8 The Determinacy of Equilibria 25 Years Later[1]

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8.1 INTRODUCTION

The paper by G. Debreu, 'Economies with a Finite Set of Equilibria', appeared in *Econometrica* in 1970. As is commonly known, the paper has had since then a great impact on the style and the substance of economic theorizing. If I had been more energetic I would have verified this impact in the Index of Citations. Yet, whatever the latter says, it is bound to be a considerable underestimate of the impact. Often, the sign of the success of a paper is its being referred to only implicitly, without authors even being aware of the fact. In our case, for example, any user of the term 'generic' is indebted to Debreu's paper. At any rate, the research directly inspired by the paper is enormous. By now it includes, at least, two full-length systematic books (Balasko, 1988, and Mas-Colell, 1985).

As a personal note, I would like to add that the paper appeared at a time where it could mark a young researcher like myself most decisively. And it did. As a young mathematical economist in Minnesota I saw that gold had been discovered in California and I did not hesitate to join the rush (where I encountered not only Debreu but also Stephen Smale.)

I would say that the contributions of the 1970 paper were three, the first two of a methodological character and the last substantive:

(1) The paper shaped the basic conceptual apparatus for the analysis of the *determinacy* of equilibrium. To start with, it reinvigorated the traditional desideratum that the equilibrium of an economic model be determinate. But, more important, it gave a precise mathematical expression to the notion of a determinate equilibrium: local uniqueness and robustness (i.e. persistency). It also formulated the quest for determinacy as the desideratum that the property hold not for every but for 'almost every' economy. That is, exceptions are allowed in

rare cases which, by definition and common sense, we think of as pathological. This leads to the notion of a *generic* property, now of standard use in economics.

(2) The paper also proposed that differential topology, in general, and transversality theory, in particular, were the appropriate mathematical tools for the study of the determinacy problem. In spite of the nature of this contribution being more technical, it should be added that these techniques had all the right resonances for economists. Indeed, they can be viewed as the modern version of the counting equations and unknowns, a procedure most familiar to the economist of yore.

(3) Finally, the substantive contribution of the paper was the claim, and the proof, that the classical Walrasian model of the general equilibrium is generically determinate.

In the following pages I limit myself to presenting a few remarks on the two methodological aspects. On the substantive aspect I would say, however, that much as the theory has been clarified and extended (and the frontiers of its validity – narrower than it was first suspected – drawn), I do not believe we have got to the bottom of the matter. I think, for example, that we do not yet understand in a very deep way the contrast, as it concerns the determinacy property, between classical Walrasian theory and game theory.

Let me also note that although I am classifying the methodological innovations of Debreu's paper into two, the reality is that in that paper, as well as in the areas of mathematics that influenced it, the two aspects are intimately connected. So the current exercise definitely belongs to the conceptual splitting variety.

8.2 ON THE CONCEPT OF DETERMINACY

The new framework for the analysis of the determinacy problem has much enriched our box of tools; yet its practical use has revealed some conceptual puzzles. I comment on three of these.

The first has to do with the information content of the term 'local uniqueness'. For an economic model imbedded, mathematically speaking, in a finite dimensional world, and under natural compactness hypotheses, the notions of 'local uniqueness' and of 'finiteness of the number of equilibria' coincide. But in infinite dimensional models (and these have a significant presence in economics), this is typically not so. The question arises: which desideratum should we then privilege, local uniqueness or finiteness?. Intuitively speaking, the first seems more fundamental because it is at the base of comparative static analysis. But there is a problem: the notion of local uniqueness is topological and therefore it depends on the choice of topology. Thus, in an equilibrium model with infinitely many commodities it may depend on the topology on prices. Different natural choices may give different conclusions. Clearly, additional thinking is needed on what we should mean by local uniqueness in these situations and how the appropriate notion is connected with, or emerges from, the comparative-statics analytical needs. At any rate, as long as we are in a finite dimensional context the problem does not arise: there is a standard topology and the finiteness of the number of equilibria implies its local uniqueness.

The second comment has to do, also, with informational content. Suppose that we have the finiteness of the number of equilibria and its local uniqueness. We should still ask: how finite is finite?, or, how local is the local uniqueness? It is, obviously, not the same to have a few equilibria (say 3 or 5) as to have a few thousands. In fact, in the latter case a result on finiteness may be misleading: a great density of equilibria may be better modelled by a continuum than by the notion of a finite set. The conceptualization as a continuum, for example, opens the door to refinement since it allows one to distinguish sizes of continua by means of dimension.

The above is not an abstract issue. It arises in economics. The simplest example (which is trivial but not degenerated) may well be the following. Consider an economy with a number n of states of the world but no insurance possibilities whatsoever (i.e. there are no financial instruments allowing the transfer of wealth across states). For every state of the world the spot markets of these economies have three equilibria. Then, every assignment of spot equilibrium prices to states is an overall equilibrium price vector. The number of such vectors is 3^n, which is a large number if n is not very small! See Mas-Colell (1991) for further elaboration.

My third remark concerns the possibility of a conflict between the desiderata of local uniqueness/finiteness and of robustness, that is, of persistency under small perturbations of parameters.

The robustness desideratum is the expression of a judgment not to take seriously a theory in which predictions are fragile and may change drastically with small misspecifications of the model. However, situations have been encountered (not so much in equilibrium theory as in game theoretical models) where a given theory yields no locally unique equilibrium predictions (not even generically), yet the equilibrium set is robust, at least in the weak sense of upper hemicontinuity (and even generically). The research drive for predictibility as local uniqueness points then towards refinements programmes, that is, to bringing in new equilibrium considerations so as to size down the equilibrium set. We can often do so, but typically we lose along the way any sort of robustness we may have had. We succeed in predicting perhaps even a unique equilibrium, but at the cost that a perturbation of the basic data can easily make it vanish. A researcher committed to the robustness desideratum would then ask if it may not be more honest to predict the occurrence of a large set, but at least a persistent one. Wouldn't this be more convincing than the spurious, fragile precision accomplished by the refinement programme?.

I cannot offer an answer to this conflict, but there is a clear lesson from the discussion: If you are swayed by the judgement underlying the robustness desideratum then you should give up the logic of refinements. If local uniqueness is of the essence of what you are after, then you should give at least a thought to the possibility that the theory needs complete reconstruction from the bottom up.

8.3 ON THE USE OF TRANSVERSALITY METHODS

Thanks to Debreu's 1970 contribution, smoothness becomes once again an acceptable, even respectable, hypothesis in economics. Also, the Implicit Function Theorem and Sard's Theorem (the key technical tools of transversality theory) have become familiar instruments for economic theorists (the second, Sard's theorem, was completely absent from economics before that date).

I will make three comments on these technical aspects. Only the last has any significance.

(1) It is not easy to comprehend, from today's perspective, why differentiability methods became so unpopular in the period that spans from 1950 to 1970. In fact, we still have some remnants of this and one can ocasionally encounter resistance to smoothness hypotheses, or perhaps simply a lingering judgement that an environment of smoothness hypotheses carries with it a whole different order of restrictiveness than, say, an environment of continuous, or of Lipschitz, hypotheses. Yet the research on determinacy (local uniqueness, etc) shows, in my view very persuasively, that this is misguided. With smoothness hypotheses a simple and elegant theory emerges. Without them it is very difficult to prove anything. It is hard to resist the conclusion that from

the vantage of theory a differentiable world (properly interpreted for every model at hand: it may sometimes mean piecewise differentiable) is the world that fits the problem.

(2) In today's theoretical research the appeal to Sard's theorem has become routinized. But it is still technique-intensive. Establishing genericity results is usually hard work, sometimes very hard work. Yet, often, but not always, the result is quite predictable. One soon develops a feel for what is and what is not generic. But, of course, a 'feel' is not the same thing as a metatheorem. It would be nice indeed if there could be a shortcut!

(3) Finally, I will conclude with a historical curiosity or paradox. Debreu introduced into economics the methods of transversality theory, in general, and Sard's Theorem in particular, so as to establish that, generically, Walrasian equilibria are locally unique. However, the use of Sard's theorem is not required for this purpose. Since this is not widely known I would like to offer some explanation.[2]

Consider an exchange economy with L commodities and I consumers. We assume that each consumer has a utility function u_i: $\mathbb{R}^L_{++} \to \mathbb{R}_+$ which is smooth, strictly quasiconcave, strictly increasing, has no critical point, has all sets of the form $u_i^{-1}([\alpha,\infty))$ closed relative to $\mathbb{R}^L$, and gives rise to a differentiable demand function $\varphi_i(p,w_i)$. The description of an economy is completed by a vector $\omega = (\omega_{11}, \ldots, \omega_{L1}, \ldots, \omega_{11} \ldots, \omega_{L1}) \in \mathbb{R}^{LI}_{++}$ of initial endowments for the different consumers. In what follows, utility functions are kept fixed and possible economies are indexed by ω. For every ω we then have an aggregate excess demand function $f(p;\omega) = \sum_{i=1}^{i=I} [\varphi_i(p,p\cdot\omega_i) - \omega_i]$; note for later reference that this function is well defined even if some components of initial endowments are negative; what is required is that $p \gg 0$ and $p\cdot\omega_i > 0$ for every i.

The economy represented by ω (from now on we will just say 'the economy ω') is called *regular* if every price equilibrium of this economy is regular, that is, if whenever $f(p;\omega) = 0$ we have that rank $D_pf(p;\omega) = L - 1$. By the Implicit Function theorem every regular equilibrium is locally isolated. A price equilibrium which is not regular is called *critical*. An economy which is not regular (that is, having at least one critical equilibrium) is called *critical*. We then have:

Theorem (Debreu, 1970): The set of critical economies $C \subset \mathbb{R}^{LI}_{++}$ is null (i.e. it has LI-dimensional Lebesgue measure zero).

Proof: Note that it suffices to show that for every vector of total endowments $\bar{\omega} \in \mathbb{R}^L_{++}$ the set of critical economies $C_{\bar{\omega}} = \{\omega \in C \cap \mathbb{R}^{LI}_{++} : \sum_{i=1}^{i=I} \omega_i = \bar{\omega}\}$ is null in the $L(I\text{-}1)$ dimensional set $E = \{\omega \in \mathbb{R}^{LI}_{++} : \sum_{i=1}^{i=I} \omega_i = \bar{\omega}\}$, that is, it has $L(I\text{-}1)$ Lebesgue measure zero. Therefore from now on we fix the total endowments $\bar{\omega} \in \mathbb{R}^L_{++}$. The proof proceeds then in two steps. In the first, we construct a more convenient parameterization of the space of economies. In the second, the proof is established by means of an argument relying on polynomial functions. There is no appeal to Sard's Theorem.

Step 1

Denote then by U' the relative interior of the Pareto frontier in utility space, that is, $U' = \{ u \in \mathbb{R}^I_{++} : u = (u_1(x_1), \ldots, u_I(x_I))$ where $(x_1, \ldots, x_I)$ is some Pareto optimal allocation of the total endowments $\bar{\omega}$ $\}$. It is well known that under the stated conditions on utility functions and total endowments, U' is a differentiable manifold of dimension $I-1$ (e.g. Mas-Colell, 1985, ch. 5). Normalizing by $p_L = 1$, to every $u \in U'$ we can associate a unique vector of supporting prices $p(u) \in \mathbb{R}^L_{++}$ and a unique Pareto optimal allocation $x(u) \in \mathbb{R}^{LI}$. Denote also $w_i(u) = p(u)\cdot x_i(u) > 0$. The maps $p(\cdot), x(\cdot)$, and $w_i(\cdot)$ are smooth.

Let ω' stand for a $(L-1)$ $(I-1)$ vector of initial endowments which excludes the last good and the last consumer. The set of such vectors is $E' = \{\omega' \in \mathbb{R}^{(L-1)(I-1)}_{++}: \sum_{i=1}^{i=I} \omega'_{\ell i} < \bar{\omega}_\ell$ for $\ell = 1, \ldots, L-1\}$.

For every pair $(u,\omega') \in U' \times E'$ denote then by $\omega(u,\omega') \in \mathbb{R}^{LI}$ the vector where $\omega_{\ell i}(u,\omega') = \omega'_{\ell i}$ for $\ell < L$ and $i < I$, $\omega_{\ell I}(u,\omega') = \bar{\omega}_\ell - \sum_{i=1}^{i=L-1} \omega_{\ell i}$ for $\ell < L$, and $p(u)\cdot\omega_i(u,\omega') = w_i(u) > 0$ for every i. In words: for every consumer i the endowment $\omega_{Li}(u,\omega')$ of the last good is determined so that at the Pareto prices associated with u, $p(u)$, the consumer attains the corresponding level of wealth $w_i(u)$. The map $\omega(\cdot)$ is smooth.

Note that the dimensions of $U' \times E'$ is $I-1 + (L-1)(I-1) = L(I-1)$ and define:

$$C^* = \{(u,\omega') \in U' \times E': \text{rank } D_p f(p(u); \omega(u,\omega')) < L - 1\}.$$

In Step 2 we show that $C^* \subset U' \times E'$ is null, that is, it has $L(I-1)$ Lebesgue measure zero. Let us assume for the moment that this is true.

We claim that $C_{\bar{\omega}} \subset \omega(C^*)$. Indeed, suppose that $f(p;\omega) = 0$, $\omega_{\ell i} > 0$ for every ℓ and i, and $\sum_{i=1}^{i=I} \omega_i = \bar{\omega}$. Then by the first welfare theorem if we let $u_i = u_i(\varphi^i(p,p\cdot\omega_i))$, for every i, and $\omega'_{\ell i} = \omega_{\ell i}$ for $\ell <$

$L-1$ and $i < I-1$, we have that $u = (u_1, \ldots, u_I) \in U'$, $\omega' = (\omega'_{11}, \ldots \omega'_{L-1,1}, \ldots, \omega'_{1,I-1}, \ldots, \omega'_{L-1,I-1}) \in E'$, $\omega = \omega(u,\omega')$, and $p = p(u)$. Thus, $D_p f(p;\omega) = D_p f(p(u);\omega(u,\omega'))$ and, therefore, if rank $Df(p;\omega) < L-1$ (which necessarily occurs for some p if $\omega \in C_{\bar{\omega}}$) it follows that rank $D_p f(p(u);\, \omega(u,\omega')) < L - 1$ and so, that $(u,\omega') \in C^*$ and $\omega = \omega(u,\omega')$. Hence $C_{\bar{\omega}} \subset \omega(C^*)$, as claimed.

Because $\omega(\cdot)$ is differentiable the $L(I-1)$ Lebesgue measure of $\omega(C^*) \cap E$, hence of $C_{\bar{\omega}}$, must be zero.

Step 2

We now show that for every $u \in U'$ the set $C^*_u = \{\omega' \colon (u,\omega') \in C^*\} \subset \mathbb{R}^{(L-1)(I-1)}$ is null, i.e. has $(L-1)(I-1)$ Lebesgue measure zero. Because dim $U' = I-1$, Fubini's Theorem implies that C^* has $L(I-1)$ Lebesgue measure zero, the desired conclusion.

From now on we fix a value of $u \in U'$. Define then a function $g\colon \mathbb{R}^{(L-1)(I-1)} \to \mathbb{R}$ by letting $g(\omega')$ equal the the value of the determinant of the $L - 1 \times L - 1$ northwest submatrix of $D_p f(p(u);\omega(u,\omega'))$. Of course, $C^*_u \subset g^{-1}(0)$. We shall show that $g^{-1}(0)$ is null.

Denote $p = p(u)$, $x_i = x_i(u)$, $w_i = w_i(u)$. Let also $x'_i = (x_{1i}, \ldots, x_{L-1,i})$ and $x' = (x'_1, \ldots, x'_I)$. Then for any $\omega' \in \mathbb{R}^{(L-1)(I-1)}_{++}$ and $\ell < L$ we have

$$f_\ell(p(u),\omega(u,\omega')) = \textstyle\sum_{i=1}^{i=I} [\varphi_{\ell i}(p,w_i) - \omega'_{\ell i}]$$

and so,

$$g(\omega') = \det [\, \textstyle\sum_{i=1}^{i=I}(S_i(p,w_i) - (x'_i - \omega'_i)^T c_i(p,w_i))]$$

where the $S_i(p,w_i)$ are $L-1 \times L-1$ substitution matrices, and the $c_i(p,w_i)$ are $L-1$ row vectors of wealth terms.

Observe that in the above expression neither the $S_i(p,w_i)$ matrices nor the $c_i(p,w_i)$ vectors depend on ω'. Thus the expression tells us that $g(\cdot)$ is a polynomial formula in ω'. Moreover, this polynomial is not identically equal to zero. Indeed, $g(x') = \det [\sum_{i=1}^{i=I} S_i(p,w_i)] \neq 0$, because the matrix $\sum_{i=1}^{i=I} S_i(p,w_i)$ is negative definite (in words: when $\omega'_i = x'_i$ for every i there are no wealth effects terms and the substitution terms aggregate nicely). But for a polynomial that is not identically zero it is always true that $g^{-1}(0)$ is null. Hence, the conclusion. Q.E.D

To conclude, I will say that it is actually very good that Debreu did not proceed in the above manner. It is 'too clever'. It uses more of the economic structure than needed. If something is true for very general, almost purely mathematical, reasons then one should know that this is so. If, in addition, this becomes known to us by means of the use of a new and far-reaching technique, then it is not hard to conclude that by 'being clever' we would in fact be missing something and ending up poorer.

Notes

1. This is a lecture that I delivered at the University of Bonn on 4 July 1991 on the occasion of the seventieth anniversary of G. Debreu (the celebration was organized by W. Hildenbrand). Its content has a substantial overlap with the comments I delivered in Moscow at the Congress of the IAE.
2. The point to be made has also been made in the same way by Balasko (1992). Our research on this topic has been independent. I have arrived at this proof by the exploitation of a key polynomial trick in a manner similar to the one presented in Mas-Colell (1988) (in these lectures I appealed to the 'polynomial trick' used here to show that generically economies have a regular equilibrium). For a previous usage of the polynomial trick, see Mas-Colell (1973).

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