# Three Observations on Sunspots and Asset Redundancy

**Andreu Mas-Colell**

In this paper I make the following three observations (of which the second is perhaps the most significant):

- Suppose we have a sequential, finite-horizon, *complete* market economy. Let some uninsurable publicly observed sunspot signals occur after trade takes place in period 1 but before it takes place in period 2. Nontrivial sunspot equilibria are then possible. (*Nontrivial* means that the realizations of the equilibria are not equilibria of the model without sunspots.) This will come as no surprise, but no example making precisely this point seems to be available. The seminal reference on sunspot equilibria is Cass and Shell 1983 (see also Shell 1987). Various examples can be found in Cass and Shell 1983 and also in Aumann, Peck, and Shell 1988, Balasko 1987, Balasko, Cass, and Shell 1988, Guesnerie and Laffont 1988, Maskin and Tirole 1987, and Peck and Shell 1988.
- The presence of redundant assets may help to immunize an otherwise complete economy against sunspot contamination.
- Redundant assets may actually influence the disequilibrium properties of spot equilibria; for example, the presence of a redundant asset may help to guarantee the tâtonnement stability of a spot equilibrium. This last observation is not related to sunspot issues.

The power and the failures of the Invisible Hand have been at the center of Frank Hahn's research. In a modest way these three observations deal also with this grand topic. The presentation lacks, I am afraid, the flair that Frank would undoubtedly have given to it. But the thought of what he could do with them persuades me to dedicate these three observations to him.

## 1 Nontrivial Sunspot Equilibria

Our initial datum is a conventional Walrasian exchange economy at date $t = 1$. There are $\ell$ commodities and two (types of) consumers, $i = 1, 2$, with endowments $\omega_i \in R^{\ell}_{+}$ and utility functions $u_i$: $R^{\ell}_{+} \to R$ satisfying the usual conditions (including strict concavity and differentiability). Price vectors are denoted by $p$. A Walrasian price vector $p$ and allocation $\bar{x} = (\bar{x}_1, \bar{x}_2)$ are defined, in the familiar way, by the following conditions:

$\bar{x}_1 + \bar{x}_2 \leq \omega_1 + \omega_2$,

$x_i$ maximizes $u_i(x_i)$ subject to $p \cdot x_i \leq p \cdot \omega_i$.

Note that nothing prevents us, either here or in later developments, from interpreting some of the $\ell$ commodities as state-contingent Arrow-Debreu commodities. There is no need, however, to keep track of this separately. The key fact is that *our basic economy is a complete one.*

Suppose now that at the beginning of period 1 an extrinsic, payoff-irrelevant, publicly observable state $s \in S$, for short a sunspot, occurs with probability $\pi(s)$ (see Aumann et al. 1988, Maskin and Tirole 1987, and Peck and Shell 1991 for models without public observability). We take $\# S < \infty$. Under the hypothesis of rational expectations the equilibrium is then an assignment $p(s),x(s)$ such that for every $s$, $p(s),x(s)$ is Walrasian for the original economy. If the latter has several equilibria, then the *ex post* equilibrium realization may depend on the sunspot. This is an obvious and, for current purposes, not terribly interesting fact. After all, *ex post* the presence of sunspots is undetectable. In this paper we concern ourselves with nontrivial sunspot equilibria, defined as those where with positive probability the *ex post* realization of the equilibrium is not an equilibrium of the original economy.

For the sunspot equilibrium function to be something other than a selection of the equilibria of the original economy, we need to extend our primitive model to include some decision variables at $t = 0$ that are payoff-relevant and therefore capable of being influenced by price expectations. Perhaps the simplest way to proceed is to assume that there is a given list of assets tradable at $t = 0$ and having returns at $t = 1$.

Suppose first that the assets available at $t = 0$ can have payoffs contingent on the sunspot signal. Assume in fact that there is an Arrow security for every $s$ (that is, security $s$ pays \$1 if and only if $s$ occurs). Then, if every trader has free access to the securities markets, complete insurance is possible and risk aversion rules out the possibility of (even trivial) sunspot equilibria (Cass and Shell 1983). Cass and Shell (1983), Balasko (1987), and Balasko et al. (1988) have shown that if access to the asset markets is asymmetric (i.e., if only some consumers can use them) then nontrivial sunspot equilibria are possible. Guesnerie and Laffont (1988) arrive at the same conclusion under the hypothesis that access is symmetric but the original economy is not itself complete. The case where sunspot-contingent contracts are possible is therefore well understood (but see the remark at the end of this section).

In this paper we consider the situation where assets cannot be sunspot-contingent but otherwise the original economy is complete (thus, this case is complementary to that of Guesnerie and Laffont 1988). Informally, the interest of this case can be argued as follows. Suppose that we live in a fairly perfect world (i.e., one where complete markets emerge) but suppose also that the world is exposed to a multitude of sunspot signals—so many, in fact, that they cannot even be listed *a priori* and, as a consequence, the task of developing insurance institutions that will automatically guarantee sunspot-freeness is unfeasible. This "perfect" world will have to be more reactive. Insurance institutions (perhaps contingent markets, but see the next section for another suggestion) will develop as particular sunspot variables find their way into the equilibrium, and with the targeted intention of counteracting and nullifying them. Obviously, it would be neater if the system did not have to worry about all this because the underlying completeness made it immune to (nontrivial) sunspots. It will, probably, come as no surprise that this is not so. Nonetheless, it will be useful to have a class of examples, and this section is devoted to providing it.

We proceed to extend the original model by including asset trade at $t = 0$. The assets are sunspot-independent and pay in goods at $t = 1$. To be specific, we assume there are two assets with return vectors $r_1, r_2 \in R^{\ell}_{+}$. (Because of the budget constraint, a single asset would not allow any trade.) A sunspot equilibrium is now a four-tuple $(\bar{q}, \bar{y}, \bar{p}, \bar{x})$ composed of an asset price vector $\bar{q} \in R^2$, asset trade vectors $\bar{y}_i \in R^2$ $(i = 1,2)$, a spot equilibrium price function $\bar{p}(s) \in R^{\ell}$ $(s \in S)$, and consumption allocations $\bar{x}_i(s) \in R^{\ell}_{+}$ $(s \in S, i = 1,2)$ satisfying the following:

(a) $\sum_i \bar{x}_i(s) \leq \sum_i \omega_i$, $s \in S$, $\sum_i \bar{y}_i \leq 0$.

(b) For every $i = 1,2$, $\bar{y}_i, \bar{x}_i$ solve

$$\text{Max} \sum_s \pi(s) u_i(x_i(s))$$

$$\text{s.t. } \bar{p}(s) \cdot x_i(s) \leq \bar{p}(s) \cdot (\omega_i + r_1 \bar{y}_{i1} + r_2 \bar{y}_{i2}),\ s \in S, \text{ and } q \cdot y_i \leq 0.$$

The sunspot equilibrium is nontrivial if, for some $s \in S$, $\bar{x}(s)$ is not an equilibrium allocation of the original economy.

Without the sunspots, the asset structure is redundant (the original economy is complete). The assets are not needed at equilibria, although there may be asset trades. More precisely, there may be a continuum of consumption equivalent asset trades and every equilibrium consumption can be reached with zero asset trades (see the next section for more on redundancy). This is not an essential feature of our analysis, the important point is that the economy, extended with the asset trades, is complete. This may happen because (as in our case) the original economy is complete, or as a consequence of the asset trading possibilities.

In the presence of sunspots the option to trade in assets becomes very relevant, so much so that nontrivial sunspot equilibria are now a possibility. In particular, we have the following proposition:

PROPOSITION 1 Suppose that the basic economy $(u_i,\omega_i)$, where $\omega_i \gg 0$ and $i = 1,2$, admits three linearly independent equilibrium price vectors $p_1, p_2, p_3$ (this implies $\ell > 2$). Suppose also that $\#S > 2$. Then we can find endowments $\omega_i'$ $(i = 1,2)$ and asset-return vectors $r_1, r_2$ such that the economy $(u_i,\omega_i')$, $i = 1, 2$, admits nontrivial sunspot equilibria.

*Proof* The idea is to construct the $\omega_i'$ and the vectors of asset returns in such a way that after equilibrium asset trade we get the original economy $\omega_i$ and, depending on the sunspot signal, some of the equilibrium prices $p_1, p_2, p_3$.

Without loss of generality we let $S = \{s_1,s_2,s_3\}$ and the three states be equiprobable. Let $\lambda_j^i$ $(i = 1,2, j = 1,2,3)$ be the marginal utility of income of consumer $i$ at the equilibrium corresponding to $p_j$. Choose a vector $a = (a_1,a_2,a_3) \neq 0$ such that $\sum_j \lambda_j^i a_j = 0$ for $i = 1,2$. Because the price vectors $p_j$ are linearly independent, we can find $r \in R^\ell$ such that $p_j \cdot r = a_j$ for every $j$. Of course, $r \neq 0$. We can express $r = r_1 - r_2$ with $r_1 \geq 0$ and $r_2 \geq 0$. These are going to be the two assets. Note that because we can choose $a$ to be arbitrarily small the returns $r$ are also small—sufficiently small, in fact, for

$$\omega_1' = \omega_1 + r \geq 0, \omega_2' = \omega_2 - r \geq 0.$$

We claim that the following describes a nontrivial sunspot equilibrium for the economy $(u_i,\omega_i')$, where $i = 1, 2$: Take $q = (1,1)$, $y_1 = (1,-1)$, $y_2 = (-1,1)$, $p(s_1) = p_1$, $p(s_2) = p_2$, and $p(s_3) = p_3$. Indeed, $\sum_j \lambda_j^i p_j(r_1 - r_2) = 0$ for $i = 1, 2$ guarantees that we have an equilibrium. For the nontriviality, note that, for some $j$,

$$p_j \cdot \omega_j' = p_j \cdot \omega_j + p_j \cdot r = p_j \cdot \omega_j + a_j \neq p_j \cdot \omega_j. \quad \blacksquare$$

We end this section with a few remarks on the hypothesis of proposition 1.

The proof makes clear that we need to randomize at least three price vectors, and therefore we need at least three states. The next section will further clarify this.

We do not know if there is an example with $\ell = 2$.

The need of multiple equilibria from some endowment is clear. Sunspots can matter only if they induce randomness. However, the *ex post* economy is nonrandom, and therefore randomness can only come from a nonuniform equilibrium selection for the *ex post* economy.

The need of multiple equilibria is not so obvious if sunspot-contingent assets are possible. The following statement may be true, but, so far as we know, it is unproven, and a proof does not seem easy: "Suppose that the utility functions of the $N$ consumers at $t = 1$ are such that the equilibrium is unique for any distribution of endowments. Then for any asset structure at $t = 0$ (sunspot-contingent assets allowed) every equilibrium is sunspot-free." Note that the statement is true in the two extreme cases (where there are no sunspot-contingent assets and where there are enough of them to allow complete insurance). The problem is in the intermediate cases. [Note added in May 1990: The above question has been settled by Hens (1990). Alas, for the more natural interpretations of the expression "the equilibrium is unique for any distribution of endowments" the statement is false. See Hens 1990 for a thorough analysis.]

## 2 Redundant Assets and Sunspots

In this section I shall argue, by means of a trivial example, that the existence of redundant assets may help to immunize an economy against the presence of sunspots. The reason is that with more assets there are more possibilities that the economy finds a way to coordinate itself so that trade takes place before the sunspot signal occurs—which is like saying that the agents of the economy agree not to look at the sunspot.

As in the previous section, at $t = 1$ there are spot markets for $\ell$ goods. We do not, however, require now that the market structure be complete at $t = 1$. It is possible that, for example, one of the characteristics defining a commodity be a set of (payoff-relevant, non-sunspot) states of the world, and that each of these leads to a separate budget constraint. We need not be explicit about the particular spot market structure.

At time $t = 0$, there is trade in $J$ assets. We assume that the family of payoff vectors $\{r_1, \ldots, r_J\} \subset R^\ell$ has maximal possible rank, i.e., the rank equals the minimum of $J$ and $\ell$.

Depending on the spot market structure, the assets may or may not be redundant. In section 1 all the assets were redundant because the system of spot markets was complete to begin with. More generally, suppose that we had $G$ physical goods and $M$ payoff-relevant states. Then $\ell = GM$, and if there is a budget constraint per state we have redundancy only if $J > M$.

The number of traders is $N$.

As in section 1, at the beginning of period 1 the publicly observable sunspot state $s \in S$ occurs. The cardinality of the set $S$ is not restricted in any way (except that for simplicity we take it to be finite). A sunspot equilibrium $(\bar{q}, \bar{y}, \bar{p}, \bar{x})$ is defined as in section 1. (Note that if the spot market structure requires several budget constraints, even after the occurrence of $s$, then the budget restriction in part (b) of the definition in section 1 must be modified accordingly.)

The example in section 1, or for that matter any trivial (but nonconstant across sunspots) sunspot equilibrium, shows that sunspot equilibria may not be Pareto-optimal. If we could create a complete set of sunspot-contingent assets, then the Pareto-optimality of equilibria would be restored. Proposition 2, which is a version of the First Fundamental Theorem, shows that there is another possibility: if $J = \ell$, then any sunspot equilibrium is optimal (that is, it is sunspot-free). Note that *$\ell$ may be quite smaller than S*, which means that even if $J = \ell$ and, from the standpoint of the original economy, there is much redundancy, we may still be quite far from completeness inclusive of sunspots (that is, insurance contracts against sunspots may remain unavailable).

There is no mystery in this result. Think of the $J$ assets as future contracts. Then the hypothesis $J = \ell$ says that any trade can be guaranteed by future contracts. Hence, even if there are no exogenously given insurance contracts on sunspot signals, there is nonetheless an endogenous insurance mechanism: future contracting before the signal occurs. In a related guise all this is very familiar. The traditional interpretation of the Arrow-Debreu equilibrium is in terms of contracting at time 0 for any future delivery of goods (which could be contingent on the occurrence of some payoff-relevant state). Clearly, any subsequently occurring sunspot signal will not affect anything. All trade has taken place at time 0. As Arrow (1953) observed, if the equilibrium is recast as taking place sequentially then there is much asset redundancy in the traditional interpretation (only one contingent commodity for every payoff-relevant state is needed). In turn, my point is simply that this redundancy is not entirely useless: it makes sunspot infection more difficult.

Formally,

PROPOSITION 2 If $J \geq \ell$ then every sunspot equilibrium is Pareto-optimal.

*Proof* As indicated, no more is involved than the usual proof of the First Fundamental Theorem. Suppose that $(\bar{q},\bar{y},\bar{p},\bar{x})$ is a sunspot equilibrium and that the allocation $x$ Pareto-dominates $\bar{x}$. Because the rank of $\{r_1,\dots,r_J\}$ is $\ell$, we can find $y = (y_1,\dots,y_N)$ such that $\sum_i y_i = 0$ and $x_i = \sum_j r_j y_{ij}$. If $\bar{q} \cdot y_i \leq 0$ then agent $i$ could attain the consumption $x_i$ at prices $(\bar{q},\bar{p})$. Hence $\bar{q} \cdot y_i \geq 0$ for all $i$ and $\bar{q} \cdot y_i > 0$ for at least one $i$. But this contradicts $\sum_i y_i = 0$. ∎

## 3 More on Redundancy

In the previous section we saw by means of an example how the presence of redundant assets may prevent the occurrence of some unsuitable phenomena. This point can be deepened and made much more general. Even in a context of complete markets (where the set of equilibrium allocations is invariant to the addition or deletion of redundant assets) the properties of particular equilibria may depend on the precise set of available assets. As an illustration, it is shown in this section that an Arrow-Debreu equilibrium that with probability 1 is *ex post* tâtonnement stable and unique can become *ex post* unstable and nonunique with positive probability if a redundant asset is dropped.

Suppose that at $t = 1$ one of two states $s = 1, 2$ can occur. In state 2 there is a single physical commodity; in state 1 there are two. We have two traders with utility functions of the form

$$u_i(x_i,y_i) = v_i(x_i) + y_i,$$

where $x_i \in R^2_+$ and $y_i \in R_+$ are, respectively, the consumptions in states 1 and 2 (obviously, these functions can be viewed as von Neumann-Morgenstern utility indicators). We assume that $v_i(\cdot)$ is strictly concave, increasing, and continuously differentiable in $R^2_+$. The endowment vectors are denoted $(\omega_i, z_i) \in R^2_+ \times R_+$. The following fact is well known:

LEMMA 3 Given $v_i(\cdot)$, $\omega_i$, $i = 1, 2$, there is a $k > 0$ large enough that if $z_i > k$, $i = 1, 2$, then

$$\{(u_i(x_i) + y_i, \omega_i, z_i)\}_{i=1}^2$$

has a single Arrow-Debreu equilibrium.

*Proof* Let $\omega_i < re$, $e = (1,1)$, for $i = 1, 2$ and $t > \partial_j v_i(x_i)$ for $i = 1, 2, j = 1$, 2, and $x_i \leq re$. Take then $k > 4rt$. There can be only an equilibrium allocation with $y_i > 0$ for $i = 1, 2$. This is simply due to the quasi-linearity of the utility functions. Suppose there was an equilibrium allocation $(\bar{x}_i, \bar{y}_i)$ with $\bar{y}_1 = 0$. Normalizing to $p_3 = 1$, the corresponding equilibrium price vector is denoted $p = (p_1, p_2)$. We must have

$$k \leq p \cdot \omega_1 + z_1 \leq p \cdot \bar{x}_1 \leq p \cdot (\omega_1 + \omega_2) \leq 2rp \cdot e.$$

Hence, $p \cdot e \geq k/2r > 2t$. On the other hand, for each $j = 1, 2$ there should be an $i$ with $\bar{x}_{ij} > 0$. From the first-order conditions, $\lambda_i p_j = \partial_j v_i(x_i)$ and $\lambda_i \geq 1$. Therefore, $p_j \leq \partial_j v_i(x_i) < t$, and so $p \cdot e < 2t$. This contradiction shows that there cannot be more than one equilibrium. ■

Denote by $f_i(p)$, $p \in R^2_{++}$, the excess demand of consumer $i$ in the spot economy of state 1 determined by the utility functions $v_i(\cdot)$ and the endowments $\omega_i$. Letting $e = (1,1)$, suppose that $f_1(e) + f_2(e) = 0$ and $\partial v_i(\omega_i + f_i(e)) = e$. What restrictions does all this impose on $f(p) = f_1(p) + f_2(p)$? As it is well known (see Shafer and Sonnenschein 1982 or section 5.5 of Mas-Colell 1985), the answer is "almost none." In particular, we may choose $f(\cdot)$ to have as many zeroes as we wish, and we can also predetermine the matrix $\partial f(e)$. Thus, for example, the equilibrium price vector $e$ could be unstable.

On the other hand, if the $z_i$ $(i = 1, 2)$ are large enough, lemma 3 ensures that the overall Arrow-Debreu economy has a single equilibrium, which, normalizing to $p_3 = 1$, cannot be any other than $(p_1, p_2) = e$.

We have the following situation. Consider three assets with returns $r_1 = (1,0,0)$, $r_2 = (0,1,0)$, and $r_3 = (0,0,1)$. Thus, $r_1$ and $r_2$ are contingent assets on state 1 but with returns in different commodities. If the asset structure is $\{r_1, r_2, r_3\}$ one of the two first assets is redundant but we can sustain the Arrow-Debreu equilibrium very nicely. In fact, we can even guarantee that in the *ex post* spot economy of state 1 the equilibrium price vector $e$ is a no-trade equilibrium, hence unique and stable. However, if we drop one redundant asset the Arrow-Debreu equilibrium can be reached only with zero trade in assets, which means that in the *ex post* spot economy of state 1 the equilibrium price vector $e$ may not be particularly nice (it could be nonunique, unstable, etc.). There is, therefore, a definitive loss in deleting the redundant asset.

A similar point has been made by Samuelson (1974) in the context of lump-sum taxation. There the redundancy issue is that the same wealth transfer can be accomplished by in-kind transfers denominated in any commodity.

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