A Note on Cost-Share Equilibrium and Owner-Consumers

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The contributions of J. Farrell (*Econ. Letters* **19**, 1985, 303–306) and A. Mas-Colell and J. Silvestre (*J. Econ. Theory* **47**, 1989, 239–256) have studied the implications of two unanimity principles when a firm is owned by the consumers. Interestingly, both approaches lead to what J. E. Roemer and J. Silvestre (The proportional solution for economies with both public and private ownership, mimeo, University of California, Davis, 1991) call proportional allocations. We compare these approaches and analyze how well they serve as equilibrium foundations for proportional allocations. In a nutshell, the conclusion is that Farrell's unanimity principle is simpler than Mas-Colell and Silvestre's, but (assuming that the cost function is convex) the latter always works (i.e., equilibrium exists and yields proportional allocations) while the former may not (equilibrium may not exist, in other words, proportional allocations may not be supportable as equilibria). *Journal of Economic Literature* Classification Numbers: 022, 514, 612, 614.   © 1991 Academic Press, Inc.

1. INTRODUCTION

The set up of this note is that of a single consumption good produced with cost function $C(y)$ and to be distributed among N consumers. The technology is, in principle, owned by the consumers themselves (or, equivalently, the owners are also consumers). Until Section 8 we assume that $C(\cdot)$ is differentiable. For simplicity, we take the consumers to have quasilinear utility functions $u_i(y_i) + x_i$ and endowments ω_i only in the numeraire good. The indirect utility and individual demand functions are denoted $v_i(p)$ and $f_i(p)$, respectively. We write $F(p) = \sum_{i=1}^{N} f_i(p)$.

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The problem we discuss is a classical one: How much of the good to produce and how to distribute it?

Two recent contributions (Farrell [1] and Mas-Colell and Silvestre [4], see also Manning [3]) have studied the implications of two different unanimity-based principles. The motivation of Mas-Colell and Silvestre [4] is normative: it aims at exhibiting a mechanism where owner-consumers unanimously agree to support a Pareto optimal level of production. The motivation of Farrell [1] is more positive: it aims at studying the possibility and consequences of unanimous agreement. But, interestingly, both approaches lead to the class of allocations called *proportional* by Roemer and Silvestre [5].

An allocation is proportional if (i) it is Pareto optimal, and (ii) costs are shared proportionally to consumption. In the present context of quasilinear utilities, proportional allocations exist and are unique under standard conditions. Indeed, total production and individual consumptions of the non-numeraire good are uniquely determined by Pareto optimality. The numeraire contributions of each consumer are then determined by the proportional cost-sharing condition. For general utilities, proportional allocations exist provided the cost function is convex (see Theorem 1 in Roemer and Silvestre [5]).

The main purpose of this note it to compare the approaches of Farrell [1] and Mas-Colell and Silvestre [4], and to determine how well they serve as equilibrium foundations for proportional allocations. In a nutshell, our conclusion will be that Farrell's unanimity principle is simpler than Mas-Colell and Silvestre's but (assuming a convex $C(\cdot)$) the latter always works (i.e., equilibrium exists and yields the proportional allocation) while the first may not (i.e., equilibrium yields the proportional allocation but it may not exist; in other words, the proportional allocation may fail to be supportable as an equilibrium).

2. Cost-Share Mechanisms

It will be useful to present first the cost-share mechanism for the provision of a public good studied in Kaneko [2] and Mas-Colell and Silvestre [4].

Let there be a public good producible with cost function $G(q)$. Individual valuations are given by $w_i(q)$. A vector of shares $(s_1, ..., s_N)$ is defined to be a *balanced linear cost-share equilibrium* (or ratio equilibrium) if there is a $\bar{q}$ which solves Max $w_i(q) - s_i G(q)$ for every i. Of course, we require $s_1 + \cdots + s_N = 1$.

A cost-share equilibrium yields an optimum (see the above references; the proof is immediate). Suppose that all the functions are C^1 and that

$\bar{q} > 0$. Then we have $w_i'(\bar{q}) = s_i G'(\bar{q})$ for all i and $G'(\bar{q}) = \sum_i w_i'(\bar{q})$ (which is the necessary condition for optimality). Therefore, we can conclude that the equilibrium allocation has the following proportionality property: $s_i = w_i'(\bar{q})/\sum_j w_j'(\bar{q})$ for all i. In words, the equilibrium cost-share of agent i is equal to the ratio of his marginal valuation to society's marginal valuation.

3. FARRELL EQUILIBRIUM

We now describe the approach taken by Farrell [1]. We go back to the model of Section 1 and suppose that while the agents, as consumers, are price takers, they must, as owners, agree on the price level p. Formally, p can be viewed as a public good enjoyed by the owners, and it is thus natural to apply to the profit-sharing problem the cost-sharing mechanism of the previous section. But what is $G(p)$, i.e., the "cost" of the level p of public good? At p the forthcoming demand is $F(p)$. The cost of production is $C(F(p))$ but there is also a revenue $pF(p)$. So we let the net cost (or surplus!) be $G(p) = C(F(p)) - pF(p)$, i.e., the negative of profits.

Suppose that $(\bar{p}, \overline{s_1}, ..., \overline{s_N})$ is a balanced, linear cost-share equilibrium for this problem, i.e., for every i, $\bar{p}$ solves $\text{Max}(v_i(p) - \bar{s_i} G(p))$. This is precisely the solution singled out by Farrell: the profit shares are distributed so that owners can unanimously agree on a price. From Section 2 we can then conclude (assuming $\bar{p} > 0$ and $\overline{s_i} > 0$ for all i):

(i) For every i, $\overline{s_i} = v_i'(\bar{p})/\sum_j v_j'(\bar{p})$. Because $v_i'(\bar{p}) = -f_i(\bar{p})$, and $\sum_j v_j'(\bar{p}) = -F(\bar{p})$, this means that profits are distributed proportionally to consumption.

(ii) The allocation generated by $(\bar{p}, \overline{s_1}, ..., \overline{s_N})$ is Pareto optimal among all the allocations in which consumers have the same marginal utility for the produced good. Because this is a necessary condition for overall optimality, it follows that the allocation is in fact a first best optimum, i.e., $\bar{p} = C'(F(\bar{p}))$. We can also prove this directly. Indeed, at the equilibrium $-F(\bar{p}) = \sum_j v_j'(\bar{p}) = G'(\bar{p})$. On the other hand, from the definition of $G(p)$ we get $G'(p) = C'(F(p)) F'(p) - pF'(p) - F(p)$. Hence, $\bar{p} = C'(F(\bar{p}))$. The conclusion we (and, of course, Farrell) reach is therefore that the equilibrium notion yields the *proportional allocation*, as defined in Section 2.

4. A DIGRESSION

This section is inspired by R. Manning [3].
Suppose that we modify the model of the previous sections as follows. Of

the N consumers only the first $M < N$, with aggregate excess demand function $0 < F_0(p) \leqslant F(p)$, are owners, in the sense that only their unanimous agreement is required in order to guide production. Let us proceed as in Section 3 and view p as a public good for the M owners. As before, the cost function is $G(p) = C(F(p)) - pF(p)$. If $(\bar{p}, \overline{s_1}, ..., \overline{s_M})$ is the cost-share equilibrium for this problem then (assuming interiority):

(i)   for every i,

$$\bar{s}_i = \frac{v_i'(\bar{p})}{\sum_{j=1}^{M} v_j'(\bar{p})} = \frac{f_i(p)}{F_0(p)},$$

i.e., the profit share of the ith owner equals his relative consumption on the total consumption of the set of owners, and

(ii)   the allocation generated is Pareto optimal for the owners in the class of allocations which are price supported (i.e., every consumer is on his demand curve) and only involve redistributions of profits.

At the equilibrium we must have $-F_0(\bar{p}) = \sum_{j=1}^{M} v_j'(\bar{p}) = G'(\bar{p})$. From the definition of $G(p)$ we get $G'(p) = C'(F(p)) F'(p) - pF'(p) - F(p)$. Hence $-F_0(\bar{p}) = -F(\bar{p}) + [C'(F(\bar{p})) - \bar{p}] F'(\bar{p})$. Denote $\alpha(\bar{p}) = F_0(\bar{p})/F(\bar{p})$ and $\eta(\bar{p}) = -\bar{p}F'(\bar{p})/F(\bar{p})$ (this is the elasticity of demand). Then $\alpha(\bar{p}) = 1 + [C'(F(\bar{p}))/\bar{p} - 1] \eta(\bar{p})$, or $C'(F(\bar{p}))/\bar{p} = (\alpha(\bar{p}) - 1)/\eta(\bar{p}) + 1$. This formula has been derived by R. Manning [3]. It measures the degree of overall inefficiency (i.e., the gap between $C'(F(\bar{p}))$ and $\bar{p}$) created by restricting ownership to a subset of agents. (Note that $\alpha(\bar{p}) = 1$ implies $C'(F(\bar{p})) = \bar{p}$.)

5. ANOTHER DIGRESSION

We have seen how the Farrell unanimity problem amounts to an instance of the cost-share mechanism applied to the population of owners, the price p as public good, and a certain specification of the cost function $G(p)$.

Suppose we now let $G(p)$ be an unspecified C^1 cost function and assume that $(\bar{p}, \overline{s_1}, ..., \overline{s_N})$ is an (interior) cost-share equilibrium with respect to this $G(p)$. Then $-f_i(\bar{p}) = v_i'(\bar{p}) = \bar{s}_i G'(\bar{p})$ for every i and so $G'(\bar{p}) = \sum_i v_i'(\bar{p}) = -F(\bar{p})$. Hence, $\bar{s}_i = f_i(\bar{p})/F(\bar{p})$ for all i. What this says is that no *matter what (differentiable) cost function $G(p)$ we wish to specify, as long as we insist on unanimity any equilibrium will distribute the costs proportionally to consumption.*

For illustration let us consider a specification of $G(p)$ different (and in a sense dual) to the one in Section 3: Put $G(p) = C(\Phi(p)) - p\Phi(p)$, where $\Phi(p)$ is such that $C'(\Phi(p)) = p$. Here consumer-owners assume that, given

the price p, production will be run at the level where price equals marginal cost. Thus the first order conditions for Pareto optimality are automatically satisfied (in contrast to Section 3, where they were satisfied only at equilibrium) but, again in contrast to Section 3, at an arbitrary p feasibility is not assured (i.e., $\Phi(p) \neq F(p)$). We now check that, interestingly, feasibility does obtain at equilibrium. Differentiating we get $G'(p) = C'(\Phi(p)) \Phi'(p) - p\Phi'(p) - \Phi(p) = -\Phi(p)$, which is just the familiar property of competitive profit functions. However, at $\bar{p}$ we have $-F(\bar{p}) = G'(\bar{p})$. Hence, $F(\bar{p}) = \Phi(\bar{p})$, i.e., feasibility obtains.

6. PROPORTIONAL ALLOCATIONS NEED NOT BE FARRELL EQUILIBRIA

We take up again the thread of Section 3 and pose the existence question for Farrell equilibria. If an equilibrium exists then it must yield the (unique) proportional allocation. Hence, there is a single candidate for equilibrium, namely $(\bar{p}, \overline{s_1}, ..., \overline{s_N})$ where $\bar{p} = C'(F(\bar{p}))$ and $\overline{s_i} = f_i(\bar{p})/F(\bar{p})$. The issue is, therefore, if, given the shares $\overline{s_i}$, the price $\bar{p}$ will enjoy the unanimous agreement of owners; that is, will $\bar{p}$ maximize $v_i(p) + s_i[pF(p) - C(F(p))]$ for every i? Note that as far as the first order conditions are concerned we are fine because

$$v_i'(\bar{p}) + \overline{s_i}[F(\bar{p}) + \bar{p}F'(\bar{p}) - C'(F(\bar{p})) F'(\bar{p})] = -f_i(\bar{p}) + \overline{s_i}F(\bar{p}) = 0.$$

But alas!, the next example shows that the second order conditions may not be satisfied (and therefore the proportional allocation is not supportable as a Farrell equilibrium). This should not come as a complete surprise. Indeed, the function $v_i(p)$ is convex and while the second term of the objective function $s_i(pF(p) - C(F(p)))$ may (or may not) have the right curvature, the fact remains that critical points may easily be minima rather than maxima.

EXAMPLE 1. Let $N = 2$, $u_1(y_1) = \frac{3}{2}y_1 - \frac{1}{4}y_1^2$, $u_2(y_2) = 4y_2 - \frac{1}{2}y_2^2$, with corresponding demand functions $f_1(p) = 3 - 2p$, $f_2(p) = 4 - p$. Marginal cost is constant and equal to 1. The candidate equilibrium is $\bar{p} = 1$, $s_1 = \frac{1}{4}$, $s_2 = \frac{3}{4}$. Figure 1 plots the objective function of agent 1 for $\overline{s_1} = \frac{1}{4}$. Observe that $\bar{p} = 1$ is a local minimum. For agent 2, $\bar{p} = 1$ is a local maximum. It is always true that at the candidate equilibrium at least one agent is at a local maximum. But this is it. It is not difficult to give an N agent example where $\bar{p}$ is a strict local minimum for $N - 1$ of them. Moreover, allowing non-linear demand we can prespecify the consumption shares at $\bar{p}$. In other words, we can construct the example so that there is nearly share-unanimity in moving away from $\bar{p}$ (in any direction!).
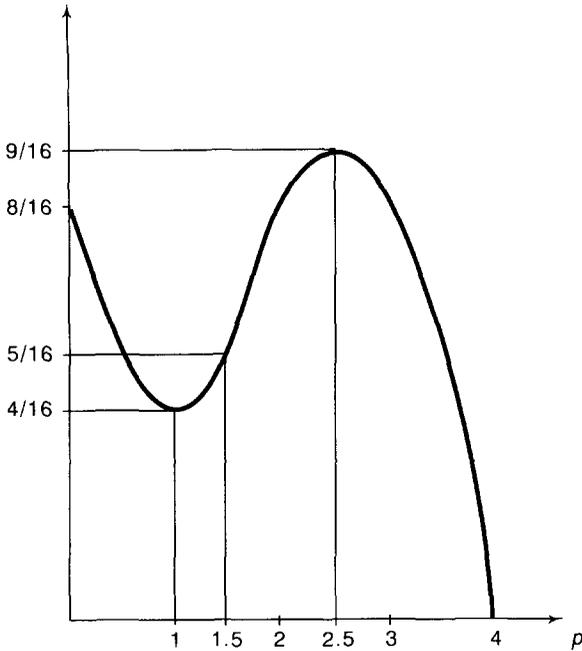
$$v_1(p) + (1/4)(p-1)F(p)$$

FIG. 1.   The objective function of the first agent, example 1, attains a local minimum at $\bar{p} = 1$.

We conclude this section with some remarks.

Remark.   In the variation studied in Section 5, the situation is even worse. If $C(\cdot)$ is convex then the proportional allocation is *always* a unanimous minimum. With increasing returns, however, no clear-cut conclusion is possible.

Remark.   It would be interesting to investigate under what conditions on the distribution of preferences the proportional allocation would be sustainable (at least "locally") as a Farrell equilibrium. Perhaps some hypothesis on the positive correlation of demand and demand elasticities would be of help. For example, if individual demand functions are of the form $f_i(p) = \alpha_i(2 - p)$, i.e., they are linear with the same "vertical" intercept, and marginal cost is equal to one, then sustainability obtains (Proof. Every individual objective function is quadratic, with positive derivative for $p < 1$ and with zero derivative at $p = 1$. Hence any critical point must be a maximum.)

Remark.   We could ask the same existence question in the context of Section 4, i.e., if only part of the population are a priori owners. The

answers are similar. It is important to realize, however, that if the number of a priori owners is small, then the exact unanimity issue loses some of its meaning. This is because all agents want to remain close to the monopoly price even when they cannot exactly agree.

Remark. The implication of Example 1 is that there may not be any vectors of profit shares $s = (s_1, ..., s_N)$ for which the optimum price cannot be vetoed by some shareholder. It is natural to ask how the situation changes if we take a majority voting rather than a unanimity approach. Two questions seem particularly interesting:

(a) Given s, is there a p which is a Condorcet winner, i.e., there is no strict majority of *shares* against p? We feel that the answer to this question should be "not necessarily." But we have not succeeded in constructing a counterexample.

(b) Can the optimal price $\bar{p}$ be supported as a Condorcet winner for some s? We have a partial answer to this. Suppose that $\bar{s}$ are the proportional shares. If $s_i > \bar{s}_i$ (resp. $s_i < \bar{s}_i$) then profit (resp. consumption) considerations prevail and agent i would prefer a p slightly larger (resp. smaller) than $\bar{p}$. Therefore, if we take an s such that $s_i \neq \bar{s}_i$ for all i and $\sum_{\{i: s_i > \bar{s}_i\}} s_1 = \frac{1}{2}$ (if $N \geqslant 3$ such an s exists), it follows that there is no strict majority (with a more elaborate construction we could typically make sure that there is not even a weak majority) for any *small* change from $\bar{p}$. For the corresponding global question we have no answer.

7. BALANCED LINEAR COST-SHARE EQUILIBRIA

The approach of Mas-Colell and Silvestre [4] differs from Farrell [1] in three respects. First, the unanimity is sought over all feasible distribution of outputs (an N-dimensional problem) rather than only on those arising as vectors of demands at different prices (a 1-dimensional problem). Second, the share parameters are on cost rather than profit. Third, the shares are supplemented as equilibrating variables by side compensation parameters. We again get proportional allocations as outcomes. The decision problems are now more complex but there is substantial theoretical payoff: a general existence theorem. We refer to Mas-Colell and Silvestre [4] for a more precise motivation of the equilibrium concept to be presented. The key idea is to look at the entire distribution of output as a public good vector and then specialize to the corresponding decision problem the multiple good version of the cost-share mechanism discussed in Section 2.

An output distribution vector $\bar{y} = (\bar{y_1}, ..., \bar{y_N}) \geqslant 0$ is a *linear cost-share equilibrium* if there are vectors $s \in R^N$, $a_i \in R^N$, $i = 1, ..., N$, such that

$\sum_i s_i = 1$, $\sum_i a_i = 0$, and, for all i, $\bar{y}$ maximizes $u_i(y_i) - (a_i \cdot \bar{y} + s_i C(\sum_j y_j))$. Of course, $a_i \cdot y + s_i C(\sum y_i)$ is interpreted as the share of the cost to be paid by i. If, in addition, we have $a_i \cdot \bar{y} = 0$ for all i, then we say that the equilibrium is *balanced*.

Balanced linear cost-share equilibria (BLCSE) have the following properties (see Mas-Colell and Silvestre [4]):

(i)   a BLCSE is a Pareto optimum (the argument is as in Section 2);

(ii)   if the cost function is convex, then a BLCSE exists;

(iii)   a BLCSE (assumed interior) yields the proportional allocation;

(iv)   if the cost function is convex then any proportional allocation can be supported as a BLCSE.

An implication of (iv) and of Example 1 (in the previous section) is that a proportional allocation may be sustainable as a BLCSE but not as a Farrell equilibrium. In Example 1 the parameters that sustain the proportional allocation as a BLCSE are $a_1 = (\frac{3}{4}, -\frac{1}{4})$, $a_2 = (-\frac{3}{4}, \frac{1}{4})$, $s = (\frac{1}{4}, \frac{3}{4})$.

Interestingly, if C is not convex it may happen that the proportional allocation is sustainable as a Farrell equilibrium but not as a BLCSE. The next example shows this possibility.

EXAMPLE 2.   Let $N = 2$, $u_1(y_1) = 12y_1 - \frac{11}{2}y_1^2$, $u_2(y_2) = 30y_2 - \frac{29}{4}y_2^2$. The cost function is $C(z) = z$ for $z \leqslant 1.2$ and $C(z) = 1 + z$ for $z > 1.2$ (Note: this cost function is discontinuous. None of the conclusions of the example would be modified if the function is made smooth in a neighborhood of $z = 1.2$).
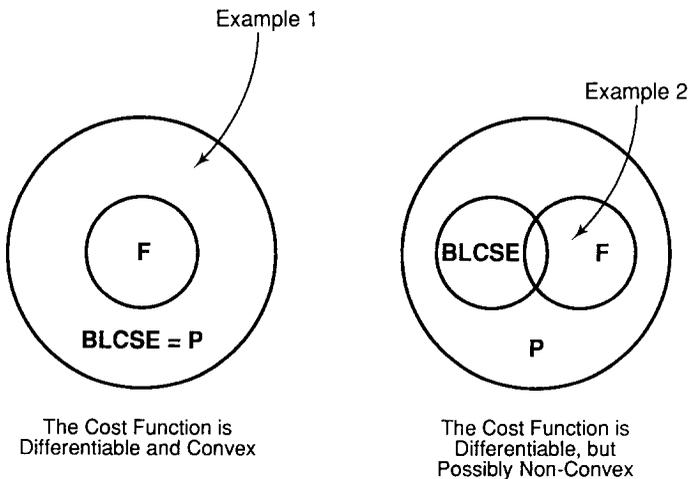

FIG. 2.   The relationship among proportional allocations (P), Farrell equilibria (F), and balanced linear cost-share equilibria (BLCSE) with differentiable cost functions.

The proportional allocation is $\bar{y} = (1, 2)$, $\bar{x} = (\omega_1 - \frac{4}{3}, \omega_2 - \frac{8}{3})$. Our first claim is that this cannot be supported as a BLCSE. Indeed, if it was then one easily derives from the first order conditions that $a_1 = (\frac{2}{3}, -\frac{1}{3})$, $s_1 = \frac{1}{3}$. This gives a value of $\frac{31}{6}$ for the objective function of agent 1 evaluated at $\bar{y} = (1, 2)$. But the same objective function has a value of $\frac{33}{6}$ when evaluated at $y = (1, 0)$. Contradiction.

The second claim is that the proportional allocation can be supported as a Farrell equilibrium for the price $\bar{p} = 1$ and the profit shares $\bar{s} = (\frac{1}{3}, \frac{2}{3})$. The verification of this is straightforward but rather tedious (since the objective functions are not concave, one has to check a variety of cases). We will skip it.

Figure 2 illustrates the implications of this section.

8. THE ROLE OF THE DIFFERENTIABILITY OF COST

Without differentiability of the cost function Farrell equilibria or BLCSE still yield Pareto optimal allocations (the arguments of Section 2 do not depend on differentiability) but they may not be proportional (nor need a Farrell equilibrium be a BLCSE). Examples 3 and 4 will illustrate these possibilities. In Mas-Colell and Silvestre [4] it is shown that the existence of a BLCSE does not depend on the differentiability of the cost functions (as long as the cost function is convex). Neither does the fact that a proportional allocation is always sustainable as a BLCSE.

EXAMPLE 3. There may be BLCSE which do not yield proportional allocations. Let $N = 2$, $u_1(y_1) = 4y_1 - y_1^2$, $u_2(y_2) = 10y_2 - 2y_2^2$, and $C(z) = z$ for $z \leqslant 3$, $C(z) = 3 + 10(z - 3)$ for $z \geqslant 3$. Consider the allocation $\bar{y} = (1, 2)$, $\bar{x} = (\omega_1 - 1.8, \omega_2 - 1.2)$. It is obviously not proportional. Although tedious, it is not hard to verify that it is a BLCSE for the parameters

$$a_1 = (\tfrac{4}{3}, -\tfrac{2}{3}), \qquad a_2 = (-\tfrac{4}{3}, \tfrac{2}{3}), \qquad s = (\tfrac{6}{10}, \tfrac{4}{10}).$$

EXAMPLE 4. There may be Farrell equilibria which do not yield allocations sustainable as BLCSE (and are, therefore, nonproportional, since the cost function is convex). Let $N = 2$, $u_1(y_1) = 2.02 y_1 - 0.01 y_1^2$, $u_2(y_2) = 2.08 y_2 - 0.02 y_2^2$, and $C(z) = z$ for $z \leqslant 3$, $C(z) = 3 + 10(z - 3)$ for $z \geqslant 3$. Consider the allocation $\bar{y} = (1, 2)$, $\bar{x} = (\omega_1 - 0.1, \omega_2 - 2.9)$. The first claim is that this corresponds to a Farrell equilibrium for $\bar{p} = 2$ and $\bar{s} = (\frac{19}{30}, \frac{11}{30})$. We skip the verification (which, as usual, is simple but tedious). The second claim is that this allocation cannot be supported by a BLCSE. This is not hard to verify (we should have $s_1 = \frac{1}{30}$ and $a_{12} = -\frac{1}{2}a_{11}$;
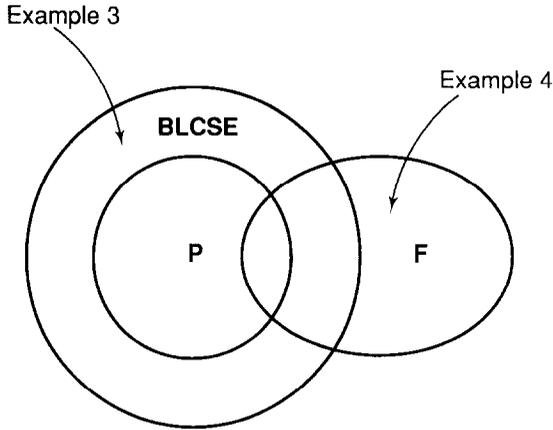
Example 3

FIG. 3. The relationship among proportional allocations (P), Farrell equilibria, and balanced linear cost-share equilibria (BLCSE) with cost functions that are convex but possibly non-differentiable.

assuming that $\bar{y}$ solves the maximization problem of agent 1 then one gets $a_{11} \geqslant \frac{5}{3}$ and $a_{11} \leqslant \frac{2}{3}$, which is obviously impossible).

Figure 3 illustrates the implications of this section for cost where the cost function is convex but not differentiable.

9. THE ROLE OF THE QUASILINEARITY OF UTILITIES

In Mas-Colell and Silvestre [4] or Roemer and Silvestre [5] the quasilinearity of utility functions is never used. Hence all results concerning BLCSE and proportional allocations remain valid without quasilinearity.

With the concept of Farrell equilibrium there is a bit of a problem. Without quasilinearity, total demand, and therefore total profits, depends not only on the price but also on the profit shares themselves. This complicates matters. It is however still possible to define a concept of Farrell equilibrium for which the results of Section 3 go through (but not the interpretation as a cost-share problem in the sense of Section 2).

10. SEVERAL PRODUCED GOODS

The analysis can be extended to the case of a firm producing M goods, as in Mas-Colell and Silvestre [4] and Roemer and Silvestre [5]. The existence of Balanced Linear Cost-Share Equilibria is in particular guaranteed under convexity.

But the existence of Farrell equilibria becomes very problematic. The new difficulty is that the argument in Section 3 yields the necessary condition (in obvious notation)

$$\bar{s}_i = \frac{f_{ij}(\bar{p})}{F_j(\bar{p})}, \qquad \text{for all} \quad \text{agents } i \text{ and all produced goods } j;$$

i.e., the ratio of individual to total consumption of each produced good must be equal across goods, a condition unlikely to be met.

The difficulty would disappear were each good produced by a distinct firm. (Obviously, this requires a separable cost function.) Then it would be possible to distribute the profits of firm j in proportion to the individual consumptions in the jth good.

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