of spot economies) the answer should be positive.

V. Regular sequential equilibrium and indeterminacy

The purpose of this section is to show that a certain class of "robust" equilibria is indeterminate, in the sense that some parameters can be perturbed while maintaining overall equilibrium. This will almost settle the issue because from its definition it will be most intuitively plausible that economies having equilibria in the class abound. The next section will then exhibit a state-independent example.

From now on an asset structure \( r \) is given.

We begin by defining the concept of spot equilibrium selection relative to an economy \( \mathcal{E} \) and an asset trade \( y \). This is simply a (measurable) price function

\[
p(\mathcal{E}, y, \cdot) : S \to R^G_+ \text{ such that, for a.e. } s \in S, p(\mathcal{E}, y, s) \text{ is an equilibrium price vector for the spot economy } (u_i(s, \cdot), \omega_i(s) + \sum_j r_j(s)y_j), i \in I, \text{ that is, there are }
\]

\[x_i(y, s) \in R^G_+ \text{ such that } \sum_i x_i(y, s) = \sum_i \left( \omega_i(s) + \sum_j r_j(s)y_j \right), \text{ and, for every } i, x_i(y, s) \text{ maximizes } u_i(s, v) \text{ on the budget set } p(\mathcal{E}, y, s) \cdot \left[ v - \omega_i(s) - \sum_j r_j(s)y_j \right] \leq 0. \text{ Not that we do not require that } y \text{ be an allocation, i.e., we allow } \sum_i y_i \neq 0.
\]

A spot equilibrium selection \( p(\mathcal{E}, y, \cdot) \) is regular if we can (locally) solve in terms of \( \mathcal{E}, y \). Precisely, we require the existence of open sets \( \mathcal{E} \in \mathcal{M}', y \in Y' \)
and an extension \( p : \mathcal{M}' \times Y' \times S \to R^G_{++} \) such that: (a) for all \((E', y')\), \(p(E', y', \cdot)\) is a spot equilibrium selection, (b) for all \(E'\) and a.e. \(s\), \(p(E', \cdot, s)\) is a \(C^1\) function, (c) the values of \(p(\cdot)\) and its partial derivatives with respect to \(y\) are bounded uniformly on \(\mathcal{M}' \times Y' \times S\) and, for a.e. \(s\), jointly continuous on \(\mathcal{M}' \times Y'\).

Suppose, as an example, that (for \(r = 0\)) the spot price equilibrium correspondence has the typical form indicated in Fig. 1. Then the regular selections would be those that in neighborhoods of \(s_1\) and \(s_2\) always take values in the upper and lower branch, respectively, of the graph. Clearly, there are many of them.

![Fig. 1](image_url)

Next we will define the concept of a \textit{sequentially regular equilibrium} \((\bar{q}, \bar{y}, \bar{p}, \bar{x})\) for the economy \(E\). In different contexts this notion has been used by Mandler (1989a, b) and Roberts (1980) (see also Mas-Colell 1982). The idea of a sequentially regular equilibrium is that the equilibrium is robust both ex-ante and ex-post (and, more generally, at any point of the equilibrium trajectory).

We say that the equilibrium \((\bar{q}, \bar{y}, \bar{p}, \bar{x})\) for the incomplete economy \((r, E)\) is \textit{sequentially regular} if it satisfies the following three conditions:

(a) The spot equilibrium selection (relative to \(E\) and \(f\)) given by \(\bar{p} : S \to R^G_{++}\) is regular. Let \(p : \mathcal{M}' \times Y' \times S \to R^G_{++}\) be a corresponding extension.

(b) Denote by \(a_i(E, s, p, w_i)\) the function which to every \(p \in R^G_{++}\) and \(w_i > 0\) assigns the corresponding marginal utility of wealth for type \(i\) at the economy \(E\) and state \(s\). Then the first-order conditions for utility maximization corresponding to the asset trades variables \(y_{ij}, i \in I, j = 1, \ldots, J-1\), is given by (putting \(q_j = 1\) and \(y_{ij} = - \sum_{j=1}^{J-1} q_j y_{ij}\)):

\[
\hat{p}(E, s, \cdot) \cdot (r_i(s) - q_j r_j(s)) a_i(E, s, \hat{p}(E, y, s)),
\]

\[
\hat{p}(E, y, s) \cdot (\omega(s) + \sum_j r_j(s) y_{ij}) \right) d\mu(s) = 0 ,
\]

all \(i \in I\) and \(j = 1, \ldots, J-1\). [*]

We require this system of \(I(J-1)\) equations to have full rank with respect to the \(I(J-1)\) variables \(y_{ij}, i \in I, j = 1, \ldots, J-1\). This means, in particular, that we can differentiably solve \(y(E, q)\) for \(q\) in a neighborhood \(Q\) of \(\bar{q}\). Of course, \(y(E, q) = \bar{y}\).
(c) The function $\hat{y}(\mathcal{E}, q) = \sum_i y_i(\mathcal{E}, q)$ is a sort of (locally defined) excess demand functions for assets. The definition of sequential regularity is completed by requiring that $\partial_q \hat{y}(\mathcal{E}, q)$ be of maximal possible rank, i.e., $(J-1)$.

It is clear from the definition that the family of economies $\mathcal{E}'$ having a sequentially regular equilibrium for $(r, \mathcal{E}')$ is open. Indeed, the values and the derivatives of the system of equations \[*\] are continuous with respect to $y$ and $\mathcal{E}$. Therefore, by the Implicit Function Theorem (IFT, use the version C.3.3 in Mas-Colell 1985) we can solve for $y(\mathcal{E}', q)$ with, again, the values and derivatives being continuous with respect to $\mathcal{E}'$. Hence, using the IFT once more, property (c) of the definition of sequential regularity implies that we can solve $\hat{y}(\mathcal{E}', q) = 0$ for $q(\mathcal{E}')$. Of course, $q(\mathcal{E}) = \hat{q}$. Then, for $\mathcal{E}'$ near $\mathcal{E}$, we obtain the regular sequential equilibrium $(q(\mathcal{E}'), y(\mathcal{E}', q(\mathcal{E}'))$, $\bar{p}(\mathcal{E}', y(\mathcal{E}', q(\mathcal{E}'))$, $\bar{x}(\mathcal{E}', y(\mathcal{E}', q(\mathcal{E}'))$, $\cdot))$ where $\bar{x}(\cdot)$ has the obvious meaning.

Let us now define the set of economies $\mathcal{M}_r$ as those $\mathcal{E} \in \mathcal{M}$ satisfying the following two properties:

(a) There is a sequentially regular equilibrium $(\hat{q}, \hat{y}, \bar{p}, \bar{x})$ for $(r, \mathcal{E})$.

(b) There is a regular spot equilibrium selection $p(\mathcal{E}, \hat{y}, \cdot)$ and a set of states $S' \subset S$ such that $\mu(S') > 0, \mu|S'$ is not a finite union of atoms, and $\|p(\mathcal{E}, \hat{y}, s) - \bar{p}(s)\| \leq \varepsilon$ for some $\varepsilon > 0$ and a.e. $s \in S'$.

In other words, property (b) says that besides the spot equilibrium selection $\bar{p}(\cdot)$ generated by the overall equilibrium there is another one, relative to the asset trade equilibrium $\hat{y}$, which is different from $\bar{p}(\cdot)$ in a set which is not too small. See Fig. 2. Property (b) is also clearly open (the same $S'$ will do for a whole neighborhood of $\mathcal{E}$) and therefore we conclude that $\mathcal{M}_r$ is an open set.

![Fig. 2](image-url)

We shall prove that every $\mathcal{E} \in \mathcal{M}_r$ is indeterminate, i.e., admits a continuum of equilibria. The intuitive reason why this is so can be easily grasped. A sequentially regular equilibrium $(\hat{q}, \hat{y}, \bar{p}, \bar{x})$ is, by construction, very robust. So much so that for any sufficiently small $S'' \subset S'$ we can replace $\bar{p}(s)$ by $p(\mathcal{E}, \hat{y}, s)$, for all $s \in S''$, without losing the equilibrium, that is, all the variables of the system adjust slightly (and uniformly on $s$) so as to restore equilibrium. Hence, for every small $S''$ (and there are a continuum of these sets) we obtain a distinct equilibrium.

**Proposition 3.** Every $(r, \mathcal{E})$ with $\mathcal{E}$ in the open set $\mathcal{M}_r$ admits a continuum of equilibria.
Proof. The economy $\mathcal{E} \in \mathcal{M}$, remains fixed for the entire proof. We thus suppress reference to it in the notation.

Let $(q, y, \bar{p}, \bar{x})$ be a sequentially regular equilibrium for $(r, \mathcal{E})$. Choose a spot equilibrium extension $\bar{p} : Y' \times S \rightarrow R^Q_+$. Denote by $p : Y' \times S \rightarrow R^Q_+$ another spot equilibrium selection and let $S' \subset S$ be such that $\mu(S') > 0$, $\mu|S'$ is not a finite union of atoms and $\mu|S'$ is not the sum of finitely many atoms) for every $\delta > 0$ the set $\{B \in \mathbb{B}' : 0 < \mu(B) < \delta\}$ has, at least, the cardinality of the continuum. If $\mu|S'$ has an atomless part this is clear. On the other hand if $\mu|S'$ is formed by countably many atoms then there is a countable set $B \in \mathbb{B}'$ such that $0 < \mu(B') < \delta$ for every subset $B'$ of $B$.

Define $\bar{p} : \mathbb{B}' \times Y' \times S \rightarrow R^Q_+$ by $\bar{p}(B, y, s) = \bar{p}(y, s)$ if $s \notin B$ and $\bar{p}(B, y, s) = p(y, s)$ if $s \in B$. Note that for every $B, p(B, \cdot, \cdot)$ is a spot equilibrium selection.

Suppose that in the equation system [*] we replace $\bar{p}(y, s)$, that is $\bar{p}(\mathcal{E}, y, s)$, by $\bar{p}(B, y, s)$. The values and the derivatives of the equation system are continuous with respect to $B$. Also, $\bar{p}(\phi, y, s) = p(y, s)$. Therefore, by property (b) of sequential regularity and the IFT we can solve the system for $y(q, B)$ where $(q, B) \in Q \times \mathbb{B}'$. $Q$ open and $\mathbb{B}'$ an open neighborhood of $\phi$. In turn, the values and derivatives with respect to $q$ of $y(q, B)$ are jointly continuous with respect to $(q, B)$. Therefore, by property (c) of sequential regularity we can solve $\bar{y}(q, B) = 0$ for $q(B)$ on an open neighborhood $B''$ of $\emptyset$.

To every $B \in \mathbb{B}''$ there corresponds the distinct equilibrium

$$(q(B), y(q(B), B), \bar{p}(B, y(q(B), B), \cdot), x(B, y(q(B), B), \cdot))$$

where $x(\cdot)$ has the obvious meaning. Because $\mathbb{B}''$ is an open neighborhood of $\phi$ (in the $L_1$ norm of the characteristic function) there is $\delta > 0$ such that $\{B \in \mathbb{B}' : 0 < \mu(B) < \delta\} \subset \mathbb{B}''$. Hence $\mathbb{B}''$ has, at least the cardinality of the continuum. $\square$

VI. An Example

It remains to argue that $\mathcal{M}$, is nonempty. The definition of the set makes this fact most plausible. But we need to exhibit an example. As it turns out this is very easy: in essence it suffices to take any state-independent economy where the spot economy has several equilibria and indifference curves are sufficiently flat in the neighborhood of an equilibrium.

Specifically, let $I = 2$, and $u_i(s, \cdot) \in U, \omega_i(s)$ be independent of $s$. Abusing notation slightly, we denote $u_i(s, \cdot) = u_i(\cdot), \omega_i(s) = \omega_i$. Assume that $(u_i, \omega_i), i = 1, 2$, have been chosen so that in any state there are at least two spot equilibria $\bar{p}, \bar{p}'$ which, in addition, are regular, i.e., the derivative of the spot excess demand function at equilibrium has maximal rank. The spot allocation corresponding to $\bar{p}$ is denoted $\bar{x}$. We will also assume (after all we only want an example) that is
in a neighborhood of \( \bar{x} \), both \( u_i \) coincide with \( \ln v(x_i) \) where \( v \) is homogeneous of degree one (of course, this implies that \( \bar{x}_1, \bar{x}_2 \) are collinear). We can put \( \bar{\rho} = \partial \ln v(\omega_1 + \omega_2) \).

Let the asset structure \( r \) be given. We claim that there is a state-independent incomplete equilibrium of the form \((\bar{q}, 0, \bar{\rho}, \bar{x})\), i.e., the equilibrium asset trades are null. Indeed, the state-independent allocation \( \bar{x} \) is Pareto optimal and it can be reached with spot prices \( \bar{\rho} \) and any \( q \). But this means, that if \( \bar{q} \) clears the asset markets (conditional on the spot prices being \( \bar{\rho} \) at all states) then the optimal asset trading plan must be \( y_i = 0 \) for \( i = 1, 2 \). Note: because of the linear independence condition on asset's returns the excess demand functions for assets (given the expected prices \( \bar{\rho} \)) are single-valued.

Suppose for a moment that on a neighborhood of \( \bar{x} \), the indifference curves were linear, say that \( v(x_i) = \bar{\rho} \cdot x_i \). Formally, this is then an economy with a single consumption good and the asset demand functions \( y_i(q) \) generated from system \([=*]\) (where, formally, \( \bar{\rho}(y, s) = \bar{\rho} \)) has all the conventional properties of a demand function. In particular, they are differentiable and 0 is a regular value of \([=*]\). Since \( y_i(\bar{q}) = 0 \) for \( i = 1, 2 \) there are no wealth effects at the equilibrium. Henceforth, letting \( \dot{y}(q) = y_1(q) + y_2(q) \), the derivative \( \partial \dot{y}(\bar{q}) \) is composed only of substitution effects and is consequently of maximal rank.

As for the general case, we can express

\[
\bar{\rho}(y, s) = \partial \ln v \left( \sum_{i=1,2} \left( \omega_i + \sum_j r_j(s) y_{ij} \right) \right),
\]

\[
\alpha_i \left( s, \bar{\rho}(y, s), \bar{\rho}(y, s) \cdot \left( \omega_i + \sum_j r_j(s) y_{ij} \right) \right) = \frac{1}{\bar{\rho}(y, s) \cdot \left( \omega_i + \sum_j r_j(s) y_{ij} \right)}.
\]

We see that \( \alpha_i(\cdot) \) is a \( C^1 \) function of \( \bar{\rho}(\cdot) \) not involving \( v(\cdot) \). Also, if \( v(\cdot) \) is almost linear (i.e., it is \( C^2 \)-on-compacta close to \( \bar{\rho} \cdot x_i \)) then \( \bar{\rho}(\cdot) \) is almost a constant (more precisely it is \( C^1 \) close uniformly on \( s \) to the constant \( \bar{\rho} \)). Therefore we conclude that if \( v(\cdot) \) is almost linear then, by continuity of the system \([=*]\), \( \dot{y}(q) \) is well defined and \( \partial \dot{y}(\bar{q}) \) has maximal rank, which is what we wanted.

VII. Unique spot equilibrium selections

Our indeterminacy result depends on the possibility of multiple equilibrium in the spot markets. In this section we shall see that this is crucial: under general hypotheses if at all asset trades the spot equilibrium is unique then generically the overall equilibria are locally isolated (and therefore finite in number). One should not derive too much comfort from this result for at least two reasons. In the first place the multiplicity of spot equilibria is not a pathological situation. In the second place (I owe this observation to O. Hart) the last period of this model should be thought as a reduced form of a multiperiod model. It may be very difficult to insure the uniqueness of equilibrium in this situation, (a systematic study of uniqueness conditions for incomplete market models is missing, see Hart 1975, and Mas-Colell 1988, for a variety of observations). A similar observation can be made for our hypothesis \( l > 1 \). It helps to capture in two periods models phenomena that with \( l = 1 \) would still happen with more than two periods.
In order to prove the following proposition we need to strengthen our hypotheses on the asset structure. As Geanakoplos-Polemarchakis (1986) we say that \( r \) is a \textit{numeraire asset structure} if for every \( s \), all the \( r_j(s) \) are non-zero in a single and the same coordinate. The role of this hypothesis is to avoid the well-known discontinuities in the span of asset returns.

**Proposition 4.** Suppose that for fixed preferences the spot equilibrium price vector (normalized to have unit norm) is unique for any \( \omega_i: S \rightarrow \mathbb{R}^g_{++}, i \in I, \) and \( s \in S \). Denote it by \( \bar{p}(\omega(s), s) \) and assume that it is \( C^1 \) as a function of \( \omega(s) \) (with the value of the derivatives uniformly bounded on \( s \)). Then given a numeraire asset structure \( r \) there is an open and dense set of endowment functions \( \omega = (\omega_1, \ldots, \omega_I) \) such that the corresponding incomplete economy has a finite number of equilibria.

**Proof.** Clearly, the proof will consist in an application of the transversality theorems.

For the variables \( q, y \) the equilibrium conditions are given by the system of equations [\(^*\)] supplemented by \( \sum_i y_i = 0 \). If for a given economy this system of \((I+1)(J-1)\) equations (we drop \( y_j \)) in \((I+1)(J-1)\) unknowns (we fix \( q_I = 1 \)) always has full rank (i.e., has zero as a regular value) then the equilibria are locally isolated. By the boundary hypotheses on utilities the condition is evidently open and therefore we only need to worry about density. By the transversality theorem we will be done if we can show that the system of equations [\(^*\)] has full rank with respect to suitable perturbations of \( \omega \) (we are then free to use the variables \( y_i \) as perturbation parameters for the system \( \sum_i y_i = 0 \)).

To find appropriate perturbations we can deal with each \( i \) separately.

Denote \( \bar{p}(\omega, y, s) = \bar{p}(\omega_1(s) + \sum r_j(s)y_{1j}, \ldots, \omega_I(s) + \sum r_j(s)y_{Ij}, s) \).

By the hypotheses made the functions \( h_j(s) = \bar{p}(\omega, y, s) \cdot (r_j(s) - \tilde{q}_j r_j(s)) : S \rightarrow \mathbb{R}, j = 1, \ldots, J - 1 \) are linearly independent. Therefore, the \((J - 1) \times (J - 1)\) matrix with generic \( j' \) entry \( \int h_j(s) h_{j'}(s) du(s) \) has full rank.

Let \( v_j(s) \in \mathbb{R}^g \) be the income effect vector of trader \( i \) at state \( s \). Note that the derivative of \( \bar{p}(\omega, y, s) \) when \( \omega_i(s) \) is perturbed in the direction \( v_i(s) \) is null.

For every \( j = 1, \ldots, J - 1 \) denote

\[
\beta_j(s) = \frac{h_j(s)}{\partial_{\omega_i} \alpha_i \left[ s, \bar{p}(\omega, y, s), \bar{p}(\omega, y, s) \cdot \left( \omega_i(s) + \sum_j r_j(s)y_{ij} \right) \right]}
\]

Then the functions \( \beta_j(s)v_j(s) : S \rightarrow \mathbb{R}^g \) give a set of appropriate \( J - 1 \) perturbations of \( \omega_i \). Indeed, evaluating the \( i \)-th \((J-1)\) block of the derivative of \([\ast]\) with respect to \( \omega_i \) we get that the derivative of the \( ij \) equation when \( \omega_i \) is perturbed in the direction \( \beta_j v_j \) is precisely \( \int h_j(s) h_{j'}(s) du(s) \) (remember that \( \bar{p}(\omega, y, s) \cdot v_i(s) = 1 \)).

Because all these perturbations do not alter the spot price vector (i.e., \( \partial_{\omega_i} \bar{p}(\omega, y, s)(\bar{u}_i) = 0 \) for all \( i \)) the ensuing overall \( I(J - 1) \times I(J - 1) \) matrix is formed by \((J - 1) \times (J - 1)\) nonsingular diagonal blocks. Hence it is nonsingular. \( \square \)
VIII. Asymptotics on the number of assets

Suppose now that the economy $\mathcal{E}$ remains fixed and that the sequence of asset structures $r_n$ "tends to completeness". We do not need to be too precise here about the meaning of this. It suffices to say that in our setting such sequences, suitably defined, exist (although, as recently emphasized by Zame 1988, more generally this may become a nontrivial problem; see also Green-Spear 1987) and that for any sensible definition the following implication holds: if $(q_n, y_n, p_n, x_n)$ is an incomplete equilibrium for $(r_n, \mathcal{E})$ then the utility vector

$$\left(\int u_1(s, x_{n1}(s))d\mu(s), \ldots, \int u_f(s, x_{nf}(s))d\mu(s)\right)$$

accumulates at the utility Pareto frontier of the complete economy $\mathcal{E}$. Suppose, without loss of generality (extract a subsequence if necessary), that it actually converges to a point in such frontier. Underlying it there can be only an allocation $x$, supported by a price function $p$. By continuity, $x_n \to x$ in measure and, therefore, $p_n \to p$ in measure (where we normalize so that $\|p_n(s)\| = \|p(s)\|$ for all $s$). For some multipliers $\alpha_n > 0$ we should have $q_{nj} = \alpha_n \int p_n(s) \cdot r_{nj}(s) d\mu(s)$ for all $j$. Hence, $q_n \cdot y_n = 0$ implies $\int p_n(s) \cdot (x_n(s) - \omega_j(s))d\mu(s) = 0$ for all $i$. From this we conclude $\int p(s) \cdot (x_j(s) - \omega_j(s))d\mu(s) = 0$, that is, $x$ is a complete equilibrium for $\mathcal{E}$.

The argument of the previous paragraph says that relative to the asset structure the incomplete equilibrium correspondence is upper hemi-continuous at completeness (a fact already known from Green-Spear 1987, and Zame 1988). Thus, pleasantly enough, there is no fundamental discontinuity. If $\mathcal{E}$ is regular as a complete economy and $r_n$ tends to completeness then eventually the equilibrium for $(r_n, \mathcal{E})$ falls into a finite number of sets with small radius. It is natural to wonder if this conclusion can be strengthened too: "eventually the set of incomplete equilibria for $(r_n, \mathcal{E})$ is finite." If so this would mean that the bad set $\mathcal{M}_{r_n}$ while open, is however, increasingly small as $r_n$ approaches completeness. Hela, the next example will show that the strengthened conclusion is false.

Example. Let $I = 2$ and take a spot economy $(u_i, \omega_i), i = 1, 2$ having three linearly independent regular equilibria $p_1, p_2, p_3$ (normalized to have unit length) with the property that the values of the $i$-th trader marginal utility of wealth at each $(p_h, p_h, \omega_h), h = 1, 2, 3$ are $\alpha_i = (1, 2, 3)$, for $i = 1$, and $\alpha_2 = (3, 2, 1)$ for $i = 2$. It is fairly clear that such a combination of values is possible. Take $S = [0, 1]$ and let $\mathcal{E}$ be state-independent. The constant price function $p_2(s) = p_2$ is then a complete equilibrium. It is simply verified that it is regular in the sense of Sect. III (more precisely, in the sense of the Proof of Proposition 1).

Consider now an asset structure $r$ where: (a) $r_1$ is orthogonal to the other $r_j$, i.e., if $r_1(s) \neq 0$ then $r_j(s) = 0$ for all $j \neq 1$, and (b) for $s \in S_1 = \{s \in S : r_1(s) \neq 0\}$, $r_1(s)$ is constant and $p_h \cdot r_1(s) = 1$ for every $h = 1, 2, 3$ (since the vectors $p_h$ are linearly independent such a nonzero $r_1(s)$ exists). Obviously we can approach completeness with asset structures satisfying these properties. As in Sect. VI one verifies that $y_i = 0, i = 1, 2$ and $p_2(s) = p_2$ still yields an equilibrium. But there are many more of them. Take any $S'_1 \subset S_1$ with $\mu(S'_1) = \frac{1}{2} \mu(S_1)$. Then $y_i = 0, i = 1, 2$ and the price function defined by $p(s) = p_2$ for $s \notin S_1, p(s) = p_1$ for $s \in S'_1$ and $p(s) = p_3$ for $s \in S_1 \setminus S'_1$ also constitute an equilibrium (check that the first-order conditions for asset trade are satisfied for the two traders). Since there is a continuum of distinct $S'_1$, our point is made.
IX. Asymptotics with a finite number of states

If the number of states is finite then the Geanakoplos-Polemarchakis theorem holds (see 1986): generically the number of incomplete equilibria is finite. But the contrast we have drawn in this paper is far from a pathology of the continuum. To convince the skeptics we show in this section that in the finite case the contrast takes the following asymptotic form. Suppose that the number of states increases in some suitably regular manner towards a continuum limit. Then for an open set of economies the number of equilibria remains bounded in the complete case and becomes unbounded in the incomplete case.

The Proposition to be established is only meant to be illustrative. It is by no means the most general possible.

It will be convenient to place ourselves in a more restricted setting. From now on we let \( S = [0, 1] \) and \( \mu \) be Lebesgue measure. Also we add to the requirements defining the set of characteristics \( \mathcal{M} \) the condition that both \( \omega: [0, 1] \rightarrow R_+^G \) and \( u: [0, 1] \rightarrow U \) be continuous. Similarly, for \( r: [0, 1] \rightarrow R^G \).

Given \( r \) denote by \( \mathcal{M}_r \subset \mathcal{M} \) the set of \( \mathcal{E} \in \mathcal{M} \) which are defined in all respects as the elements of the \( \mathcal{M} \), except that spot equilibrium selection (and its extensions) are required to satisfy a further condition, namely if \( p(\mathcal{E}, *, *) : Y^* \times [0, 1] \rightarrow R_+^G \) is one such then the set \( \{ s \in [0, 1] : \text{for some } y \in Y^* \text{, } p(\mathcal{E}, y, *) \text{ is discontinuous at } S \} \) is finite. We call this property "almost continuity". It is not difficult to verify by means of the same arguments used so far that \( \mathcal{M}_r \) is nonempty (see Sect. VI) and open. Also the analog of Proposition 3 holds: every \( \mathcal{E} \in \mathcal{M}_r \) is indeterminate and the continua of equilibria can be chosen so that every equilibrium spot price function (and its extension) satisfies the almost continuity property.

Suppose that the economy \( \mathcal{E} \) is regular as a finite economy, i.e., \( \mathcal{E} \in \mathcal{M}^* \) (as in Proposition 1 this set is open and dense), but that the incomplete economy \( (r, \mathcal{E}) \) is indeterminate, in the sense that \( \mathcal{E} \in \mathcal{M}_r \). There is a nonempty, open set of such economies. Since \( \mathcal{E} \) and \( r \) are continuous function from \( [0, 1] \) to \( (U \times R^G_+) \) we view a finite state approximation as a sequence of step function \( (r_n, \mathcal{E}_n) \) converging uniformly to \( (r, \mathcal{E}) \) and associated, respectively, with the partition of \( [0, 1] \) into subintervals of length \( \frac{1}{n} \). In the following when we refer to allocations or price functions for \( (r_n, \mathcal{E}_n) \) we always assume that they are constant in every element of the partition.

**Proposition 5.** Let \( E^c_n, E^i_n \) be, respectively, the set of complete and incomplete equilibrium allocations for \( \mathcal{E}_n \) and \( (r_n, \mathcal{E}_n) \). Then:

\[
\limsup \# E^c_n < \infty \quad \text{and} \quad \liminf \# E^i_n = \infty.
\]

**Proof.** (a) First we show that \( \limsup \# E^c_n < \infty \). This is an obvious consequence of Proposition 1 (or, rather, of its proof). The number of equilibria is finite and locally constant in a neighborhood of \( \mathcal{E} \). This neighborhood will eventually include \( \mathcal{E}_n \). Hence the result. We note that exactly the same proof works for the more general setting of Sects. II–VI.

(b) Next we show that \( \liminf \# E^i_n = \infty \). We know that \( \mathcal{E} \) admits a continuum of distinct sequentially regular equilibria (each one satisfying the additional al-
most continuity property). Hence it suffices to show that if \((\vec{q}, \vec{y}, \vec{p}, \vec{x})\) is one of them then \((q_m, y_m, p_m, x_n) \rightarrow (\vec{q}, \vec{y}, \vec{p}, \vec{x})\) where \((q_m, y_m, p_m, x_n)\) is an equilibrium for \((r_m, \mathcal{E}_n)\) and \(n\) is large enough. Letting \(\mathcal{M}' = \cup \mathcal{E}_n \cup \mathcal{E}\) be our universe of economies, the proof follows by an application of the IFT which is identical to the one we used in Sect. V to establish the openness of the set of economies having a regular sequential equilibria. The only "detail" we must watch for is to make sure that the equilibrium extension \(\tilde{p}(\mathcal{E}_n, y, s)\) is compatible with the partition structure (i.e., \(\tilde{p}(\mathcal{E}_n, y, \cdot)\) is constant on subintervals of the partition). Otherwise, when we go from \(\mathcal{E}\) to \(\mathcal{E}_n\) we may not be solving for a true equilibrium. It is in order to guarantee this "detail" that we appeal to be specialized structure of this section. Indeed, suppose that \(\tilde{p}: \mathcal{M}' \times Y' \times [0, 1] \rightarrow \mathbb{R}^4_+\) is an extension satisfying the conditions of Sect. V and the almost continuity property. For every \(n\) and \(s\) let \(s_n\) be the midpoint of the subinterval of the \(n\) partition which includes \(s\). Define then a new extension \(\tilde{p}': (\mathcal{M}' \cap \mathcal{M}') \times Y' \times [0, 1] \rightarrow \mathbb{R}^4_+\) by \(\tilde{p}'(\mathcal{E}_n, y, s) = p(\mathcal{E}_n, y, s_n)\) (and, of course, \(\tilde{p}'(\mathcal{E}_n, y, s) = \tilde{p}(\mathcal{E}_n, y, s)\)). Because of the almost continuity property \(\tilde{p}'\) is an admissible extension (i.e., satisfies the required continuity properties). From here the proof proceeds exactly as in Sect. V.

An interesting question to ask is if analogs of the second part of Proposition 5 would hold for other models where some sort of indeterminacy has been detected (e.g., in the Howitt-McAfee 1988, model would the number of equilibria diverge to infinity if the horizon is truncated at an arbitrarily large date?).

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