On Aggregate Demand in a Measure Space of Agents*

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Preface (1991): This paper first appeared in May 1973 as Working Paper IP-183 of the Center for Research in Management Science of the University of California - Berkeley. In spite of the high regard that this author had for it, it remained unpublished. It is on a topic very close to Trout Rader's interest and it belongs to a line of the literature which grew considerably after 1973 (see W. Trockel: *Market Demand: An Analysis of Large Economies with Non-convex Preferences*, 1989 *Lecture Notes in Economics and Mathematical Systems* 223, Springer-Verlag). It is dedicated to Trout Rader with much sympathy.

We are concerned in this paper with an instance in equilibrium theory of the general phenomenon of smoothing by aggregation.

It is known that without a convexity assumption on preferences the demand correspondence of a consumer will not, in general, be convex valued. However, if the economy is large, actually if there is a continuum of agents, one gets an aggregate demand correspondence which is convex valued (see R. Aumann (1964)). This is a consequence of Liapunov's theorem, and the fact that the correspondences being integrated (i.e., aggregated) are demand ones is immaterial; by integrating a correspondence with respect to an atomless measure one always obtains a convex set. Recently, G. Debreu (1972, p. 614) has suggested that, precisely because of the economic structure of the problem, one may expect, under reasonable regularity conditions, the aggregate demand to be a function. This is, indeed, very plausible; of course, not every economy (not even atomless) will have this property; the contention is, rather, that classes of economies having it are significant ones and can be distinguished in a natural manner.

We want to report here a result which gives support to this intuition. It is not a difficult one; its key features are the reduction of the problem to one dealing with finite-dimensional objects and the exploitation of smoothness hypothesis on preferences. Representing an economy by a measure over a space of agents' characteristics (see W. Hildenbrand (1970)) the result says, very roughly, that if the

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measure is supported by and evenly spread over the set of agents which preferences are representable by polynomial utility functions, then the economy generates a continuous demand function. Obviously, this does not go very far but at least it gives some flavor of the sort of the theorems one would like to have; we shall defer more detailed comments to the remarks after the precise statement (1) and to the final section (3).

As it is clear from the above paragraphs we are emphasizing an interpretation in terms of large economies. We want to remark, however, that the results given (especially the theorem in 2) could be interpreted as well in an uncertainty of preferences framework.[1]

1 Statement of the Proposition

For the following concepts and description of an economy see W. Hildenbrand (1970). The commodity space, as well as the price domain, will be $P = \{x \in \mathbb{R}^\ell : x \gg 0\}$.[2] The set of continuous, monotone preference relations on P is denoted by $\mathcal{R}$;[3] it is endowed with the topology of closed convergence of sets;[4] the space of preferences-endowments pairs is $\mathcal{E} = \mathcal{R} \times P$ (with the product topology). The demand correspondence[5] $h : \mathcal{E} \times P \to P$ is defined by $h(R, \omega, p) = \{x \in P : (x, y) \in R$ for every $y \in P$ such that $py \le p\omega\}$. An economy will be described by a probability measure ν on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$[6] satisfying $\int_{\mathcal{E}} \omega(\cdot) d\nu < \infty$. Let $\mathcal{V}$ be the set of such measures endowed with the weak convergence topology. For every $\nu \in \mathcal{V}$ the aggregate demand correspondence h^ν: $P \to \mathbb{R}^\ell$ is given by $h^\nu(p) = \int_{\mathcal{E}} h(\cdot, p) d\nu$. It is well defined.

The set of polynomial functions f: $\mathbb{R}^\ell \to \mathbb{R}$ shall be denoted by $\mathcal{P}$ and endowed with the topology of uniform convergence on compact sets; $\mathcal{P}^* = \{f \in \mathcal{P}$: $Df(x) \gg 0$ for every $x \in \overline{P}\}$. Let $\Phi : \mathcal{P}^* \to \mathcal{R}$ assign to every $f \in \mathcal{P}^*$ the preference relation which it represents. One has (see 1.18, 1.19 in Mas-Colell (1972)):

(1) Φ *is continuous.*

Let $\mathcal{P}_n$ (resp. $\mathcal{P}_n^*$) be the subspace of $\mathcal{P}$ (resp. $\mathcal{P}^*$) formed by the polynomials of degree less than or equal to $n > 0$. Obviously, by adapting some uniform notational convention, every $f \in \mathcal{P}$ can be identified with its (ordered) set of coefficients and so, if s_n designates the cardinality of the set of coefficients of a n-th degree polynomial, then, for every n, a continuous mapping $\pi_n : \mathbb{R}^{s_n} \to \mathcal{P}_n$ can be defined by assigning to every $a \in \mathcal{R}^{s_n}$ the n-th degree polynomial whose coefficients equal a. Let $\pi'_n : \mathbb{R}^{s_n} \times P \to \mathcal{P}_n \times P$, $\Phi' : \mathcal{P}_n^* \times P \to \mathcal{E}$ be given by, respectively, $\pi'_n(a, \omega) = (\pi_n(a), \omega), \Phi'(f, \omega) = (\Phi(f), \omega)$.

[1] This observation is motivated by remarks of Professor R. Radner.

[2] As usual, $x > 0$ means $x \neq 0$ and $x \geq 0$; $x \gg 0$ means $x^i > 0$ for every i.

[3] A preference relation R on P is a subset of $P \times P$ which is complete, reflexive, and transitive; R is monotone if $x - y \in P$, $x, y \in P$ implies $(y, x) \notin R$; R is continuous if it is closed (rel. to $P \times P$).

[4] The fact that we are taking the consumption set to be open is immaterial for the definition of this topology.

[5] We shall allow correspondences to be empty valued.

[6] If X is a topological space we denote by $\mathcal{B}(X)$ the Borel σ-field generated by the open subsets of X.

For every positive integer r let μ_r be a probability measure on $I\!R^r$ which is absolutely continuous with respect to Lebesgue measure and reciprocally.[7] Our results do not depend on the particular ones chosen. Define then a measure m on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ by $m = \sum_{n=1}^{\infty} \frac{1}{2^n} m_n$, $m_n = \mu_{s_n+\ell}(\Phi' \circ \pi'_n)^{-1}$. By the continuity of the functions involved all this is well defined.

PROPOSITION. *If $\nu \in \mathcal{V}$ is absolutely continuous with respect to m then, for every $p \in P$, $\#h^\nu(p) \leq 1$ (i.e., h^ν is a function in the domain $\{p \in P : h^\nu(p) \neq \phi\}$).*

Remarks.

1. It is easily checked that $m(\mathcal{E}) > 0$, hence the Proposition is nonvacuous.
2. *Mutatis mutandis*, the statement and proof of the Proposition can be adapted to cover the situation (appropriate to an uncertainty interpretation where preferences, but not income, are random) in which every agent has the same ω and the demand correspondence is defined for every p and ω.
3. Informally, the hypothesis of the Proposition can be divided into two parts; the first, which is very strong and difficult to interpret, says that the economy (i.e., ν) is fully concentrated on the set of agents whose preferences are representable by polynomial utility functions; the second, which is very natural, says that it is not the case that positive weight is given to a family of agents for which, for every n, the Lebesgue measure of the set of pairs of "endowments-coefficients of polynomial utility functions of the n-th degree" corresponding to agents in this family is zero.

2 Proof of the Proposition

We shall prove first a purely formal result and show then how it implies the Proposition.

For every $(p, w) \in P \times (0, \infty)$ define $\gamma(p, w) = \{x \in \overline{P}: px \leq w\}$. Let $f: \overline{P} \to I\!R$ be a C^2 strictly increasing[8] function. Define $g: \overline{P} \times I\!R^\ell \to I\!R$ by $g(x, a) = f(x) + ax$ and, for every $a \in I\!R^\ell, g_a: \overline{P} \to R$ by $g_a(x) = g(x, a)$. For every $p \in P$ form the open subset of $I\!R^{2\ell}$, $A(p) = \{(a, \omega) \in I\!R^\ell \times P: Dg_a(x) \gg 0$ for every $x \in \gamma(p, p\omega)\}$. For every $(p, a, \omega) \in P \times A(p)$, let $h'(p, a, \omega) = \{x \in P: x$ is a maximum of g_a on $\gamma(p, p\omega)\}$.

THEOREM. *For every $p \in P$ the set $C(p) = \{(a, \omega) \in A(p): \#h'(p, a, \omega) > 1\}$ has (2ℓ–dimensional) Lebesgue measure zero.*

Proof of the Theorem. Take a $p \in P$. From now on this p will be kept fixed. Define a correspondence $\hat{h}: A(p) \to P$ by $\hat{h}(a, \omega) = \{x \in P: x$ is a critical point of $g_a \mid Bdry\ \gamma(p, p\omega)\}$.[9]

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[7] That is to say, $\mu_r(A) = 0$ if and only if the Lebesgue measure of A is zero.

[8] In other words, $Df(x) \gg 0$ for every $x \in \overline{P}$.

[9] This is an "extended demand correspondence" in the same sense of the "extended equilibrium" of S. Smale (1972).

If $F: X \to Y$ is a correspondence, $G(F)(= \{(x,y) \in X \times Y,\ y \in F(x)\})$ denotes its graph. A C^1 function $F: S \to \mathbb{R}^r, S \subset \mathbb{R}^t, r \geq t$, is regular if, for every $x \in S$, $DF(x)$ has full rank.

(2) *There is a set* $J \subset A(p)$ *such that*

(i) J *has Lebesgue measure zero;*

(ii) *If* $(a,\omega) \in A(p) \sim J$ *and* $x \in \hat{h}(a,\omega)$, *then there are open neighborhoods of* (a,ω) *and* x, $B_1(a,\omega,x)$, $B_2(a,\omega,x)$, *respectively, and a* C^1 *function* $v_{a,\omega,x}$: $B_1(a,\omega,x) \to B_2(a,\omega,x)$ *such that* $G(v_{a,\omega,x}) = G(\hat{h}) \cap B_1(a,\omega,x) \times B_2(a,\omega,x)$.

Proof: Define the C^1 function Ψ: $A(p) \times P \times (0,\infty) \to \mathbb{R}^{\ell+1}$ by $\Psi(a,\omega,x,\lambda) = (Dg_a(x) - \lambda p, px - p\omega)$; for every $(a,\omega) \in A(p)$ let $\Psi_{a,\omega}$: $P \times (0,\infty) \to \mathbb{R}^{\ell+1}$ be given by $\Psi_{a,\omega}(x,\lambda) = \Psi(a,\omega,x,\lambda)$. Then $(a,\omega,x) \in G(\hat{h})$ if any only if, for some $\lambda > 0$, $(x,\lambda) \in \Psi_{a,\omega}^{-1}(0)$.

The map Ψ is regular since for every $(a,\omega,x,\lambda) \in A(p) \times P \times (0,\infty) \subset \mathbb{R}^{2\ell}$ the following submatrix of the Jacobian matrix of Ψ has full rank:

$$[D_{x^1}\Psi, D_{a^1}\Psi, \ldots, D_{a^n}\Psi] = \begin{bmatrix} D^2_{x^1x^1}g & 1 & 0 & \ldots & 0 \\ & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ D^2_{x^1x^\ell}g & 0 & 0 & \ldots & 1 \\ p^1 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Therefore $\Psi^{-1}(0)$ is a C^1 2ℓ-manifold and by Sard's Theorem (see S. Sternberg (1964, p. 47)) there is a set $J \subset A(p)$ of null measure such that if $(a,\omega) \notin J$ then zero is a regular value of $\Psi_{a,\omega}$; by the Implicit Function Theorem this yields the second conclusion of the lemma and finishes its proof.

Let $E = G(h' \mid A(p) \sim J)$. Then, by (2), $\{G(v_{a,\omega,x})$: $(a,\omega,x) \in E\}$ is an open covering of E (in the relative topology). Take a countable subcovering $\{G(v_{a_n,\omega_n,x_n})\}_{n=1}^{\infty}$ and call $v_n = v_{a_n,\omega_n,x_n}$, $U_n = B_1(a_n,\omega_n,x_n)$. Denoting the natural numbers by Z, let, for every $i \in Z$, $T_i = \{j \in Z$: $U_i \cap U_j \neq \phi$ and $v_i \mid U_i \cap U_j \neq v_j \mid U_i \cap U_j\}$. For every $i,j \in Z$ such that $j \in T_i$, define a C^1 function φ_{ij}: $U_i \cap U_j \to \mathbb{R}$ by $\varphi_{ij}(a,\omega) = g(v_i(a,\omega),a) - g(v_j(a,\omega),a)$.

(3) *For every* $i,j \in Z$ *such that* $j \in T_i, \varphi_{ij}^{-1}(0)$ *has Lebesgue measure zero.*

Proof: It is sufficient to show that φ_{ij} is a regular map, then $\varphi_{ij}^{-1}(0)$ will be a C^1 manifold of dimension $< 2\ell$ and, therefore, a null set. Let $(a,\omega) \in U_i \cap U_j$, $x_i = v_i(a,\omega)$, $x_j = v_j(a,\omega)$. Since $j \in T_i$, it follows, by construction and (2(*ii*)), that $x_i \neq x_j$. Let $x_i^1 \neq x_j^1$. For $k = i,j$ we have $D_{a^1}g(v_k(a,\omega),a) = D_x g(v_k(a,\omega),a) \circ D_{a^1}v_k(a,\omega) + x_k^1$, but since $pD_{a^1}v_k(a,\omega) = 0$ (because $pv_k(\cdot,\omega) = p\omega$) and $D_x g(v_k(a),a) = \lambda p$ for some $\lambda \in \mathbb{R}$, the first term of the sum vanishes so that $D_{a^1}g(v_k(a,\omega),a) = x_k^1$ $(k = i,j)$ and, therefore, $D_{a^1}\phi_{ij}(a,\omega) \neq 0$. This ends the proof of (3).

The proof of the theorem is now completed: let $(a,\omega) \in A(p) \sim J$ and $x', x'' \in h'(p,a,\omega), x' \neq x''$; of course, $g_{a,\omega}(x') = g_{a,\omega}(x'')$. For some $i,j \in Z$, $(a,\omega,x') \in G(v_i)$, $(a,\omega,x'') \in G(v_j)$ which implies that $j \in T_i$ and $(a,\omega) \in \phi_{ij}^{-1}(0)$. Therefore $C(p) \subset J \cup \bigcup_{i,j\in Z, j\in T_i} \phi_{ij}^{-1}(0)$, the last set being null by (2) and (3).

The Proposition follows immediately from the Theorem: take a $p \in P$ and let $\mathcal{E}' = \{(R,\omega) \in \mathcal{E} : \#h(R,\omega,p) > 1\}$; it is not hard to see that $\mathcal{E}'$ is an $\mathcal{F}_\sigma$ set, hence measurable. Therefore, by Fubini's Theorem and the Theorem above, $\mu_{s_n+\ell}(\phi' \circ \pi_n')^{-1}(\mathcal{E}') = 0$ for every n. Thus, $m(\mathcal{E}') = 0$ and so $\nu(\mathcal{E}') = 0$. **Q.E.D.**

3 Final Comments and Extensions

The result in this paper can be straightforwardly extended in a variety of directions, but the two lines of further research which we regard as more interesting do not seem quite so simple. They are suggested by the following two comments on the limitations of the Proposition.

(i) The hypotheses of the Proposition are far from guaranteeing that the aggregate demand function (if it exists) be differentiable or even Lipschitzian. Clearly, this limits the appeal of the result, since with the much weaker assumption of atomlessness of the economy one already has that the aggregate demand correspondence is convex-valued.

(ii) No characterization independent of the choice of utility function is available for the class of preferences representable by polynomial utility. This is troublesome because it makes the results difficult to interpret. The class of preferences representable by analytic utility functions can be easily and naturally described without reference to utility. One could perhaps surmise that some analog of the Proposition (i.e., some consequence of the Theorem in Part 2) holds for economies fully concentrated in the set of agents with preferences in this class.

We finish by mentioning, in an informal manner, an application of the Theorem in Part 2 to approximation problems. Consider the following elementary proposition: "...every finite (pure exchange) economy whose agents have convex preferences can be approximated by economies yielding continuous excess demand functions." Theorem 2 in (Mas-Colell, 1972) (see, also, G. Debreu (1972, p. 613)) implies that the Proposition can be improved by substituting C^1 for continuous. By applying the Theorem in 2 one obtains a different extension, namely, if approximation is understood in the sense of weak convergence for measures (and the word "finite" is dropped) then the above statement remains true without the convexity of preferences assumption. A somewhat more significant question remains open: can every pure exchange economy (nonconvex preferences allowed) be approximated (in the sense of weak convergence for measures) by economies yielding C^1 excess demand functions?

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