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EGALITARIAN SOLUTIONS OF LARGE GAMES: II. THE ASYMPTOTIC APPROACH

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This is the second of two papers devoted to the study of egalitarian solutions for nontransferable utility (NTU) games with a large number of players. This paper is concerned with the egalitarian solutions of finite games as the number of players increases. We show that these converge to the egalitarian solution of the limit game with a continuum of players as defined in our previous paper. The same convergence holds for the underlying potential functions. These asymptotic results are particularly significant since they provide the definitive justification for our definitions in the limit continuum case.

I. Introduction. This is the second of two papers devoted to the study of egalitarian solutions of large games. We refer to the introduction of our first paper Hart and Mas-Colell (1995), *to be referred from now on as* [HM, I], for the motivation and the laying out of the general program of research.

In the first paper the set of players was modelled as a continuum. This paper deals with the asymptotic approach. Namely, we consider a sequence of finite games converging to a game with a continuum of players. We show that the finite egalitarian solutions converge to the continuum egalitarian solutions, as defined and studied in [HM, I]. Moreover, the underlying potential functions—a fundamental technical tool associated with the egalitarian solutions—converge as well to the continuum potential (again, as defined and studied in the previous paper).

These results are significant, first, in establishing a very basic convergence fact that may be viewed as a generalization to the NTU-case of the classical asymptotic results for the Shapley TU-value (e.g., see Aumann-Shapley, 1974). Second, and most important, the results lend support to our previous paper to the extent that they show that our choice for the potential function in the continuum case was indeed the right one. In addition to all the other justifications in [HM, I], we now have the validation of the finite games' approximations.

The continuum potential is related to partial differential equations (PDEs). It turns out to be the "viscosity solution" to a Hamilton-Jacobi equation (see, e.g., Fleming, 1969, 1986; and Lions, 1982). It is interesting to note that asymptotic results such as ours have been proved in the PDE literature. These results deal, essentially, with the convergence of the PDE solutions, i.e., of the potentials (see, e.g., Crandall-Lions, 1984; and Souganidis, 1985). However, we obtain here a further result: the convergence of the *derivatives* of the potentials (which are precisely the egalitarian solutions and constitute our object of central interest). It is conceivable that our method of proof for this, based on probabilistic considerations, may be of more general interest.

The paper is organized as follows: §II contains the basic model and assumptions. Section III presents the potentials and the egalitarian solutions for the finite games as

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well as the limit continuum game. The main result is stated in §IV, and proved in the remaining §§V–VII.

Some standard notations: e^i stands for the ith unit vector, while e denotes $(1,\dots,1)$. For $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)$ in R^n, we let $\|x\| := \sum_{i=1}^n |x_i|$, $x\cdot y := \sum_{i=1}^n x_i y_i$, $x*y := (x_1y_1, x_2y_2,\dots,x_ny_n)\in R^n$ and $x/y := (x_1/y_1, x_2/y_2,\dots,x_n/y_n)\in R^n$ (the latter defined when $y_i\neq 0$ for all i). For a finite set S and $i\in S$ we write $S\setminus i$ for $S\setminus\{i\}$. For a set $A\subset R^n$, we denote by ∂A, Int A and co A its boundary, interior and convex hull, respectively. Finally, $|N|$ is the number of elements of the finite set N.

II. The basic model: Description and assumptions.

The basic framework is that of nontransferable utility (NTU) games with finitely many, say n, types of players. We consider a sequence of such finite games together with the (limit) continuum game.

Formally, regard $x=(x_1,x_2,\dots,x_n)\in R^n_+$ as a vector of total masses of each of the n types: a *profile*. Let $\bar{x}\gg 0$ be the profile of the grand coalition; i.e., $\bar{x}_i$ is the total mass of type i. It is convenient to assume that $\bar{x}_i$ is a positive integer for all i (for instance, $\bar{x}=(1,\dots,1)$).

The *continuum game* is given, as in §II of [HM, I], by a point-to-set map[1] $V: R^n_+ \to R^n$. The interpretation is that each $a\in V(x)$ represents a feasible allocation for a coalition of profile x, in *per-capita* terms; i.e., each one of the x_i players of type i gets a payoff a_i. Thus, we only consider type-symmetric allocations. When some type i is missing, i.e., $x_i=0$, we will use the standard convention of *not* restricting the corresponding payoff coordinates[2] (thus, $x_i=0$ and $a\in V(x)$ imply $a'\in V(x)$ whenever $a_j=a'_j$ for all $j\neq i$). In the notations of [HM, I], the continuum game is $(\bar{x},V)$; the map V is called a "game form" or "characteristic function."

The *sequence of finite games* is indexed by the positive integer $r=1,2,\dots,$ and it is obtained by regarding each player as having mass $1/r$. The rth game (N_r,V_r) is as follows: the set of players N_r consists of $r\bar{x}_i$ players[3] of type i (hence $|N_r|=\sum_i r\bar{x}_i = r\|\bar{x}\|$). For a coalition $S\subset N_r$, let $z_S\in R^n_+$ denote its *profile*: S contains rz^i_S players of type i (thus, the profile of N_r is $\bar{x}$). Then *the set of type-symmetric feasible payoff vectors of* S is exactly $V_r(S) := V(z_S)$. (As we shall see below, we need not consider the non-type-symmetric allocations.) Let L_r denote the $1/r$*-non-negative lattice*, i.e., $L_r := \{z\in R^n_+ : rz_i$ is an integer for all $i\}$; then V_r depends on the values of V on L_r. We note here that one could well have more general sequences. For instance, by allowing different r_i's for different types. We have chosen the simpler case since it suffices to support our claims, and the technical details are already complex enough.

The assumptions on V are essentially the same as in [HM, I]. The only addition is that the most basic ones, which we now denote (A.1)*, (A.2)*, (A.3)* and (A.3w)*, are assumed for *all* profiles in R^n_+ (rather than R^n_{++} only); note, however, that (A.4)–(A.9) still apply to R^n_{++} only. We list them now; for discussion and interpretation, the reader is referred to §II in [HM, I].

(A.1)*. *For every $x\in R^n_+$, the set $V(x)$ is closed, convex and comprehensive; moreover, $V(x)\neq\emptyset$ and $\neq R^n$. Also, if $x_i=0$ and $a\in V(x)$, then $a'\in V(x)$ for any a' with $a'_j=a_j$ for all $j\neq i$.*

(A.2)*. *There exists a $\theta>0$ such that $\theta e=(\theta,\theta,\dots,\theta)\in V(x)$ for all $x\in R^n_+$.*

Let $\hat{V}(x) := \{b\in R^n : b\le x*a$ for some $a\in V(x)\}$, and $\hat{V}_0(x) := \hat{V}(x)\cap R^n_+$.

(A.3)*. *For all $x, x'\in R^n_+$, we have $\hat{V}_0(x)+\hat{V}_0(x')\subset \hat{V}_0(x+x')$.*

[1] Note that here V is defined on R^n_+, not only on R^n_{++} as in [HM, I] (more on this below).

[2] This is just a matter of technical convenience; it has no bearing on the results.

[3] Recall that $\bar{x}_i$ was assumed to be an integer.

As in [HM, I], a weaker assumption actually suffices:

(A.3w)*. *For all* $x, x' \in R^n_+$, *we have* $\hat{V}_0(x) + \{x' * \theta e\} \subset \hat{V}_0(x + x')$.

Recall that the assumptions below refer to R^n_{++} only. For any $x \in R^n_{++}$ and $p, q \in R^n_+$ recall also the definition of the support functions: $v(x, p) := \sup\{p \cdot a : a \in V(x)\}$ and $\hat{v}(x, q) := \sup\{q \cdot b : b \in \hat{V}(x)\}$. Of course, v and $\hat{v}$ may well take the value $+\infty$. We thus define their effective domains $D := \{(x, p) : v(x, p) < \infty\}$ and $\hat{D} := \{(x, q) : \hat{v}(x, q) < \infty\}$. Finally, let $\hat{D}^+ := \{(x, q) \in \hat{D} : \hat{v}(x, q) = q \cdot b$ for some $b \in \hat{V}_0(x)\} \subset R^n_{++} \times R^n_+$.

(A.4) *There is a compact set* $C \subset \{q \in R^n_{++} : \Sigma_i q_i = 1\}$ *such that* $\hat{D}^+ \subset R^n_{++} \times$ Cone C, *where* Cone $C := \{\beta q : q \in C, \beta \geq 0\}$.

(A.5) *There exists a constant* $K < \infty$ *such that* $\|b\| \leq K\|x\|$ *for all* x *and all* $b \in \hat{V}_0(x)$.

Define $\hat{v}_0 : R^n_{++} \times C \to R$ by $\hat{v}_0(x, q) := \text{Max}\{q \cdot b : b \in \hat{V}_0(x)\}$. Of course, $\hat{v}_0(x, q) = \hat{v}(x, q)$ for any $(x, q) \in \hat{D}^+$.

(A.6) *The support function* $\hat{v}_0(x, q)$ *is uniformly Lipschitz on any bounded (not necessarily compact) subset of* $R^n_{++} \times$ Cone C; *i.e., for any* $\beta < \infty$ *there is* $K < \infty$ *such that* $|\hat{v}_0(x, q) - \hat{v}_0(x', q')| \leq K\|(x, q) - (x', q')\|$ *for all* $x, x' \in R^n_{++}$ *with* $\|x\|, \|x'\| \leq \beta$ *and all* $q, q' \in C$.

(A.7) *For every* $x \in R^n_{++}$ *the support function* $\hat{v}(x, q)$ (*or, equivalently,* $v(x, p)$) *is strictly subadditive with respect to* q, *that is* $\hat{v}(x, q) + \hat{v}(x, q') > \hat{v}(x, q + q')$ *whenever* q, q' *are not collinear.*

(A.8) *The gradient* $\nabla_x \hat{v}_0(x, q)$ *exists for all* $(x, q) \in R^n_{++} \times$ Cone C. *Moreover*:

(a) *It is uniformly bounded on any bounded (not necessarily compact) subset of its domain* $R^n_{++} \times$ Cone C; *i.e., for every* $\beta < \infty$ *there is* $K < \infty$ *such that* $\|\nabla_x \hat{v}_0(x, q)\| \leq K$ *for all* $x \in R^n_{++}$ *with* $\|x\| \leq \beta$ *and all* $q \in C$.

(b) *It is uniformly Lipschitz on any bounded (not necessarily compact) subset of its domain which is bounded away from the origin of* R^n_{++}; *i.e., for every* $0 < \rho \leq \beta < \infty$ *there is* $K < \infty$ *such that* $\|\nabla_x \hat{v}_0(x, q) - \nabla_x \hat{v}_0(x', q')\| \leq K\|(x, q) - (x', q')\|$ *for all* $x, x' \in R^n_{++}$ *with* $\rho \leq \|x\|, \|x'\| \leq \beta$ *and all* $q, q' \in C$.

(A.9) *The domain* $\hat{D}^+$ *has nonempty interior and the function* $\hat{v}(x, q)$ *is* C^2 *on* $\hat{D}^+$. *Moreover,* $\nabla_{qq} \hat{v}(x, q)$ *has full possible rank* $n - 1$, *and its minimal nonzero eigenvalue is positive and bounded away from zero in any bounded (not necessarily compact) subset of* $\hat{D}^+$ *that is bounded away from the origin of* R^n_{++}, *i.e., on any set of the form* $\hat{D}^+ \cap (\{x \in R^n_{++} : \rho \leq \|x\| \leq \beta\} \times C)$ *for some* $0 < \rho \leq \beta < \infty$.

From now on we will always assume (A.1)*–(A.3)* and (A.4)–(A.9). Note that any V defined on R^n_{++} and satisfying (A.1)–(A.9) (as in [HM, I]) may be extended to R^n_+ so that (A.1)*–(A.3)* hold (for example, take $V(x) = \{\theta e\} - R^n_+$ for all $x \in \partial R^n_+$). The convergence results, as we shall see, do not depend on which particular extension is chosen for the nonstrictly positive profiles (as long, of course, as (A.1)*–(A.3)* hold). Indeed, the main theorem shows that the limit depends only on the restriction of V to R^n_{++}. Finally, recall from [HM, I] that one can easily construct examples—in particular, economic ones—that satisfy all the assumptions here.

III. Egalitarian solutions and potentials. Let (M, W) be a finite NTU-game: M is a finite set and, for all $S \subset M$, the set[4] $W(S) \subset R^S$ is closed, convex, comprehensive, $\neq \emptyset$ and $\neq R^S$. The egalitarian solution was introduced by Shapley (1953) for the TU-case (it is the classical Shapley TU-value), generalized to NTU games by Harsanyi (1963) as the first step in the construction of his NTU-value, and studied by Kalai-Samet (1985) as a solution on its own right.

[4]One may work indistinctly with $W'(S) = W(S) \times R^{M \setminus S} \subset R^M$, as we do in the continuum case (see (A.1)*).

Formally, as in Hart and Mas-Colell (1989), the *egalitarian solution* $\mathrm{Eg}(M, W)$ may be defined as follows: Let $P : 2^M \to R$ be a real function satisfying[5] $P(\emptyset) = 0$ and $DP(S) \in \partial W(S)$ for all $S \subset M$, where $DP(S)$ denotes the vector of marginal contributions $DP(S) = (D_i P(S))_{i \in S} := (P(S) - P(S \setminus i))_{i \in S}$. By Theorem 6.2 in Hart and Mas-Colell (1989), there exists exactly one such P, called *the potential function*; moreover, the resulting payoff vector $DP(M)$ of the grand coalition M is the *egalitarian solution* of the game (M, W). Thus, the requirement that the marginal contributions be always efficient uniquely determines the potential and, *a fortiori*, the egalitarian solution.

How does this translate into our setup? Fix r, and consider the finite game (N_r, V_r). It is straightforward to check that the potential, being unique, must preserve all symmetries of the game; *a fortiori*, so will the egalitarian solutions. Therefore, the potential of a coalition depends only on its profile, and we define the *potential* of (N_r, V_r) as a real function[6] $P_r : L_r \to R$ such that $P_r(0) = 0$ and, for all $x \in L_r$, $x \leq \bar{x}$,

$$DP_r(x) = \left(D_i P_r(x)\right)_{i=1}^n := \left(P_r(x) - P_r(x^{(i)})\right)_{i=1}^n \in \partial V(x),$$

where $x^{(i)} := (x_1, \ldots, x_{i-1}, x_i - 1/r, x_{i+1}, \ldots, x_n) = x - (1/r)e^i$ if $x_i \geq 1/r$ and $x^{(i)} := x$ otherwise[7]. Equivalently, we may write $x * DP_r(x) \in \partial \hat{V}(x)$. Thus, the payoff of each player of type i in the coalition (of profile) x equals his marginal contribution to the potential (dropping a player of type i yields a coalition of profile $x^{(i)}$). The potential P_r for the game (N_r, V_r) is uniquely determined by the above condition.

We consider next the limit, i.e., the continuum game $(\bar{x}, V)$. Our previous paper [HM, I] is devoted to the study of the potential and the egalitarian solution for such a game. The basic result there is the following.

THEOREM A (HART AND MAS-COLELL, 1995). *For every $x \in R^n_{++}$ (note: R^n_{++}, not R^n_+), let $\Gamma_0(x)$ be the set of absolutely continuous paths $\mathbf{x} : [0, 1] \to R^n_{++} \cup \{0\}$ such that $\mathbf{x}(0) = 0$ and $\mathbf{x}(1) = x$. Define*

$$P(x) := \min_{\mathbf{x} \in \Gamma_0(x)} \int_0^1 v(\mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt.$$

Then the function P is well defined, it is a Lipschitz function, and its gradient $\nabla P(x)$ satisfies $\nabla P(x) \in \partial V(x)$ whenever P is differentiable at x (hence, almost everywhere).

This function P is called the (*variational*) *potential*. If P is differentiable at x, then the *egalitarian solution* of the game (x, V) is defined as the gradient $\nabla P(x)$ of P at x. If P is not differentiable at x—which happens on a negligible set—we have not yet formally defined the corresponding egalitarian solution. The continuum approach of [HM, I] does not lead to a precise candidate. Settling on the right definition is however essential for some applications—most notably, for the Harsanyi NTU-value, where one cannot ignore the nondifferentiability points (see Hart and Mas-Colell (1996)). The asymptotic approach of the present paper provides the correct answer, which we now present.

We define formally for every $x \in R^n_{++}$ the *egalitarian solution* $\mathrm{Eg}(x, V)$ *of the game* (x, V) as

$$\mathrm{Eg}(x, V) := (\nabla^c P(x) + R^n_+) \cap \partial V(x),$$

[5] $P(S)$ here is denoted $P(S, W)$ in Hart and Mas-Colell (1989).

[6] Recall that each player has mass $1/r$ and L_r is the $1/r$-lattice in R^n_+.

[7] Note that when $x_i = 0$ there are no players of type i, thus the ith marginal contribution is irrelevant; we put $x^{(i)} = x$ for convenience only.

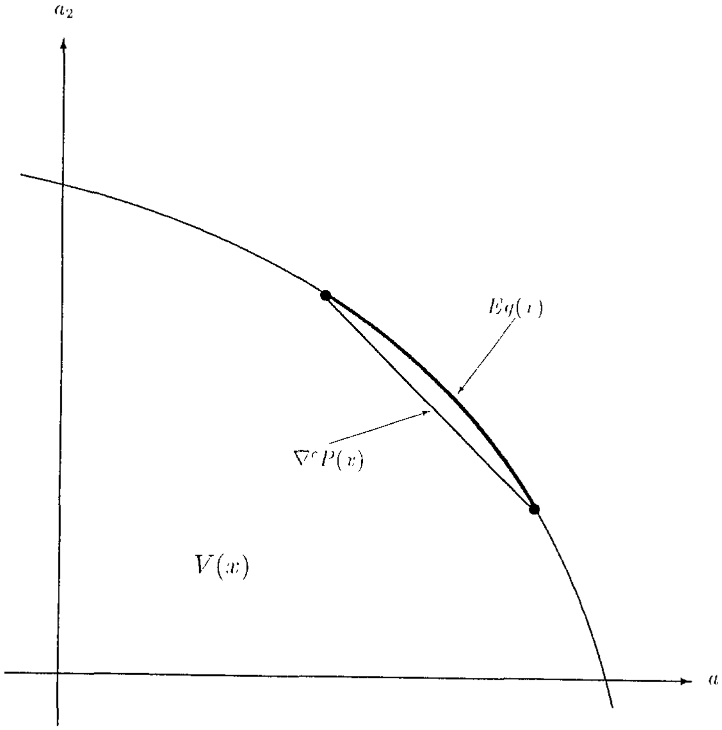

FIGURE III.1.

where $\nabla^c P(x) := \text{co}\{\lim_m \nabla P(x_m): x_m \to x \text{ and } P \text{ is differentiable at } x_m\}$ is Clarke's (1983) generalized gradient of P at x. When P is differentiable at x, the generalized gradient at x reduces to the gradient at x (recall Corollary VII.4.3 in [HM, I]), and thus the egalitarian solution $\text{Eg}(x, V)$ of the game (x, V) is just the point $\nabla P(x)$.[8] In general (see Figure III.1), the set $\text{Eg}(x, V)$ consists of all payoff vectors $a \in R^n$ that are efficient for x (i.e., $a \in \partial V(x)$), and are greater than or equal to, in all coordinates, some generalized gradient of P at x. Note that an extreme generalized gradient lies on $\partial V(x)$, but in general the convex hull $\nabla^c P(x)$ contains interior points (unless $\partial V(x)$ is flat in that region—such as TU or hyperplane games, whose boundary is flat everywhere).

This definition should be easy to understand. The egalitarian solution at a nondifferentiability point of the potential P is set-valued, it is related to the egalitarian solutions—i.e., to the gradients of the potential P—at nearby points where P is differentiable, and it guarantees efficiency.

IV. Statement of the main result. We can now state the main result of this paper.

MAIN THEOREM. *Assume* (A.1)*, (A.2)*, (A.3)* (*or* (A.3w)*) *and* (A.4)–(A.9). *Then, for any* $\bar{x} \gg 0$:

(a) $\lim_{r \to \infty} (1/r) P_r(\bar{x}) = P(\bar{x})$.

(b) *If P is differentiable at* $\bar{x}$, *then* $\lim_{r \to \infty} DP_r(\bar{x}) = \nabla P(\bar{x})$ $(= \text{Eg}(\bar{x}, V))$.

(c) $\lim_{r \to \infty} \text{dist}(DP_r(\bar{x}), \text{Eg}(\bar{x}, V)) = 0$.

[8] We will ignore the distinction between the point $\nabla P(x)$ and the one-element set $\{\nabla P(x)\}$. No confusion should arise.

We write "dist" for the *distance* between a point and a set (i.e., $\text{dist}(y, Z) := \inf\{\|y - z\| : z \in Z\}$). If P is differentiable at $\bar{x}$, then, by (b), the sequence $DP_r(\bar{x})$ converges to $\nabla P(\bar{x})$, which is $\text{Eg}(\bar{x}, V)$. If P is not differentiable at $\bar{x}$, then we do *not* claim that the sequence converges. But we show (this is (c)) that any limit point must be in $\text{Eg}(\bar{x}, V)$.

The proof of the Main Theorem will be divided into three parts (following some preliminaries in §V): "$\limsup(1/r)P_r(\bar{x}) \leq P(\bar{x})$" in §VI, "$\liminf(1/r)P_r(\bar{x}) \geq P(\bar{x})$" in §VII (together yielding (a)), and finally "$\lim \text{dist}(DP_r(\bar{x}), \text{Eg}(\bar{x}, V)) = 0$" in §VIII (which yields (b) and (c)).

As a complement to part (c), note that the example in §VIII of [HM, I] of a nondifferentiable variational potential shows also that $DP_r(\bar{x})$ may well converge to a point on $\partial V(\bar{x})$ which is *not* a generalized gradient of P at $\bar{x}$. Indeed, for $\bar{x} = (1, 1)$, the egalitarian solutions $DP_r(\bar{x})$ of the finite approximations equal $(1/3, 1/3)$ for all r (by symmetry and efficiency), which does not belong to $\nabla^c P(\bar{x})$ (see Figure VIII.3 in [HM, I]).

V. Preliminaries. In this section we will use the positivity property (in game theoretic terms, "individual rationality") of the egalitarian solutions (for finite games as well as for the continuum game) to obtain the "localization to the positive orthant": namely, the property that only $V_0(x) = V(x) \cap R^n_+$ really matters.

We deal first with finite games. Let thus (M, W) be a finite NTU-game, as in §III. The following lemma shows that if the marginal contribution of a player i to every coalition is always at least ξ, then the egalitarian solution will give him at least ξ.

LEMMA V.1. *Let (M, W) be a finite NTU-game, and $i \in M$ a player. Assume there exists a scalar ξ such that $W(S) \supset W(S \setminus i) \times (-\infty, \xi]$ for all coalitions $S \subset M$ containing i. Then $D_i P(S) \geq \xi$ for all $S \subset M$, $S \ni i$.*

PROOF. By induction. Assume that $D_i P(S \setminus j) \geq \xi$ for all $j \in S$, $j \neq i$ (this holds vacuously if $S = \{i\}$). Define $a_j := P(S \setminus i) + \xi - P(S \setminus j)$, and note that $a_i = \xi$ and $a_j = D_j P(S \setminus i) + \xi - D_i P(S \setminus j) \leq D_j P(S \setminus i)$ for $j \neq i$ (by the induction hypothesis). Thus, $(a_j)_{j \in S \setminus i} \leq DP(S \setminus i) \in \partial W(S \setminus i)$, hence $(a_j)_{j \in S \setminus i} \in W(S \setminus i)$ by comprehensiveness. The assumption on player i's contribution yields $a = (a_j)_{j \in S} \in W(S)$. Now $D_j P(S) - a_j = P(S) - P(S \setminus i) - \xi = D_i P(S) - \xi$ for all $j \in S$; if $D_i P(S) < \xi$, then $DP(S) \ll a \in W(S)$, contradicting $DP(S) \in \partial W(S)$. □

This lemma implies that in a so-called "0-monotonic," or "individually superadditive," game (i.e., when $W(S) \supset W(S \setminus i) \times W(i)$ for all $i \in S \subset M$), the egalitarian solutions are "individually rational" (i.e., $D_i P(S) \geq \xi_i$ for all $i \in S \subset M$, where $W(i) = (-\infty, \xi_i]$).

Applying Lemma V.1 to the game (N_r, V_r) with $\xi = \theta$ (recall (A.3w)*) yields

COROLLARY V.2. *For every $r = 1, 2, \ldots$ and every $x \in L_r$ we have $x^* DP_r(x) \geq \theta x$.*

(NOTE. We have multiplied both sides by x_i so as to include the case $x_i = 0$).

We state now our "localization to the positive orthant" result. It is immediate since $\theta > 0$.

COROLLARY V.3. *Let V and V' satisfy* (A.1)*–(A.3)* *and* (A.4)–(A.9). *If $V_0(x) = V'_0(x)$ for all $x \in R^n_+$, $x \leq \bar{x}$, then*[9] *$P_r(x) = P'_r(x)$ for all $x \in L_r$.*

A similar result holds for the continuum potential: see §V.6 in [HM, I], in particular Proposition V.6.5, which we now state.

[9]All notations with ′ refer, of course, to V', e.g., P'_r is the potential of (N_r, V'_r), where V'_r is derived from V'.

PROPOSITION V.4. *Let V and V' satisfy* (A.1)–(A.9). *If $V_0(x) = V'_0(x)$ for all $x \in R^n_{++}$, then $P(x) = P'(x)$ for all $x \in R^n_{++}$.*

Corollary V.3 and Proposition V.4 allow us to modify $V(x)$ outside the nonnegative orthant. Furthermore, in [HM, I] (Proposition V.6.5(c)) it is shown that this can be done so as to get a V' that satisfies some slightly stronger and more convenient assumptions. Formally,

PROPOSITION V.5. *Assume that V satisfies* (A.1)*–(A.3)* *and* (A.4)–(A.9). *Then there exists a V' satisfying the same assumptions, and such that, in addition*:

(i) *$V'_0(x) = V_0(x)$ for all $x \in R^n_+$;*

(ii) *$\hat{D}' = \hat{D}'^{+} = R^n_{++} \times \text{Cone}\, C$;*

(iii) *$\hat{v}'(x,q) = \hat{v}'_0(x,q) = \hat{v}_0(x,q)$ for all $(x,q) \in R^n_{++} \times \text{Cone}\, C$; and $\hat{v}'(x,q) = \hat{v}(x,q)$ for all $(x,q) \in \hat{D}^+$.*

PROOF. For $x \in \partial R^n_+$ we make no change: $V'(x) := V(x)$. For $x \in R^n_{++}$ use Proposition V.6.5(c) in [HM, I]. □

Thus, neither P nor the P_r's change as we replace V by V' (in view of Corollary V.3 and Proposition V.4). Without loss of generality, we will therefore assume from now on that (ii) and (iii) are also satisfied by our original game form V. This implies in particular that we may replace $\hat{v}_0$ by $\hat{v}$ in Assumptions (A.6) and (A.8).

VI. Proof of "lim sup convergence". This section is devoted to the proof of

PROPOSITION VI.1.

$$\limsup_{r \to \infty} \frac{1}{r} P_r(\bar{x}) \le P(\bar{x}).$$

The idea of the proof is as follows: using the continuum potential function P, one constructs a superpotential function G_r for the finite game (N_r, V_r), such that $(1/r)G_r$ is close to P for r large enough. But a superpotential always majorizes the potential P_r, completing the proof.

Let us first define formally the notion of a superpotential for finite games (see §V.7 of [HM, I] for the continuum case).

DEFINITION. A function $G: 2^M \to R$ is a *superpotential* for the (finite NTU) game (M, W) if $G(\emptyset) = 0$ and $DG(S) \notin \text{Int}\, W(S)$ for all $S \subset M$, (where, as usual, $DG(S) := (G(S) - G(S \setminus i))_{i \in S} \in R^S$).

We have the following (see also Proposition V.7.3 in [HM, I]):

LEMMA VI.2. *Let G be a superpotential for the game (M, W), and let P be its (unique) potential. Then $G(S) \ge P(S)$ for all $S \subset M$.*

PROOF. By induction. If $G(S \setminus i) \ge P(S \setminus i)$ for all $i \in S$ but $G(S) < P(S)$, then $DG(S) \ll DP(S)$, a contradiction since $DG(S) \notin \text{Int}\, W(S)$ and $DP(S) \in \partial W(S)$. □

Recall that the continuum potential function P is uniformly Lipschitz on any bounded subset of R^n_{++} (Proposition V.2 in [HM, I]). Therefore, it has a unique continuous extension to the boundary of R^n_+, which we will also denote by P.

Fix $\varepsilon > 0$. Denote by K the constant of (A.5), and define $G_r : R^n_+ \to R$ by

$$G_r(x) := rP(x) + r\varepsilon\|x\| + rK\,\text{Min}\{\|x\|, \varepsilon\},$$

for all $x \in R^n_+$ (note that G_r is defined on R^n_+, not R^n_{++}). Let $\alpha > 0$ be such that $q \geq \alpha e$ for all $q \in C$ (recall (A.4)), and δ be given by Lemma VII.5.2 in [HM, I] applied with $\|\bar{x}\| + 1$ as β there, ε as ρ, and $\alpha\varepsilon^2$ as ε.

LEMMA VI.3. *For all r large enough, G_r is a (type-symmetric) superpotential for (N_r, V_r): i.e., $DG_r(x) \notin \operatorname{Int} V(x)$ for all $x \in L_r$, $x \leq \bar{x}$.*

PROOF. Fix $r > 1/\delta$. Let $x \in R^n_+$ be such that $x \leq \bar{x}$. For $x = 0$ we have $G_r(0) = 0$.

Consider first the case where $x \gg 0$, $\|x\| \geq \varepsilon$ and $x_i \geq 1/r$ (note that x is not assumed to be in L_r). We have $\|x\| - \|x^{(i)}\| = 1/r$, therefore

$$D_i G_r(x) = G_r(x) - G_r(x^{(i)}) \geq r\left[P(x) - P(x^{(i)})\right] + \varepsilon.$$

Recall the function $Q_x : R^n_{++} \to R$ of §VII.5 in [HM, I]. By Proposition V.2 and Lemma VII.5.1 there, it is continuously differentiable, $Q_x(x) = P(x)$, $Q_x(y) \geq P(y)$ for all y, and it can be also extended to the boundary of R^n_+. Therefore

$$D_i G_r(x) \geq r\left[Q_x(x) - Q_x(x^{(i)})\right] + \varepsilon = \frac{\partial}{\partial x_i} Q_x\left(x - \frac{\xi}{r} e^i\right) + \varepsilon,$$

for some $0 \leq \xi \leq 1$, by the mean-value theorem. Since $\xi/r \leq 1/r < \delta$ we obtain from Lemma VII.5.2 in [HM, I] and our choice of δ,

$$x_i D_i G_r(x) \geq x_i \frac{\partial}{\partial x_i} Q_x(x) - \alpha\varepsilon^2 + \varepsilon x_i.$$

Also $x * \nabla Q_x(x) \in \partial \hat{V}(x)$ by Lemma VII.5.1 in [HM, I]. Let $q \in C$ be such that $q \cdot (x * \nabla Q_x(x)) \geq \hat{v}(x, q)$. Since $q \cdot x \geq \alpha e \cdot x = \alpha\|x\| \geq \alpha\varepsilon$, we have $q \cdot (x * DG_r(x)) \geq q \cdot (x * \nabla Q_x(x)) - \alpha\varepsilon^2 + \varepsilon q \cdot x \geq q \cdot (x * \nabla Q_x(x)) \geq \hat{v}(x, q)$, finally implying $x * DG_r(x) \notin \operatorname{Int} \hat{V}(x)$.

Next consider the case where $x \in \partial R^n_+ \cap L_r$ and $\|x\| \geq \varepsilon$. Define $x^m := x + (1/m)e \gg 0$ for $m = 1, 2, \ldots$. For all m large enough we have

$$x^m * DG_r(x^m) \geq x^m * \nabla Q_{x^m}(x^m) - (\alpha\varepsilon^2)e + \varepsilon x^m.$$

Indeed, for coordinates i with $x_i \geq 1/r$ this has already been proved (apply the argument above to x^m), and for i with $x_i = 0$ it follows from $x_i^m \to 0$ and the uniform boundedness of $(\partial/\partial x_i)Q_{x^m}$ (recall Proposition V.2 in [HM, I]). As above, since $\|x^m\| \geq \varepsilon$, we conclude that $x^m * DG_r(x^m) \notin \operatorname{Int} \hat{V}(x^m)$. The super-additivity assumption (A.3)* (or, (A.3w)*) implies $\hat{V}(x) \subset \hat{V}(x^m)$ for all m. Also,

$$r\left[P(x^m) - P\left((x^m)^{(i)}\right)\right] \underset{m \to \infty}{\to} r\left[P(x) - P(x^{(i)})\right]$$

(recall that P has been extended continuously to the boundary). Hence, in the limit, $x * DG_r(x) \notin \operatorname{Int} \hat{V}(x)$ obtains for all $x \in L_r$ with $\|x\| \geq \varepsilon$.

Finally, for $\|x\| < \varepsilon$ we have $x * DG_r(x) \geq Kx$ (this comes from the third term of G_r), therefore $x * DG_r(x) \notin \operatorname{Int} \hat{V}(x)$ by (A.5). □

PROOF OF PROPOSITION VI.1. By Lemmas VI.2, VI.3 and the definition of G_r,

$$P_r(x) \leq G_r(x) \leq r(P(x) + \varepsilon\|x\| + K\varepsilon)$$

for all $x \in L_r$, $x \leq \bar{x}$, in particular for $x = \bar{x}$. Dividing by r and recalling that the inequality holds for all $\varepsilon > 0$ yields the result. □

VII. Proof of "lim inf convergence." This section is devoted to the proof of the second half of the convergence result $(1/r)P_r \to P$, namely

PROPOSITION VII.1.

$$\liminf_{r\to\infty} \frac{1}{r} P_r(\bar{x}) \geq P(\bar{x}).$$

One way to prove this is in parallel to the previous section: using the continuum potential P one constructs *sub*potentials for the finite games. We will, however, bring here a different proof, using probabilistic arguments. It is more complex, but it has the definite advantage that refining it will also yield the convergence of the derivatives (which is what we are ultimately interested in).

VII.1. The stochastic process $\{Z^m\}$. *Fix r.* For every $x \in L_r \setminus \{0\}$, let $p \in R^n_+$ be a supporting normal to $\partial V(x)$ at $DP_r(x)$, i.e., $p \cdot DP_r(x) = v(x, p)$. Normalize p to obtain a $\pi(x) \in \Delta^{n-1} := \{p \in R^n_+ : \Sigma_i p_i = 1\}$, the $(n-1)$-dimensional simplex in R^n, such that $\pi(x) \cdot DP_r(x) = v(x, \pi(x))$. Put $\Psi(x) := \pi(x)/x$. For $x \gg 0$, we have $\Psi(x) := \pi(x)/x \in \text{Cone } C$ by (A.4). For x on the boundary of R^n_+, we put $\pi_i(x) = 0$ and $\Psi_i(x) = 0$ whenever $x_i = 0$. As in [HM, I], we let $0 < \alpha < 1$ be such that $q \geq \alpha e$ whenever $q \in C$.

LEMMA VII.1.1. *For all $x \in L_r \setminus \{0\}$,*
(i) $\pi(x) \cdot DP_r(x) = v(x, \pi(x))$;
(ii) $P_r(x) = v(x, \pi(x)) + \Sigma^n_{i=1} \pi_i(x) P_r(x^{(i)})$; *and*
(iii) $\Psi(x) \equiv \pi(x)/x \leq (1/(\alpha\|x\|))e$ *for all* $x \gg 0$.

PROOF. (i) is just the definition of $\pi(x)$. From (i) we get

$$\sum_i \pi_i(x)\big(P_r(x) - P_r(x^{(i)})\big) = v(x, \pi(x)).$$

But $\Sigma_i \pi_i(x) = 1$, yielding (ii). For (iii): $\Psi(x) \in \text{Cone } C$, thus $\Psi(x) \geq \alpha\|\Psi(x)\|e$ and $1 = \|\pi(x)\| = \|\Psi(x) * x\| \geq \alpha\|\Psi(x)\|\,\|x\|$. Hence $\Psi(x) \leq \|\Psi(x)\|e \leq (1/(\alpha\|x\|))e$. □

We will now define a stochastic process (a discrete Markov process) on L_r. Let $M := r\|\bar{x}\|(= |N_r|)$. Put $Z^\circ \equiv \bar{x}$ and

$$\text{Prob}\Big(Z^{m+1} = Z^m - \frac{1}{r}e^i \Big| Z^0, Z^1, \dots, Z^m\Big) := \pi_i(Z^m).$$

Denote $\Pi^m := \pi(Z^m)$.

The basic properties of $\{Z^m\}$ are:

LEMMA VII.1.2. *For all $m = 0, 1, \dots, M$:*
(i) $Z^m \in L_r$;
(ii) $\|Z^m\| = \|\bar{x}\| - m/r = \|\bar{x}\|(1 - m/M)$;
(iii) $P_r(Z^m) = v(Z^m, \Pi^m) + E[P_r(Z^{m+1})|Z^0, \dots, Z^m]$; *and*
(iv) $P_r(\bar{x}) = E[\Sigma^{M_0 - 1}_{m=0} v(Z^m, \Pi^m)] + E[P_r(Z^{M_0})]$ *for every* $M_0 \leq M$.

PROOF. (i) is proved by induction; note that if $Z^m_i = 0$, then $\Pi^m_i = \pi_i(Z^m) = 0$, hence $Z^{m+1}_i = Z^m_i = 0$. For (ii), note that $\|Z^{m+1}\| = \|Z^m\| - 1/r$. Part (ii) of Lemma VII.1.1 is precisely (iii). Taking expectation yields $E[P_r(Z^m)] = E[v(Z^m, \Pi^m)] + E[P_r(Z^{m+1})]$, from which (iv) follows by induction (recall that $Z^0 = \bar{x}$). □

To get some intuition for the remainder of the proof, note that if we take $M_0 = M$ in (iv) then

$$P_r(\bar{x}) = E\left[\sum_{m=0}^{M-1} v(Z^m, \Pi^m)\right],$$

since $Z^M = 0$ and $P_r(Z^M) = 0$. Therefore, the right-hand side is an expectation of a random finite sum, each realization of which resembles (hence, approximates[10]) an integral $\int v(\mathbf{x}, \dot{\mathbf{x}})$ along the particular realization of the (random) path $0 = Z^M, Z^{M-1}, \ldots, Z^2, Z^1, Z^0 = \bar{x}$. Consequently, the expectation $P_r(\bar{x})$ is minorized by the infimum of the integral along all of these paths, namely $P(\bar{x})$. The next subsections will make this approximation argument precise.

VII.2. Law of Large Numbers. In this section we will prove a Law of Large Numbers for the process $\{Z^m\}_m$. This is a common theme in the classical approaches to values of large games (e.g., see Shapley (1964) and Aumann and Shapley (1974)). However, our variables are *not* sums of independent variables and, therefore, we do not obtain asymptotically the "diagonal path" $\mathbf{x}(t) = t\bar{x}$ for $0 \le t \le 1$ (e.g., Aumann and Shapley (1974)).

Let $\varepsilon > 0$ be given and let H be a positive integer (to be determined later—see the proof of Proposition VII.3.2—as a function of ε). Put $s := \lfloor M/H \rfloor$ and divide the steps $m = 0, \ldots, M$ into H blocks each composed of s consecutive steps (plus a remainder at the end). (Think of H as fixed, whereas $r \to \infty$, thus also $M = |N_r| = r\|\bar{x}\| \to \infty$ and $s \to \infty$.)

PROPOSITION VII.2.1. *For every ε and H,*

$$\operatorname{Prob}\left\{\left\|(Z^{hs} - Z^{(h+1)s}) - \frac{1}{r}\sum_{m=hs}^{(h+1)s-1} \Pi^m\right\| < \frac{1}{H^2} \text{ for all } h = 0, 1, \ldots, H-1\right\} \ge 1 - \varepsilon,$$

for all r sufficiently large.

PROOF. Define $\hat{Z}^0 = Z^0 = \bar{x}$ and $\hat{Z}^{m+1} := Z^{m+1} - E(Z^{m+1}|Z^0, \ldots, Z^m)$ for $m = 0, \ldots, M-1$. By definition of the process $\{Z^m\}_m$, we have $E(Z^{m+1}|Z^0, \ldots, Z^m) = Z^m - \sum_{i=1}^n \Pi_i^m \cdot (1/r)e^i = Z^m - (1/r)\Pi^m$. Thus $\hat{Z}^{m+1} = Z^{m+1} - Z^m + (1/r)\Pi^m$. Note that $E(\hat{Z}_i^m|Z^0, \ldots, Z^{m-1}) = 0$ for $m > 0$. For each $i = 1, \ldots, n$, the random variables $\{\hat{Z}_i^m\}_m$ are uncorrelated (if $\mu < m$, then $E(\hat{Z}_i^\mu \cdot \hat{Z}_i^m) = E[E(\hat{Z}_i^\mu \cdot \hat{Z}_i^m|Z^0, \ldots, Z^{m-1})] = E[\hat{Z}_i^\mu \cdot E(\hat{Z}_i^m|Z^0, \ldots, Z^{m-1})] = 0 = E(\hat{Z}_i^\mu) \cdot E(\hat{Z}_i^m)$).

Writing Σ^h for $\Sigma_{m=hs}^{(h+1)s-1}$, we have

$$\Sigma^h \hat{Z}^{m+1} = Z^{(h+1)s} - Z^{hs} + \frac{1}{r}\Sigma^h \Pi^m,$$

(note that this is exactly the expression whose norm we want to show to be $< 1/H^2$ with probability $\ge 1 - \varepsilon$), and

$$\operatorname{Var}\left(\Sigma^h Z_i^{m+1}\right) = \Sigma^h \operatorname{Var}\left(\hat{Z}_i^m\right) \le s\frac{1}{r^2},$$

since $|\hat{Z}_i^{m+1}| \le 1/r$. Using Chebychev's inequality,

$$\operatorname{Prob}\left(\left|\Sigma^h \hat{Z}_i^{m+1}\right| \ge \rho\right) \le \frac{1}{\rho^2} s \frac{1}{r^2}$$

for any $\rho > 0$.

[10] After dividing by $1/r = \|\bar{x}\|/M$ for appropriate normalization.

Doing this for all $i = 1, \ldots, n$ and $h = 0, \ldots, H-1$ yields

$$\operatorname{Prob}\left(\left|\Sigma^h \hat{Z}_i^{m+1}\right| < \rho \text{ for all } i = 1, \ldots, n \text{ and } h = 0, \ldots, H-1\right)$$
$$\geq 1 - nH\frac{1}{\rho^2} s \frac{1}{r^2},$$

implying

$$\operatorname{Prob}\left(\left\|\Sigma^h \hat{Z}^{m+1}\right\| < n\rho \text{ for all } h = 0, \ldots, H-1\right)$$
$$\geq 1 - \frac{nHs}{\rho^2 r^2} \geq 1 - \frac{n\|\bar{x}\|}{\rho^2 r}.$$

(Recall that $Hs \leq M = r\|\bar{x}\|$.) Hence we may choose $\rho := 1/nH^2$, and then $r \geq n^3\|\bar{x}\|H^4/\varepsilon$. □

To get an appreciation of what Proposition VII.2.1 says, note that the conclusion is false for $H = M$ (i.e., for blocks of size $s = 1$): the (random) net change $Z^m - Z^{m+1}$ and its conditional expectation $(1/r)\Pi^m$ are not close. It is only the averages over sufficiently large blocks that become close.

VII.3. The stochastic process $\{Y^m\}$. We now introduce another stochastic process $\{Y^m\}_m$ which is derived from $\{Z^m\}_m$. The aim is that its values will be in the positive (rather than nonnegative) orthant, and that it will be close to $\{Z^m\}_m$. Its definition is as follows: Denote $\Psi^m := \Psi(Z^m) = \Pi^m/Z^m$ and then let $Y^0 := \bar{x}$ and[11]

$$Y^{m+1} := \bar{x} * \exp\left(-\frac{1}{r}\sum_{\mu=0}^{m} \Psi^\mu\right) \quad \text{for } m = 0, \ldots, M-1.$$

Note that $Y^m \in R^n_{++}$ for all $m = 0, \ldots, M$ (but Y^m is *not* on the grid L_r).

Fix $\varepsilon > 0$, and let $0 < \delta \leq (1/3)\exp(-1/\alpha\varepsilon)$. We will choose δ as a function of ε more precisely later (before Lemma VII.4.2).

LEMMA VII.3.1. *For all $m \leq M(1-\varepsilon)$: if $Z^m \gg 0$, then $\Psi^m \leq (1/(\alpha\|\bar{x}\|\varepsilon))e$ and $Y^{m+1} \geq 3\delta e$.*

PROOF. $Z^m \gg 0$ implies $Z^\mu \gg 0$ for all $\mu \leq m$. By Lemma VII.1.1(iii) and Lemma VII.1.2(ii) we have for all $\mu \leq m$,

$$\Psi^\mu \leq \frac{1}{\alpha\|Z^\mu\|}e \leq \frac{1}{\alpha\|\bar{x}\|\varepsilon}e,$$

therefore

$$\frac{1}{r}\sum_{\mu=0}^{m} \Psi^\mu \leq \frac{M}{r\alpha\|\bar{x}\|\varepsilon}e \leq \frac{1}{\alpha\varepsilon}e$$

[11] When $x \in R^n$, we write $y = \exp(x) \in R^n$ for $y_i = e^{x_i}$ for all $i = 1, \ldots, n$.

and so (recall that $\bar{x}_\iota \gg 0$ is an integer and, therefore, $\bar{x} \geq e$),

$$Y^{m+1} \geq \bar{x} * \exp\left(-\frac{1}{\alpha\varepsilon} e\right) \geq 3\delta e. \quad \square$$

The basic fact of this subsection is:

PROPOSITION VII.3.2. *For every $\varepsilon > 0$ and $\delta > 0$,*

$$\text{Prob}\{\|Z^m - Y^m\| \leq 3\delta, Y^m \geq 3\delta e \text{ and } Z^m \geq 2\delta e \text{ for all } m \leq M(1-\varepsilon)\} \geq 1 - \varepsilon,$$

for all r sufficiently large.

PROOF. Let $\gamma_1 := 1/\alpha\varepsilon$ and $\gamma_2 := \alpha\varepsilon + \|\bar{x}\| + \|\bar{x}\|/(2\alpha\varepsilon)$. Choose H to be large enough such that $1/H \leq \varepsilon$, $\|\bar{x}\|/H \leq \delta$ and $(\gamma_2/H)(e^{\gamma_1} - 1) \leq \delta$. Let $\rho_h := (\gamma_2/H)[(1 + \gamma_1/H)^h - 1]$ for $h = 0, 1, \ldots, H$; note that $\rho_h \leq \rho_H \leq \delta$. Denote by Ω_r^* the event

$$\Omega_r^* := \left\{\left\|Z^{hs} - Z^{(h+1)s} - \frac{1}{r}\Sigma^h \Pi^m\right\| < \frac{1}{H^2} \text{ for all } h = 0, \ldots, H-1\right\}$$

(recall that Σ^h stands for $\Sigma_{m=hs}^{(h+1)s-1}$). Thus, by Proposition VII.2.1, $\text{Prob}(\Omega_r^*) \geq 1 - \varepsilon$ for all r large enough. We will divide the proof into two steps.

STEP 1. For all r sufficiently large we have "$\|Y^{hs} - Z^{hs}\| \leq \rho_h$ and $Y^{hs} \geq 3\delta e$ for all h such that $hs \leq M(1-\varepsilon)$" in the event Ω_r^*.

PROOF OF STEP 1. By induction on h. For $h = 0$ we have $Y^0 = Z^0 = \bar{x}$, and thus the claim is correct. Assume that $(h+1)s \leq M(1-\varepsilon)$ and $Y^{hs} \geq 3\delta e$, $\|Y^{hs} - Z^{hs}\| \leq \rho_h$ in the event Ω_r^*; we will show that $Y^{(h+1)s} \geq 3\delta e$ and $\|Y^{(h+1)s} - Z^{(h+1)s}\| \leq \rho_{h+1}$ in the same event. For all $m = hs, \ldots, (h+1)s - 1$ we have

$$\|Z^m - Y^{hs}\| \leq \|Z^{hs} - Y^{hs}\| + \|Z^m - Z^{hs}\|$$

$$\leq \rho_h + s \cdot \frac{1}{r} \leq \rho_h + \frac{\|\bar{x}\|}{H} \leq \delta + \delta = 2\delta,$$

and $Y^{hs} \geq 3\delta e$, implying $Z^m \geq \delta e \gg 0$. Hence $\Psi^m \leq (1/(\alpha\|\bar{x}\|\varepsilon))e$ and $Y^{m+1} \geq 3\delta e$ by Lemma VII.3.1, in particular $Y^{(h+1)s} \geq 3\delta e$.

Next we have

$$\|Y^{(h+1)s} - Z^{(h+1)s}\| \leq \|Y^{hs} - Z^{hs}\|$$

$$+\left\|Z^{hs} - Z^{(h+1)s} - \frac{1}{r}\Sigma^h \Pi^m\right\|$$

$$+\left\|Y^{hs} - Y^{(h+1)s} - \frac{1}{r}\Sigma^h \Pi^m\right\|.$$

The first term is $\leq \rho_h$ by assumption and the second is $\leq 1/H^2$ by Proposition VII.2.1. For the third:

$$\left\|Y^{hs} - Y^{(h+1)s} - \frac{1}{r}\sum_h \Pi^m\right\| = \left\|Y^{hs} - Y^{hs} * \exp\left(-\frac{1}{r}\Sigma^h\Psi^m\right) - \frac{1}{r}\Sigma^h\Pi^m\right\|$$

$$\leq \left\|Y^{hs} - Y^{hs} * \exp\left(-\frac{1}{r}\Sigma^h\Psi^m\right) - Y^{hs} * \frac{1}{r}\Sigma^h\Psi^m\right\|$$

$$+ \frac{1}{r}\Sigma^h\|(Y^{hs} - Z^m) * \Psi^m\|$$

(recall that $\Pi^m = Z^m * \Psi^m$). Now $|e^{-u} - (1-u)| \leq \frac{1}{2}u^2$ for all $u > 0$, $\Psi^m \leq (1/(\alpha\|\bar{x}\|\varepsilon))e$ and $\|Y^{hs} - Z^m\| \leq \rho_h + \|\bar{x}\|/H$. Therefore, the above expression is at most

$$\|Y^{hs}\|\frac{1}{2}\left(\frac{1}{r}\cdot s \cdot \frac{1}{\alpha\|\bar{x}\|\varepsilon}\right)^2 + \frac{1}{r}\cdot s \cdot \left(\rho_h + \frac{\|\bar{x}\|}{H}\right)\cdot\frac{1}{\alpha\|\bar{x}\|\varepsilon}$$

$$\leq \|\bar{x}\|\frac{1}{2}\cdot\frac{1}{H^2\alpha^2\varepsilon^2} + \frac{1}{H\alpha\varepsilon}\rho_h + \frac{\|\bar{x}\|}{H^2\alpha\varepsilon}.$$

Putting all three terms together we get

$$\|Y^{(h+1)s} - Z^{(h+1)s}\| \leq \rho_h\left(1 + \frac{1}{H\alpha\varepsilon}\right) + \frac{1}{H^2} + \frac{\|\bar{x}\|}{2H^2\alpha^2\varepsilon^2} + \frac{\|\bar{x}\|}{H^2\alpha\varepsilon}$$

$$= \rho_h\left(1 + \frac{\gamma_1}{H}\right) + \frac{\gamma_1\gamma_2}{H^2}$$

$$\leq \frac{\gamma_2}{H}\left[\left(1 + \frac{\gamma_1}{H}\right)^h - 1\right]\left(1 + \frac{\gamma_1}{H}\right) + \frac{\gamma_1\gamma_2}{H^2}$$

$$= \frac{\gamma_2}{H}\left[\left(1 + \frac{\gamma_1}{H}\right)^{h+1} - 1\right] = \rho_{h+1},$$

which completes Step 1.

STEP 2. For all r sufficiently large we have "$\|Y^m - Z^m\| \leq 3\delta$, $Y^m \geq 3\delta e$ and $Z^m \geq 2\delta e$ holds for all $m \leq M(1-2\varepsilon)$" in the event Ω_r^*.

PROOF. Let $m \leq M(1-2\varepsilon)$ and h be such that $hs \leq m \leq (h+1)s$. Then, $(h+1)s \leq M(1-\varepsilon)$ (since $s \leq M/H \leq M\varepsilon$), and $Y^{hs} \geq Y^m \geq Y^{(h+1)s}$, $Z^{hs} \geq Z^m \geq Z^{(h+1)s}$. By Step 1 we have $Y^{hs} \geq 3\delta e$, $Y^{(h+1)s} \geq 3\delta e$, $\|Y^{hs} - Z^{hs}\| \leq \delta$ and $\|Y^{(h+1)s} - Z^{(h+1)s}\| \leq \delta$, which together with $\|Z^{hs} - Z^{(h+1)s}\| = s \cdot 1/r \leq \|\bar{x}\|H \leq \delta$ imply $\|Y^m - Z^m\| \leq 3\delta$ as well as $Y^m \geq Y^{(h+1)s} \geq 3\delta e$ and $Z^m \geq Z^{(h+1)s} \geq Y^{(h+1)s} - \delta e \geq 2\delta e$. □

VII.4. Random continuous paths. A (continuous) "path" $\mathbf{x} \in \Gamma_0(\bar{x})$ is an absolutely continuous function $\mathbf{x}: [0,1] \to R^n_{++} \cup \{0\}$ with $\mathbf{x}(0) = 0$ and $\mathbf{x}(1) = \bar{x}$. We will now construct a "random continuous path" $\mathbf{X}$, namely, a random variable with values in $\Gamma_0(\bar{x})$.

In [HM, I] we have defined $\Gamma_1(\bar{x})$ as the set of all paths $\mathbf{x} \in \Gamma_0(\bar{x})$ such that $\dot{\mathbf{x}}(t)/\mathbf{x}(t) \in \text{Cone}\, C$ and $\|\dot{\mathbf{x}}(t)\| = \|\bar{x}\|$ for a.e. t. The latter condition is just a normalization (any path with $\dot{\mathbf{x}} \gg 0$ may be reparameterized so as to satisfy this). We will find it useful here to consider unnormalized paths, i.e., paths $\mathbf{x} \in \Gamma_0(\bar{x})$ such that $\dot{\mathbf{x}}(t)/\mathbf{x}(t) \in \text{Cone}\, C$ for a.e. t. Let $\Gamma_1'(\bar{x})$ denote this set of paths; note that any result holding for paths in $\Gamma_1(\bar{x})$ will also hold in $\Gamma_1'(\bar{x})$—just reparametrize.

Let r be sufficiently large, and let Ω_r be the probability space on which the stochastic process $\{Z^m\}_m$ is defined (note that there are only finitely many possible realizations for the sequence $\{Z^m\}_m$, since it must go from $\bar{x}$ to 0 following the grid L_r; we may thus take Ω_r to be finite). Let $\Omega_r^* \subset \Omega_r$ denote the "good" event of Proposition VII.3.2; thus $\text{Prob}(\Omega_r^*) \geq 1 - \varepsilon$.

Let $\tau^m := 1 - m/M$ and $M_0 := \lfloor M(1 - 2\varepsilon) \rfloor$. For every $\omega \in \Omega_r$ define $\mathbf{X} \equiv \mathbf{X}_\omega : [0,1] \to R_+^n$ as follows: for $m = 0, 1, \ldots, M_0 - 1$ and $t \in [\tau^{m+1}, \tau^m]$ put

$$\mathbf{X}(t) := \bar{x} * \exp\left(-\frac{1}{r} \sum_{\mu=0}^{m-1} \Psi^\mu - \|\bar{x}\|(\tau^m - t)\Psi^m \right);$$

for $t \in [0, \tau^{M_0}]$, choose any path in $\Gamma_1'(Y^{M_0}(\omega))$. Hence, $\mathbf{X}(0) = 0$ and $\mathbf{X}(\tau^{M_0}) = Y^{M_0}$.

LEMMA VII.4.1. *For every* $\omega \in \Omega_r^*$:
(i) $\mathbf{X}$ *is well defined*;
(ii) $\mathbf{X} \in \Gamma_1'(\bar{x})$ *and* $\dot{\mathbf{X}}(t)/\mathbf{X}(t) = \|\bar{x}\|\Psi^m$ *for all* $m = 0, \ldots, M_0 - 1$ *and* $t \in (\tau^{m+1}, \tau^m)$;
(iii) $\int_0^1 v(\mathbf{X}(t), \dot{\mathbf{X}}(t))\, dt \geq P(\bar{x})$.

PROOF. (i) is just the continuity at τ^m, which is easily checked: $\mathbf{X}(\tau^m) = Y^m$. Property (ii) is also an immediate computation, and (iii) follows since $P(\bar{x})$ is the infimum of all such integrals. □

What we have done can be described as follows. The sequence $\{Y^m\}_m$ yields a (random) discrete path from $\bar{x}$ to 0. We could obtain a continuous path by linear interpolation. We have done instead something only slightly different: interpolate in such a way that $\dot{\mathbf{X}}/\mathbf{X}$ is constant in each interval (τ^{m+1}, τ^m).

Fix $\varepsilon > 0$ and choose $\delta := \min\{(1/3)\exp(-1/\alpha\varepsilon), \varepsilon^2\}$.

LEMMA VII.4.2. *For every r large enough we have*: "$\|\mathbf{X}(t) - Z^m\| \leq 7\delta$, $\mathbf{X}(t) \geq 3\delta e$ *and* $Z^m \geq 2\delta e$ *for all* $m \leq M_0 - 1$ *and* $t \in [\tau^{m+1}, \tau^m]$" *for every* $\omega \in \Omega_r^*$.

PROOF. Use Proposition VII.3.2 to get $\|Y^m - Z^m\| \leq 3\delta$, $\|Y^{m+1} - Z^{m+1}\| \leq 3\delta$. Together with $\|Z^m - Z^{m+1}\| = 1/r \leq \delta$ and $Y^{m+1} \leq \mathbf{X}(t) \leq Y^m$ for $\tau^{m+1} \leq t \leq \tau^m$, we get the result. □

PROPOSITION VII.4.3. *For every r large enough and every* $\omega \in \Omega_r^*$ *we have*

$$\left| \frac{1}{r} \sum_{m=0}^{M_0-1} v(Z^m, \Pi^m) - \int_0^1 v\big(\mathbf{X}(t), \dot{\mathbf{X}}(t)\big)\, dt \right| \leq \varepsilon.$$

PROOF. Let $K_6 < \infty$ be given by (A.6) for the set $\{x \in R_{++}^n : 2\delta e \leq x \leq \bar{x}\} \times C$, and $K_5 < \infty$ be given by (A.5). Then, for $m \leq M_0 - 1$ and $\tau^{m+1} \leq t \leq \tau^m$,

$$\begin{aligned} \left| v\big(\mathbf{X}(t), \dot{\mathbf{X}}(t)\big) - v(Z^m, \|\bar{x}\|\Pi^m) \right| &= \left| \hat{v}\big(\mathbf{X}(t), \dot{\mathbf{X}}(t)/\mathbf{X}(t)\big) - \hat{v}(Z^m, \|\bar{x}\|\Psi^m) \right| \\ &\leq K_6 \|\mathbf{X}(t) - Z^m\| \cdot \|\bar{x}\| \cdot \|\Psi^m\| \\ &\leq K_6 \cdot 7\delta \cdot \|\bar{x}\| \cdot \frac{n}{\alpha \|\bar{x}\| \varepsilon} = O(\varepsilon), \end{aligned}$$

by Lemmas VII.4.2 and VII.3.1 (recall also Lemma VII.4.1(ii): $\dot{\mathbf{X}}(t)/\mathbf{X}(t) = \|\bar{x}\|\Psi^m$, and $\delta \leq \varepsilon^2$). Integrating and summing over $m \leq M_0 - 1$ we have

$$\left|\int_{\tau^{M_0}}^1 v(\mathbf{X}, \dot{\mathbf{X}}) - \frac{1}{r}\sum_{m=0}^{M_0-1} v(Z^m, \Pi^m)\right| = O(\varepsilon)$$

(since $\tau^m - \tau^{m+1} = 1/M$ and $\|\bar{x}\|/M = 1/r$). As for $\int_0^{\tau^{M_0}}$, it is bounded by $(K_5/\alpha)\|\mathbf{X}(\tau^{M_0})\|$ (see the remark following Lemma V.1.1 in [HM, I]); now $\|\mathbf{X}(\tau^{M_0})\| = \|Y^{M_0}\| \leq \|Z^{M_0}\| + \|Y^{M_0} - Z^{M_0}\| \leq (1 - M_0/M)\|\bar{x}\| + 3\delta \leq 3\varepsilon\|\bar{x}\| + 3\varepsilon^2$, therefore this part is also $O(\varepsilon)$. □

Let Prob* and E^* denote the conditional probability and the conditional expectation, respectively, on the "good" event Ω_r^*; i.e., $\text{Prob}^*(\cdot) := \text{Prob}(\cdot|\Omega_r^*)$ and $E^*(\cdot) := (\cdot|\Omega_r^*)$.

PROPOSITION VII.4.4. *For every $\varepsilon > 0$ and r large enough*:

$$\frac{1}{r}P_r(\bar{x}) \geq E^*\left[\int_0^1 v(\mathbf{X}, \dot{\mathbf{X}})\right] - \varepsilon.$$

PROOF. $P_r(x) \geq 0$ for all $x \in L_r$ (this follows from Corollary V.2; a direct proof is by noting that $G(x) \equiv 0$ is a subpotential), therefore Lemma VII.1.2(iv), $\text{Prob}(\Omega_r^*) \geq 1 - \varepsilon$ and Proposition VII.4.3 yield, for r large enough,

$$\frac{1}{r}P_r(\bar{x}) \geq E\left[\frac{1}{r}\sum_{m=0}^{M_0-1} v(Z^m, \Pi^m)\right]$$

$$\geq (1-\varepsilon)E^*\left[\frac{1}{r}\sum_{m=0}^{M_0-1} v(Z^m, \Pi^m)\right]$$

$$\geq (1-\varepsilon)\left(E^*\left[\int_0^1 v(\mathbf{X}, \dot{\mathbf{X}})\right] - \varepsilon\right).$$

Now, $\int v(\mathbf{X}, \dot{\mathbf{X}}) \leq (K_5/\alpha)\|\bar{x}\|$ (again, see Lemma V.1.1 in [HM, I]) and therefore $(1/r)P_r(\bar{x}) \geq E^*[\int v(\mathbf{X}, \dot{\mathbf{X}})] - \varepsilon(1 + (K_5/\alpha)\|\bar{x}\|)$. □

We have thus proved the "lim inf convergence":

PROOF OF PROPOSITION VII.1. Immediate by Proposition VII.4.4 and Lemma VII.4.1(iii). □

VIII. Proof of "derivatives' convergence." In this section we will prove the third part of the Main Result, namely

PROPOSITION VIII.1.

$$\lim_{r \to \infty} \text{dist}(DP_r(\bar{x}), \text{Eg}(\bar{x})) = 0,$$

where $\text{Eg}(x) = (\nabla^c P(x) + R_+^n) \cap \partial V(x)$ for $x \in R_{++}^n$. Note that Proposition VIII.1 implies both (b) and (c) of the Main Theorem (see §IV).

VIII.1. The random path is almost optimal. In §VII, we have constructed for every r (large enough) a random path $\mathbf{X} \in \Gamma_1'(\bar{x})$. Propositions VII.4.4 and VI.1 imply that $\mathbf{X}$ is almost optimal. More precisely, let $J(\mathbf{X}) := \int_0^1 v(\mathbf{X}(t), \dot{\mathbf{X}}(t))\, dt$. Then (again, $\varepsilon > 0$ is given):

PROPOSITION VIII.1.1. *For every r large enough we have*

$$\text{Prob}^*\{|J(\mathbf{X}) - P(\bar{x})| \leq \varepsilon\} \geq 1 - \varepsilon$$

(*recall that* Prob* *denotes the conditional probability on the "good" event* Ω_r^*).

PROOF. Propositions VII.4.4 and VI.1 imply that

$$E^*[J(\mathbf{X})] \leq P(\bar{x}) + 2\varepsilon,$$

for all r large enough. Since $J(\mathbf{X}) \geq P(\bar{x})$ always (Lemma VII.4.1(iii)), we have $\text{Prob}^*\{J(\mathbf{X}) \leq P(\bar{x}) + \sqrt{2\varepsilon}\} \geq 1 - \sqrt{2\varepsilon}$. Because ε is arbitrary, we are done. □

An implication of Proposition VIII.1.1 is that $J(\mathbf{X})$ is almost deterministic for large r. More precisely, $J(\mathbf{X})$ converges in measure to the constant $P(\bar{x})$ as $r \to \infty$. Yet another important implication: if there is a single optimal path $\mathbf{x} \in \Gamma_1(\bar{x})$, then the random path $\mathbf{X}$ (after appropriate reparametrization) must converge in measure to $\mathbf{x}$ as $r \to \infty$; i.e., there is a limit nonrandom path (typically not the diagonal $\mathbf{x}(t) = t\bar{x}$).

Given a path $\mathbf{x} \in \Gamma_1'(\bar{x})$, let $L(\mathbf{x}) := \int_0^1 \mathbf{x}(t) * \nabla_x \hat{v}(\mathbf{x}(t), \dot{\mathbf{x}}(t)/\mathbf{x}(t))\, dt$. In [HM, I] we have shown the following (a form of Euler's equation):

PROPOSITION VIII.1.2. *If* $\mathbf{x} \in \Gamma_1'(\bar{x})$ *is an optimal path, then* $L(\mathbf{x}) \in \bar{x} * \nabla^c P(\bar{x})$.

PROOF. This is just Corollary VII.4.4 in [HM, I]. □

The next two Propositions spell out an important consequence of Proposition VIII.1.1. Not only is $J(\mathbf{X})$ almost not random for r large enough, but so is $L(\mathbf{X})$. Because $L(\mathbf{x})$ is mathematically a derivative (Proposition VIII.1.2), this provides a key link towards establishing the convergence of derivatives.

PROPOSITION VIII.1.3. *For every* $\varepsilon > 0$ *there is* $\delta \equiv \delta(\varepsilon) > 0$ *such that if* $\mathbf{x} \in \Gamma_1'(\bar{x})$ *is a* δ*-optimal path* (*i.e.*, $J(\mathbf{x}) \leq P(\bar{x}) + \delta$), *then* $\text{dist}(L(\mathbf{x}), \bar{x} * \nabla^c P(\bar{x})) < \varepsilon$.

PROOF. If not, there is $\mathbf{x}_k \in \Gamma_1(\bar{x})$ such that $J(\mathbf{x}_k) \to P(\bar{x})$, but $\text{dist}(L(\mathbf{x}_k), \bar{x} * \nabla^c P(\bar{x})) \geq \varepsilon$. By Arzela-Ascoli we can as well assume that $\{\mathbf{x}_k\}$ converges uniformly to a limit $\mathbf{x}$. This $\mathbf{x}$ must be optimal (Lemma V.1.4 in [HM, I]). Also, $L(\mathbf{x}_k) \to L(\mathbf{x})$ by Proposition VII.3.2 in [HM, I], and $L(\mathbf{x}) \in \bar{x} * \nabla^c P(\bar{x})$ by Proposition VIII.1.2 above. This contradiction proves the claim. □

PROPOSITION VIII.1.4. *For every r large enough,*

$$\text{Prob}^*\{\text{dist}(L(\mathbf{X}), \bar{x} * \nabla^c P(\bar{x})) \leq \varepsilon\} \geq 1 - \varepsilon.$$

PROOF. Proposition VIII.1.3 and Proposition VIII.1.1 (with ε replaced by $\min\{\delta(\varepsilon), \varepsilon\}$). □

VIII.2. A recursive formula for finite differences. Define a function $F: L_r \to R^n$ by $F(x) := x * DP_r(x) = (x_i[P_r(x) - P_r(x^{(i)})])_{i=1}^n$ (when $x_i = 0$, we have $F_i(x) = 0$). As in §VII.1, $\pi(x) \in \Delta^{n-1}$ satisfies $\pi(x) \cdot DP_r(x) = v(x, \pi(x))$, and $\psi(x) = \pi(x)/x$ (again, if $x_i = 0$, then $\pi_i(x) = \psi_i(x) = 0$). With a slight abuse of notation we denote $Dv(x, p) = (v(x, p) - v(x^{(i)}, p))_{i=1}^n$, and similarly for $D\hat{v}(x, q)$. The next proposition gives an analog for finite differences of the recursion formula of Lemma VII.1.1(ii). There is one key distinction however: Here we only get an inequality.

PROPOSITION VIII.2.1. *For every* $x \in L_r$, *we have*

$$F(x) \geq \sum_{i=1}^{n} \pi_i(x) F(x^{(i)}) + x * D\hat{v}(x, \psi(x)).$$

PROOF. Fix $j = 1, 2, \dots, n$, and assume $x_j > 0$ (otherwise there is nothing to prove: $0 = 0$ above). Let $p := \pi(x)$ and $q := \psi(x)$. Then $DP_r(x^{(j)}) \in \partial V(x^{(j)})$, implying $q \cdot x^{(j)} * DP_r(x^{(j)}) \leq \hat{v}(x^{(j)}, q)$. Together with $q \cdot x * DP_r(x) = \hat{v}(x, q)$, we have, therefore,

$$q \cdot \left[x * DP_r(x) - x^{(j)} * DP_r(x^{(j)}) \right] \geq \hat{v}(x, q) - \hat{v}(x^{(j)}, q) = D_j \hat{v}(x, q).$$

For $i \neq j$ we have $x_i^{(j)} = x_i$ and $x_i D_i P_r(x) - x_i^{(j)} D_i P_r(x^{(j)}) = x_i (P_r(x) - P_r(x^{(i)}) - P_r(x^{(j)}) + P_r(x^{(j)(i)})) = x_i (D_j P_r(x) - D_j P_r(x^{(i)}))$. For $i = j$, we have $x_j^{(j)} = x_j - 1/r$ and $x_j D_j P_r(x) - x_j^{(j)} D_j P_r(x^{(j)}) = x_j (D_j P_r(x) - D_j P_r(x^{(j)})) + (1/r) D_j P_r(x^{(j)})$. Substituting all this above we get

$$\sum_i q_i x_i \left(D_j P_r(x) - D_j P_r(x^{(i)}) \right) + \frac{1}{r} q_j D_j P_r(x^{(j)}) \geq D_j \hat{v}(x, q).$$

But $q * x = p \in \Delta^{n-1}$, and so $\Sigma q_i x_i = 1$. Hence we get on the left hand side, $D_j P_r(x) - \Sigma_i p_i D_j P_r(x^{(i)}) + (1/r) q_j D_j P_r(x^{(j)})$. Multiplying by x_j this becomes

$$x_j D_j P_r(x) - \sum_i p_i x_j D_j P_r(x^{(i)}) + p_j \frac{1}{r} D_j P_r(x^{(j)}) = x_j D_j P_r(x) - \sum_i p_i x_j^{(i)} D_j P_r(x^{(i)})$$

$$= F_j(x) - \sum_i p_i F_j(x^{(i)}).$$

We conclude that $F_j(x) - \Sigma_i p_i F_j(x^{(i)}) \geq x_j D_j \hat{v}(x, q)$, which is what we wanted to prove. □

Applying this to the stochastic process $\{Z^m\}_m$ defined in §VII.1, we have

COROLLARY VIII.2.2. *For every* $M_0 \leq M$,

$$\bar{x} * DP_r(\bar{x}) = F(\bar{x}) \geq E\left[\sum_{m=0}^{M_0 - 1} Z^m * D\hat{v}(Z^m, \Psi^m) \right] + E\left[F(Z^{M_0}) \right].$$

PROOF. Proposition VIII.2.1 applied at $x = Z^m$ gives

$$F(Z^m) \geq E\left[F(Z^{m+1}) | Z^0, \dots, Z^m \right] + Z^m * D\hat{v}(Z^m, \Psi^m).$$

Taking expectations yields the result by induction (recall that $Z^0 = \bar{x}$). □

COROLLARY VIII.2.3. *For every r large enough*,

$$\bar{x} * DP_r(\bar{x}) \geq (1 - \varepsilon) E^* \left[\sum_{m=0}^{M_0 - 1} Z^m * D\hat{v}(Z^m, \Psi^m) \right].$$

PROOF. For r large enough we have $\text{Prob}(\Omega_r^*) \geq 1 - \varepsilon$. The sum on the right-hand side is ≥ 0 because $Z^m \geq 0$ and $D\hat{v}(Z^m, \Psi^m) \geq 0$ (actually, $D\hat{v}(x, q) \geq \theta e$ by (A.3)*). Also $F(x) = x * DP_r(x) \geq 0$ for all x, hence $E[F(Z^{M_0})] \geq 0$. To complete the proof, use Corollary VIII.2.2. □

VIII.3. Derivatives of the random path. The random path $\mathbf{X}$ is "close" to the process $\{Z^m\}_m$. In Lemma VII.4.2 we used this to approximate $J(\mathbf{X})$. Now we will use it in the same manner to approximate $L(\mathbf{X})$. The following proposition is parallel to Proposition VII.4.3.

PROPOSITION VIII.3.1. *For every r large enough and every $\omega \in \Omega_r^*$ we have*

$$\left\| L(\mathbf{X}) - \sum_{m=0}^{M_0-1} Z^m * D\hat{v}(Z^m, \Psi^m) \right\| \le \varepsilon.$$

PROOF. Let r be large enough and $\omega \in \Omega_r^*$. For $i = 1, \dots, n$, $D_i\hat{v}(Z^m, \Psi^m) = (1/r)(\partial/\partial x_i)\hat{v}(Z^m - (\xi/r)e^i, \Psi^m)$ for some $0 \le \xi \le 1$, by the mean-value theorem. Use Lemma VII.4.2 to get, for all $m = 0, \dots, M_0 - 1$ and all $t \in [\tau^{m+1}, \tau^m]$, that $\|\mathbf{X}(t) - Z^m\| \le 7\delta$, and therefore $\|\mathbf{X}(t) - (Z^m - (\xi/r)e^i)\| \le 7\delta + 1/r \le 8\varepsilon^2$ (we take $\delta \le \varepsilon^2$), $\mathbf{X}(t) \ge 2\delta e$ and $Z^m - (\xi/r)e^i \ge \delta e$. Then

$$\left| \mathbf{X}_i(t) \frac{\partial}{\partial x_i} \hat{v}\big(\mathbf{X}(t), \dot{\mathbf{X}}(t)/\mathbf{X}(t)\big) - Z_i^m r D_i \hat{v}(Z^m, \|\bar{x}\|\Psi^m) \right|$$

$$\le |\mathbf{X}_i(t) - Z_i^m| \left| \frac{\partial}{\partial x_i} \hat{v}(\mathbf{X}, q^m) \right| + Z_i^m \left| \frac{\partial}{\partial x_i} \hat{v}(\mathbf{X}, q^m) - \frac{\partial}{\partial x_i} \hat{v}\Big(Z^m - \frac{\xi}{r} e^i, q^m\Big) \right|,$$

where $q^m := \|\bar{x}\|\Psi^m$ (see Lemma VII.4.1). Recall that $\|q^m\| \le n/(\alpha\varepsilon)$ by Lemma VII.3.1. Let K_{8a} be the constant in (A.8)(a) for $x \le \bar{x}$, and K_{8b} the constant in (A.8)(b) for $\delta e \le x \le \bar{x}$. The expression above is therefore bounded by

$$|\mathbf{X}_i(t) - Z_i^m| K_{8a} \|q^m\| + \bar{x}_i K_{8b} \left\| \mathbf{X}(t) - \Big(Z^m - \frac{\xi}{r} e^i\Big) \right\| \|q^m\|$$

$$\le 7\varepsilon^2 K_{8a} \frac{n}{\alpha\varepsilon} + \|\bar{x}\| K_{8b} 8\varepsilon^2 \frac{n}{\alpha\varepsilon} = O(\varepsilon).$$

Therefore (recall that $\tau^m - \tau^{m+1} = 1/M$),

$$\left\| \int_{\tau^{M_0}}^1 \mathbf{X} * \nabla_x \hat{v}(\mathbf{X}, \dot{\mathbf{X}}/\mathbf{X}) - \frac{1}{M} r \|\bar{x}\| \sum_{m=0}^{M_0-1} Z^m * D\hat{v}(Z^m, \Psi^m) \right\| = O(\varepsilon).$$

Moreover,

$$\left\| \int_0^{\tau^{M_0}} \mathbf{X} * \nabla_x \hat{v}(\mathbf{X}, \dot{\mathbf{X}}/\mathbf{X}) \right\| \le K_{8a} \int_0^{\tau^{M_0}} \|\mathbf{X}\| \, \|\dot{\mathbf{X}}/\mathbf{X}\|$$

by (A.8)(a). Since $\dot{\mathbf{X}}/\mathbf{X} \in$ Cone C, we have $\|\mathbf{X}\| \, \|\dot{\mathbf{X}}/\mathbf{X}\| \le (1/\alpha)\|\dot{\mathbf{X}}\|$, thus

$$\left\| \int_0^{\tau^{M_0}} \mathbf{X} * \nabla_x \hat{v}(\mathbf{X}, \dot{\mathbf{X}}/\mathbf{X}) \right\| \le (K_{8a}/\alpha) \|\mathbf{X}(\tau^{M_0})\| \le (K_{8a}/\alpha)(3\varepsilon\|\bar{x}\| + 3\varepsilon^2) = O(\varepsilon)$$

(see the computation at the end of the proof of Proposition VII.4.3). Adding the two estimates gives the result (recall that $M = r\|\bar{x}\|$). □

COROLLARY VIII.3.2. *For every r large enough,*

$$\operatorname{dist}\left(E^*\left[\sum_{m=0}^{M_0-1} Z^m * D\hat{v}(Z^m, \Psi^m)\right], \bar{x} * \nabla^c P(\bar{x})\right) \le \varepsilon.$$

PROOF. For any path $\mathbf{x} \in \Gamma_1'(\bar{x})$ we have by (A.8)(a),

$$\begin{aligned}\|L(\mathbf{x})\| &\le \int_0^1 \|\mathbf{x}(t)\| \cdot K_{8a} \|\dot{\mathbf{x}}(t)/\mathbf{x}(t)\| \, dt \\ &\le \frac{K_{8a}}{\alpha} \int_0^1 \|\mathbf{x}(t) * \dot{\mathbf{x}}(t)/\mathbf{x}(t)\| \, dt \\ &= \frac{K_{8a}}{\alpha} \|\bar{x}\|\end{aligned}$$

(recall that $\mathbf{q} := \dot{\mathbf{x}}/\mathbf{x} \in \operatorname{Cone} C$, hence $\|\mathbf{x} * \mathbf{q}\| \ge \alpha \|\mathbf{x}\| * \|\mathbf{q}\|$). Therefore, $L(\mathbf{X})$ is uniformly bounded. Apply now Propositions VIII.3.1 and VIII.1.4 (make their corresponding ε's as small as necessary) and note that $\nabla^c P(\bar{x})$ is a convex set. □

The next corollary completes the proof.

COROLLARY VIII.3.3. *For every r large enough,*

$$\operatorname{dist}(DP_r(\bar{x}), Eg(\bar{x})) \le \varepsilon.$$

PROOF. Corollaries VIII.2.3 and VIII.3.2 imply

$$\operatorname{dist}\left(\frac{1}{1-\varepsilon}\bar{x} * DP_r(\bar{x}), \bar{x} * \nabla^c P(\bar{x}) + R_+^n\right) \le \varepsilon.$$

Recall that $DP_r(\bar{x}) \in \partial V(\bar{x})$, and divide (coordinatewise) by $\bar{x} \gg 0$ to get the result. □

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