ELSEVIER Journal of Mathematical Economics 26 (1996) 51–62

JOURNAL OF Mathematical ECONOMICS

Self-fulfilling equilibria: An existence theorem for a general state space

Andreu Mas-Colell [a,*], Paulo K. Monteiro [a,b]

[a] *Harvard Institute of Economic Research, Harvard University, Cambridge, MA 02138, USA*
[b] *Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil*

Submitted May 1991; accepted May 1995

Abstract

It is shown that under a weak regularity, a two-period incomplete market economy with a finite number of assets and general (e.g. uncountable) state spaces has an equilibrium. General state spaces are important in applications. The proof proceeds by establishing the existence of self-fulfilling spot price expectations possessing a strong continuity with respect to asset trades.

JEL classification: D50; D52

Keywords: Self-fulfilling equilibria; Existence theorem; General state space

1. Introduction

Self-fulfilled equilibrium models with incomplete markets and general (that is, more than finite or countable) state spaces arise naturally in applications, e.g. in financial economics. It is thus unfortunate that there is no well developed existence theory for this situation. In this paper we shall attempt to make a first contribution towards it.

* Corresponding author: Professor A. Mas-Colell, Department of Economics, Littauer Center 326, Harvard University, Cambridge, MA 02138, USA.

Elsevier Science S.A.
SSDI 0304-4068(95)00742-3

For the case of a countable number of states, existence theorems have been provided by Green and Spear (1987), Zame (1988) and, in a somewhat different context, Hernandez (1988). All their proofs appeal to limiting arguments on the number of states. This approach is not available if the state space is a continuum; say, the interval [0, 1], at least if we do not want to resort to the introduction of extraneous noise in spot price expectations. The same is true for the functional analytical techniques that have been so successful in the treatment of the complete market case. See Mas-Colell (1991) for a discussion of these points.

In this paper we present a result for a two-period model that exploits the expected utility hypothesis on utility functions. It has been noted before that the additive separability of preferences across states facilitates existence arguments (Araujo and Monteiro, 1989a, b). In our case it allows the construction of the equilibrium by a backward recursion.

The treatment of this paper rests on a key regularity hypothesis. Namely, we shall assume that for any possible asset allocation almost every non-atomic state has a regular (i.e. robust) spot equilibrium. This assumption is weak and it allows us to proceed by constructing (self-fulfilled) price expectations functions conditional on asset allocations and having the following property: for almost every non-atomic state spot prices are continuous on asset trades (of course, the exceptional set of states may depend on the asset trade). This property is strong and of interest in itself. It is thus good to know that it is implied by a weak regularity assumption. Nonetheless, we do not claim that the regularity assumption is required for an existence result, only that it is indispensable to obtaining the expectation function with the property described and thus to our line of proof.

2. The model

We consider an exchange economy with N traders and two periods $t = 0, 1$. To focus on essentials, at $t = 0$ there will be trade in assets only. We begin by describing the economy at $t = 1$.

There is a state space $(S, \mathscr{S}, \mu)$, which we take to be a finite measure space. For example, it could be $S = [0, 1]$ with μ Lebesgue measure. We do not exclude the fact that μ may be purely atomic, but the raison d'être of this paper is the possibility that it is not.

There are ℓ physical commodities.

We make the unusual (but see Balasko, 1988) hypothesis of taking the consumption set to be the entire R^ℓ. This is not as strong as it may appear because we can allow the disutility of negative consumption to be large. At any rate, the hypothesis is not of the essence although it facilitates considerably the display of the logical structure of the model.

We let $\mathscr{U}$ be the space of utility functions $u: R^\ell \to R$, which are of class C^2, strictly monotone, differentiably strictly concave (that is, $\partial^2 u(z)$ is negative definite for all z) and has $\{z: u(z) \geq \bar{u}\}$ bounded below for all $\bar{u}$. The latter requirement substitutes for the lack of a lower bound in the consumption set. The space U is endowed with the metric of C^2 uniform convergence on compacta.

The economy at $t = 1$ is specified by a map $\mathscr{E}: S \to (\mathscr{U} \times R^{\ell})^N$, which we assume to be (Borel) measurable and satisfying the boundedness condition: "There is a compact set $K \subset \mathscr{U} \times R^{\ell}$ such that $\mathscr{E}_i(s) = (u_i(\cdot, s)\ \omega_i(s)) \in K$ for all i and a.e. $s \in S$."

At $t = 0$ there are $J < \infty$ assets. Each asset is specified by a (measurable) return function $r_j: S \to R^{\ell}$, which we assume to be bounded (i.e. there is α such that $\| r_j(s) \| \leq \alpha$ for a.e. $s \in S$). Our assets have, therefore, real returns. In Section 6 we will comment on the monetary returns case. We denote $r = (r_1, \ldots, r_J)$. We also assume that $r_1(s) \geq 0$ for a.e. $s \in S$ and $\mu(\{s: r_1(s) \neq 0\}) > 0$.

Formally there are no initial endowments of assets. We will, however, constrain short sales by having a lower bound $\hat{z}_i \in R^J$, $\hat{z}_i \ll 0$, for each i. This is as in Radner (1972), and it is necessitated in view of Hart's (1975) counterexamples.

A *self-fulfilled* (or *rational*) *expectations equilibrium* for the economy is defined in the usual way as a four-tuple $(q, \bar{\theta}, p, \bar{x})$ where:

(a) $q \in R^J$, $q \neq 0$ is an asset price vector;

(b) $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_N) \in R^{JN}$ is a vector of asset portfolios such that $\Sigma_i \bar{\theta}_i = 0$ and, for every i, $q \cdot \bar{\theta}_i \leq 0$, $\bar{\theta}_i \geq \hat{z}_i$;

(c) $p: S \to R^{\ell}$ is a non-zero, measurable, spot commodity price function;

(d) $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N)$ where $\bar{x}_i: S \to R^{\ell}$ is a (measurable) function, $\Sigma_i \bar{x}_i(s) = \Sigma_i \omega_i(s)$ for a.e. s, and for every i, $(\bar{x}_i, \bar{\theta}_i)$ solves:

$$\max \int u_i(x_i(s), s)\, \mathrm{d}\mu(s),$$

$$\text{s.t. } p(s) \cdot x_i(s) \leq p(s) \cdot \left(\omega_i(s) + \sum_j \theta_{ij} r_j(s) \right), \quad \text{for a.e. } s,$$

$$q \cdot \theta_i \leq 0, \qquad \theta_i \geq \hat{z}_i.$$

3. The main hypothesis and a theorem

For any vector of asset portfolios $\theta \in R^{JN}$, and for any $s \in S$ we have a spot exchange economy defined by $\mathscr{E}_{\theta,i}(s) = (u_i(\cdot, s), \omega_i(s) + \Sigma_j \theta_{ij} r_j(s))$. Denote by $P(\theta, s) \subset R^{\ell}$ the set of spot prices that are Walrasian equilibria for the exchange economy $\mathscr{E}_\theta(s)$. Under our hypothesis, $P(\theta, s) \neq \emptyset$ (in order to apply Debreu, 1959, one needs to specify a lower bound for the consumption set, but this can be done by the condition imposed in the utility function).

Denote by $f_{\theta,s}: R^{\ell}_{++} \to R^{\ell}$ the C^1 excess demand function associated with $\mathscr{E}_\theta(s)$. Of course, $p \in P(\theta, s)$ if and only if $f_{\theta,s}(p) = 0$. A price equilibrium $p \in P(\theta, s)$ is called *regular* if rank $\partial f_{\theta,s}(p) = \ell - 1$. This is a standard definition; see, for example, Mas-Colell (1985, Chapter 5).

It would be too strong to require that for every θ and s, every $p \in P(\theta, s)$ be regular or even the substantially weaker condition that there is a $p \in P(\theta, s)$ that is regular (see Mas-Colell and Nachbar, 1991, for variations on this theme). On the other hand, if μ is atomless, then we view as weak the requirement that, for every θ, the equilibrium set $P(\theta, s)$ contains at least one regular equilibrium for a.e. s. The key fact is not so much that regularity be demanded of one or all equilibrium prices, but that allowance be made for the existence of a small set of exceptional states (which may depend on θ). If μ contains atoms, then the requirement needs to be made only on the atomless part.

Let T be the set of atoms of S (i.e. $t \in T$ if and only if $\mu(\{t\}) > 0$). Then the hypothesis is

($\mathbb{R}$) For all θ, $P(\theta, s)$ contains a regular equilibrium for μ – a.e. $s \in S \setminus T$.

By Debreu (1970), the hypothesis will be satisfied if, for example, utilities are state-independent (i.e. $u_i(\cdot, s) = u_i(\cdot, s')$ for all $s, s' \in S$ and i), and the measure on $R^{\ell N}$ induced by $\mu | (S \setminus T)$ through the ex-post endowment map $s \mapsto (\ldots, \omega_i(s) + \Sigma_j \theta_{ij} r_j(s), \ldots)$; that is, the distribution of ex-post endowments is, for every θ, absolutely continuous with respect to Lebesgue measure.

This hypothesis can certainly be violated but, to repeat, we view this circumstance as highly pathological. Thus, if for every θ, $P(\theta, \cdot)$ is as in Fig. 1(a) or 1(b), then it holds. If for some θ, $P(\theta, \cdot)$ is as in Fig. 1(c), then it does not. Be that as it may, we have the following.

Theorem. Assume hypothesis $\mathbb{R}$. *Then a self-fulfilled expectations equilibrium exists.*

We do not claim that $\mathbb{R}$ is indispensable to establish the result, only that it is weak, that it is the key to the method of proof, and that, as the following section will make clear, it confers a kind of expectational robustness to the equilibrium.

Note also that hypothesis $\mathbb{R}$ is not really required on any positive measure subset of S with the property that all of its states generate the same preferences, endowments and asset returns. The reason is that, without loss of generality, we can identify all of these states and treat them as an atom.

4. Pointwise continuous selections

This section contains the heart of our approach. To save on notation we assume that μ is atomless.

Suppose we identify the state of the economy at $t = 0$ with the vector of portfolios $\theta = (\theta_1, \ldots, \theta_N)$. Any (measurable) function $\mathbf{p}: S \to R^l$ such that $\mathbf{p}(s) \in P(\theta, s)$ for a.e. s can be viewed as a consistent (i.e. realizable) expectation function conditional on θ. Because $P(\theta, s) \neq \emptyset$ for all θ, s such expectation functions exist. In this section we show that under hypothesis $\mathbb{R}$ we can, in addition, choose $\mathbf{p}(s)$ to depend continuously on θ where continuity has a strong meaning: if $\theta_n \to \theta$, then $\mathbf{p}_n(s) \to \mathbf{p}(s)$ for a.e. s. This is stated precisely as follows.

Proposition. Assume, first, the regularity hypothesis $\mathbb{R}$, *and, second, that* μ *is atomless. Then there is a measurable* $\mathbf{p}: R^{JN} \times S \to R^{\ell}_{++}$ *such that*

(a) *for every* θ, $\mathbf{p}(\theta, s) \in P(\theta, s)$ *for a.e.* s;

(b) $\theta \mapsto \mathbf{p}(\theta, \cdot)$ *is pointwise a.e. continuous in* S, *i.e.* $\theta_n \to \theta$ *implies* $\mathbf{p}(\theta_n, s) \to \mathbf{p}(\theta, s)$ *for a.e.* s.

To prove the proposition, the following mathematical lemma is fundamental.

Lemma 1. Suppose that A *and* B *are separable metric spaces and* G *is a topological space. Suppose also that* ν *is an atomless Borel measure in* A *and that we are given:*

(1) *a Borel set* $C \subset A \times B$ *with the property that for all* $b \in B$, $\nu\{a \in A: (a, b) \notin C\} = 0$;

(2) *for every* $c \in C$ *an open set* $V_c \subset A \times B$, *with* $c \in V_c$, *and a continuous function* $f_c: V_c \to G$.

Then there is a Borel set $D \subset C$ *and a continuous function* $f: D \to G$ *such that*:

(1′) *for all* $b \in B$, $\nu\{a \in A: (a, b) \notin D\} = 0$;

(2′) *for all* $d \in D$, *there is* $c \in C$ *such that* $d \in V_c$ *and* $f(d) = f_c(d)$.

Proof. For each $a \in A$ let $g_a: A \to [0, 1]$ be a continuous function such that, first, $\nu(g_a^{-1}(r)) = 0$ for any r and, second, for any open set $a \in U$ there is $r > 0$ with $g_a^{-1}([0, r]) \subset U$ (of course, $g_a^{-1}(0) = \{a\}$). The existence of these functions is proved in the appendix.

Without loss of generality we can suppose that for each $c = (a, b) \in C$ there is $\epsilon(c) > 0$ such that $V_c = \{(d', b') \in A \times B: g_a(d') + \rho(b', b) < \epsilon(c)\}$ where, of course, ρ is the metric in B.

Because $\{V_c: c \in C\}$ covers C, which is a separable metric subspace of $A \times B$, we can find a countable subcover $c_n = (a_n, b_n)$, $V_{c_n} = V_n$. Define $D = C \setminus \cup_n \partial V_n$, where ∂ stands for boundary, and $f: D \to G$ by $f(d) = f_{n(d)}(d)$ where $n(d)$ is chosen to be the smallest n such that $d \in V_n$.

The continuity of f follows from the fact that $n(d)$ is locally constant (indeed, $V_{n(d)}$ is open and $d \notin \cup_{m < n(d)}$ closure V_m). Property (2′) is, of course, obvious. To verify (1′) fix $b \in B$ and note that

$$\{a \in A: (a, b) \notin D\} = \{a \in A: (a, b) \notin C\} \cup \bigcup_n \{a \in A: (a, b) \in C \cap \partial V_n\}$$

$$= \{a \in A : (a, b) \notin C\} \cup \bigcup_n \{a \in A : g_{a_n}(a) + \rho(b, b_n) = \epsilon(c_n)\}$$

$$\subseteq \{a \in A : (a, b) \notin C\} \cup \bigcup_n g_{a_n}^{-1}(\epsilon(c_n) - \rho(b, b_n)).$$

Because each of these sets has measure zero conclusion (1′) is obtained. □

Proof of the proposition. Let $A = (\mathscr{U} \times R^{\ell})^N \times R^{\ell J}$, $B = R^{JN}$. To every $(a, b) = ((u_1, w_1), \ldots, (u_n, w_n)), r_1, \ldots, r_J, \theta) \in A \times B$ we assign the exchange economy $\mathscr{E}_i(a, b) = (u_i, w_i + \Sigma_j \theta_{ij} r_j)$. Denote $C = \{c \in A \times B : \mathscr{E}(c)$ has a regular equilibrium$\}$, and for every $c \in C$ let $f_c : V_c \to R^{\ell}_{++}$, $c \in V_c$, be a continuous function such that, for all c', $f_c(c')$ is a price equilibrium for $\mathscr{E}(c')$. Such a function exists because of the regularity condition on c.

We can write μ, which recall we assume is atomless, as $\mu = \mu' + \mu''$ where μ', μ'' are mutually singular, $\mu' \circ (\mathscr{E}, r)^{-1}$ is an atomless measure on A and $\mu'' \circ (\mathscr{E}, r)^{-1}$ is purely atomic. Put $S = S' \cup S''$, $\mu'(S'') = \mu''(S') = 0$. In order to define $\mathbf{p}(\theta, s)$ we need a separate argument for $s \in S'$ and for $s \in S''$.

Denote $\nu = \mu' \circ (\mathscr{E}, r)^{-1}$. Then ν is atomless and by hypothesis ℝ, the conditions of Lemma 1 are satisfied. Let D and f satisfy its conclusion. Then, given θ, we have that $(\mathscr{E}(s), r(s), \theta) \in D$ for a.e. $s \in S'$, and therefore we can define $\mathbf{p}(\theta, s) = f(\mathscr{E}(s), r(s), \theta)$. As long as $s \in S'$ this satisfies the desired properties.

For S'' let us assume, without loss of generality, that $\mu(S'') > 0$ and $(\mathscr{E}(s), r(s)) = \bar{a}$ for all $s \in S''$. By hypothesis ℝ, for every θ the economy $\mathscr{E}(\theta) = \mathscr{E}(\bar{a}, \theta)$ has a regular equilibrium. It follows that there is a countable, locally finite, open covering of R^{JN}, $\{V_n\}$ such that for every n we can find a continuous function $f_n : V_n \to R^{\ell}_{++}$ with the property that $f_n(\theta)$, $\theta \in V_n$, is an equilibrium price vector for $\mathscr{E}(\theta)$. Take now a (continuous) partition of unity $\phi_n : R^{JN} \to [0, 1]$ subordinated to $\{V_n\}$ and a measurable function $\xi : S'' \to (0, 1)$ such that $\mu(\xi^{-1}(r)) = 0$ for every r. It is not difficult to see that such a function exists (for example, if we could take $S = [0, 1]$, then the identity would do). Finally, given θ and $s \in S''$ define $\mathbf{p}(\theta, s) = f_{n(s)}(\theta)$ where $n(s)$ is chosen to be the smallest n such that $\xi(s) < \Sigma_{h=1}^{n} \phi_n(\theta)$. Note that, defined this way, we necessarily have $\theta \in V_{n(s)}$. Also, $\mathbf{p}(\cdot, s)$ will fail to be continuous at θ at most at the null set of points $\{s : \xi(s) = \Sigma_{h=1}^{n} \phi_n(\theta)$ for some $n\}$. □

We remark that for the purposes of this paper we do not require the full power of the proposition. Indeed, we could sidestep the need to obtain a selection in the set S'' of the previous proof (i.e. in the inverse image of the set of atoms of $\mu \circ (\mathscr{E}, r)^{-1}$). The reason was explained at the end of Section 3. We could identify states giving rise to the same preferences, endowments and returns. In this manner, the inverse image of the set of atoms of $\mu \circ (\mathscr{E}, r)^{-1}$ would coincide with the atoms of μ which, as we shall see in the next section, can be handled without difficulty and, more importantly, without extra restriction.

5. Proof of the theorem

Once we can resort to the selection result given to us by the proposition, the proof of the theorem will follow from more or less traditional arguments.

To begin with we take care of a preliminary task.

Lemma 2. For any compact $K \subset \mathscr{U} \times R^{\ell}$ *there is* $\alpha > 0$ *such that for any spot economy with* $(u_i, w_i) \in K$ *for every* i, *we have that* $\| x_i \| < \alpha$, *for all* i, *whenever* $\Sigma_i x_i \leq \Sigma_i w_i$ *and* $u_i(x_i) \geq u_i(w_i)$ *for all* i.

Proof. A proof is required only because of the unboundedness below of consumption sets. In contrast to the usual proof for the standard case, we will now need to exploit the convexity of preferences. It is enough to consider a sequence of economies $(u_{im}, w_{im}) \in K^N$ and $x_m \in R^{\ell N}$, such that $\Sigma_i x_{im} \leq \Sigma_i w_{im}$ and $u_{im}(x_{im}) \geq u_{im}(w_{im})$ for all i.

We can assume that $(u_{im}, w_{im}) \to (u_i, w_i)$, and $(1/\| x_{im} \|) x_{im} \to v_i$. We show first that $v_i \geq 0$ for all i. Indeed, let $\gamma > 0$ be arbitrary. Then if m is large enough we have

$$u_{im}\left(\frac{\gamma}{\| x_{im} \|} x_{im} + \left(1 - \frac{\gamma}{\| x_{im} \|} \right) w_{im} \right) \geq u_{im}(w_{im}),$$

because of the convexity of preferences. Thus, $u_i(\gamma v_i) \geq u_i(w_i)$ for all $\gamma > 0$. Since the set $\{ z : u_i(z) \geq u_i(w_i) \}$ is bounded below it follows that $v_i \geq 0$.

Relabeling, and taking a subsequence if necessary, we can suppose that $\| x_{1m} \| \geq \| x_{im} \|$ for all i, m. Take h with $v_1^h > 0$. Then $x_{1,m}^h > (v_i^h/2) \| x_m^1 \|$ for m large. Since $\Sigma_i x_{im} \leq \Sigma_i w_{im}$ is bounded, there is some $\gamma > 0$ such that

$$\sum_{i \neq 1} x_{im}^h \leq \gamma - \frac{v_1^h}{2} \| x_{1m} \|,$$

for m large enough. Hence, for some $i \neq 1$:

$$x_{i,m}^h \leq \frac{1}{N-1} \left(\gamma - \frac{v_1^h}{2} \| x_{1,m} \| \right).$$

Without loss of generality (take a subsequence if necessary) we can assume that i is independent of m. Then

$$\frac{(N-1)}{\| x_{i,m} \|} x_{i,m}^h \leq \frac{\| x_{1,m} \|}{\| x_{i,m} \|} \left(\frac{\gamma}{\| x_{1,m} \|} - \frac{v_1^h}{2} \right),$$

and so, if $\| x_{1,m} \| \to \infty$ then $0 \leq (N-1) v_i^h \leq -(v_1^h/2)$, which is a contradiction. Therefore, $\| x_{1,m} \|$ is bounded. □

For the sake of clarity we consider first the case where μ is atomless.

Denote by $\mathbf{p}(\theta, s)$ the self-fulfilled price expectation given by the proposition. This section $\mathbf{p}$ remains fixed for the rest of the proof. We normalize to $\|\mathbf{p}(\theta, s)\| = 1$.

Take $\gamma > -\Sigma_i \hat{z}_i^j$, for all j and denote $Y_i = \{\theta_i \in R^J : \hat{z}_i^j \leq \theta_{ij} \leq \gamma, \text{ all } j\}$, $Y = \Pi_i Y_i$. Let $B = \{q \in R_+^J : \| q \| \leq 1\}$. Pick a $\alpha > 0$ as in Lemma 2 where K is chosen to include every utility function $u_i(\cdot, s)$ and every endowment $\omega_i(s) + \Sigma_j \theta_{ij} r_j(s)$, $\theta_i \in Y_i$.

For every i we can define a kind of constrained asset demand correspondence $f_i : B \times Y \to Y_i$ by letting $f_i(q, \theta)$ be the solutions $\hat{\theta}_i$ to the problem:

$$\max \int u_i(x_i(s), s) \, \mathrm{d}\mu(s)$$

$$\text{s.t.} \quad p(\theta, s) \cdot (x_i(s) - \omega_i(s)) \leq \sum_j \hat{\theta}_{ij} p(\theta, s) \cdot r_j(s)$$

$$\text{and } \| x_i(s) \| \leq \alpha, \text{ for a.e. } s$$

$$\text{and } q \cdot \hat{\theta}_i \leq \frac{1 - \| q \|}{\| q \|}, \; \hat{\theta}_i \in Y_i.$$

Solutions to this problem exist. Moreover, f_i will be an upper hemicontinuous correspondence. It is obviously convex-valued.

Finally, define a fixed point map $G: B \times Y \to B \times Y$ by letting

$$G(q, \theta) = G_q(q, \theta) \times G_2(q, \theta),$$

where

$$G_1(q, \theta) = \left\{ \hat{q} \in B : \hat{q} \cdot \sum_i \theta_i \geq q' \cdot \sum_i \theta_i \text{ for all } q' \in B \right\},$$

$$G_2(q, \theta) = \prod_i f_i(q, \theta).$$

By Kakutani's fixed point theorem there is $(\bar{q}, \bar{\theta}) \in G(\bar{q}, \bar{\theta})$. Suppose that $\bar{q} \cdot \Sigma_i \bar{\theta}_i > 0$. Then $\| \bar{q} \| = 1$ and since $\bar{\theta}_i \in f_i(\bar{q}, \bar{\theta})$ we get $\bar{q} \cdot \bar{\theta}_i \leq 0$ for all i. Contradiction. Hence, $0 \geq \bar{q} \cdot \Sigma_i \bar{\theta}_i \geq q' \cdot \Sigma_i \bar{\theta}_i$ for all $q' \in B$, which yields $\Sigma_i \bar{\theta}_i = 0$. But then $\bar{\theta}_{ij} \leq -\Sigma \hat{z}_i^j < \gamma$ for all j and, therefore, $\bar{\theta}_i$ solves the (convex) utility maximization problem with the constraint $\hat{\theta}_i \in Y_i$ replaced by $\hat{\theta}_i \geq \hat{z}_i$. For every s, let $\bar{x}(s)$ maximize $u_i(x(s), s)$ subject to $p(\bar{\theta}, s) \cdot (x_i(s) - \omega_i(s)) \leq \Sigma_j \bar{\theta}_{ij} p(\bar{\theta}, s) \cdot r_j(s)$. It is now easy to see that $(\bar{q}, \bar{\theta}, p, x)$ constitutes an equilibrium. Note simply that in the expression for the utility maximization problem we have $\hat{\theta} = \bar{\theta} = \theta$ and, therefore, by construction of p, $\Sigma_i \bar{x}_i(s) \leq \Sigma_i \omega_i(s)$ for a.e. s. By Lemma 2 this implies $\| \bar{x}_i(s) \| \leq \alpha$ for a.e. s and, therefore, θ solves the utility maximization problem with the constraints $\| x_i(s) \| \leq \alpha$ removed. The desirability condition on asset 1 will guarantee that $\bar{q}^1 > 0$ and $\bar{q} \cdot \bar{\theta}_i = (1 - \| \bar{q} \|)/\| \bar{q} \|$ for all i. Because $\Sigma_i \bar{\theta}_i = 0$ this yields $\| \bar{q} \| = 1$.

We now study the general case; that is, the case where the atomless part of μ, denoted μ_1, may be different from μ. In a sense the following proof will just be a juxtaposition, i.e. a product, of the previous proof for the atomless part and of the standard proofs (e.g. Green and Spear, 1987; Zame, 1988) for the atomic part.

Denote by $p(\theta, s)$ the price selection of the proposition, taken relative to μ_1.

There are, at most, a countable number of atoms. Therefore, we can number them s_m, $m = 1,\ldots,\infty$. As above, take $\gamma > -\Sigma_i \hat{z}_i^j$ for all j and define the sets Y_i, Y accordingly.

Let $\Delta = \{p \in R_+^{\ell} : \Sigma_i p^h = 1\}$ and take a constant α from Lemma 2 as above. Denote $H = \{x \in R^{\ell} : \| x \| \leq 2\alpha\}$.

Define now the constrained demand correspondence $f_i : B \times Y \times \Delta^\infty \to Y_i \times H^\infty$ by letting $f_i(q, \theta, p_1,\ldots)$ be the solutions $(\hat{\theta}, x_{i1},\ldots)$ to the problem:

$$\max \quad \int u_i(x_i(s), s)\, \mathrm{d}\mu_1(s) + \sum_{m=1}^{\infty} u_i(x_{im}, s_m)\mu(s_m),$$

$$\text{s.t.} \quad p(\theta, s)\cdot(x_i(s) - \omega_i(s)) \leq \sum_j \hat{\theta}_{ij} p(\theta, s) r_j(s)$$

$$\text{and } x_i(s) \in H, \text{ for } \mu_1 - \text{a.e. } s,$$

$$p_m \cdot \Big(x_{im} - \omega_i(sm_)\Big) \leq \sum_j \hat{\theta}_{ij} p_m r_j(s_m)$$

$$\text{and } x_{im} \in H \text{ for } m = 1,\ldots,\infty,$$

$$\text{and } q\cdot\hat{\theta}_i \leq \frac{1 - \| q \|}{\| q \|},\ \hat{\theta}_i \in Y_i.$$

Again, every f_i is non-empty, convex-valued and upper hemicontinuous when Δ^∞, H^∞ are endowed with the product topologies. Because the x_{im} solutions are unique we abuse notation slightly and write:

$$f_i(q, \theta, p_1,\ldots) = f_{i,0}(\cdot) \times \prod_{m=1}^{\infty} f_{i,m}(\cdot).$$

Define next the fixed point map $G : B \times Y \times \Delta^\infty \to \Delta \times Y \times \Delta^\infty$ by letting

$$G(q, \theta, p_1,\ldots) = G_1(\cdots) \times G_2(\cdots) \times G_3(\cdots),$$

where

$$G_1(q, \theta, p_1,\ldots) = \Big\{\hat{q} \in B : \hat{q} :\ \sum_i \theta_i \geq q' \cdot \sum_i \theta_i \text{ for all } q' \in B\Big\},$$

$$G_2(q, \theta, p_1,\ldots) = \prod_i f_{i0}(q, \theta, p_1,\ldots),$$

$$G_3(q, \theta, p_1, \ldots) = \prod_{m=1}^{\infty} \Big\{ \hat{p}_m \in \Delta : (\hat{p}_m - p'_m) \cdot \sum_i \Big(f_{im}(q, \theta, p_1, \ldots) - \omega_i(s_m) - \sum_j \theta_{ij} r_j(s_m) \Big), \text{ all } p'_m \in \Delta \Big\}.$$

It is a routine matter to verify that if $(\bar{q}, \bar{\theta}, \bar{p}_1, \ldots)$ is a fixed point of this problem (which exists by the infinite dimensional version of Kakutani's theorem) then $\bar{q}$, $\bar{\theta}$ and the price expectations $p(\bar{\theta}, s)$, $\bar{p}_1, \ldots, \bar{p}_m, \ldots$ constitute a self-fulfilling equilibrium. This ends the proof of the theorem. □

6. Variations and extensions

To allow for consumption at $t = 0$ presents no special difficulty. Allowing for a lower bound on the consumption set at $t = 1$ should be possible although not a routine matter. The problem is that then $P(\theta, s) = \emptyset$ is a possibility. For the sake of clarity in the formalization of expectations we have preferred to stick with a consumption set equal to the entire R^{ℓ}.

The bounds on short sales of assets are required for the usual reasons. Otherwise, we have to deal with the Hart (1975) counterexamples. We could replace the short sales bound by restrictions on asset returns, such as having the returns in nominal money (the financial assets of Cass, 1984; Duffie, 1987; and Werner, 1985), or in a single commodity, (the real numeraire assets of Geanakoplos and Polemarchakis, 1986). In fact, in the financial asset case we could obtain existence with an even weaker regularity assumption (regularity for a.e. s at $\theta = 0$).

By far the most challenging extension is to allow for several periods with asset retrading in every period.

We emphasize once more that the additive separability across states is essential to our approach. Fortunately, it is also an hypothesis that makes eminent economic sense.

Acknowledgements

Paulo K. Monteiro acknowledges financial support from the CNP$_q$, Brazil.

Appendix

In this appendix we prove a purely mathematical theorem that has been used in the proof of Lemma 1 in Section 4. We do not claim originality, but we were unable to find a reference and the proof is not so trivial that it could simply be skipped. The key claim of the following theorem is the first

Theorem. Suppose T is a first countable normal Hausdorff topological space. Suppose that μ is an atomless, inner regular, σ-finite Borel measure on T. Then for any $\bar{t} \in T$ there is $g \in C_b(T)$, $g \geq 0$, such that:

(i) $\mu(g^{-1}(r)) = 0$ *for all* $r \in R$, *i.e.* $\mu \circ g^{-1}$ *is an atomless measure on* r;

(ii) *for any open neighborhood* $t \in V$ *there is* $r > 0$ *such that* $g^{-1}([0, r]) \subset V$.

Proof. The proof will be by a category argument and therefore non-constructive. We first reduce the problem to the case where μ is finite. Let A_n be measurable sets with $0 < \mu(A_n) < \infty$ and $\mu(T \setminus \bigcup_n A_n) = 0$. Define then μ' by $\mu'(A) = \Sigma_n (1/2^n)\mu(A \cap A_n)/\mu(A_n)$. It is immediately verified that μ' is atomless, inner regular and finite. Moreover, $\mu'(A) = 0$ if and only if $\mu(A) = 0$. Therefore, without loss of generality, we assume from now on that $\mu(T) = 1$.

Fix $\bar{t} \in T$ and a countable basis $\{V_n\}$ of open neighborhoods of $\bar{t}$. Let $C_b(T)$ be the space of continuous, bounded function on T endowed with the sup norm $\| \cdot \|$. Define $G = \{g \in C_b(T)\colon g \geq 0,\ g^{-1}([0, 1/2^{n+1}) \subset V_n \text{ for all } n\}$. It is easily checked that G is a non-empty, closed subset of $C_b(T)$ and, therefore, a Baire space. Note also that $g^{-1}(0) = \bar{t}$ for all $g \in G$.

For each integer m define $G_m = \{g \in G\colon \mu(g^{-1}(r)) \geq 1/m$ for some $r\}$. We shall prove that G_m is closed and has an empty interior. This will conclude the proof since, by Baire's category theorem, the set $G \setminus \bigcup_m G_m$ will then be non-empty and any g in this set will be as desired.

(a) G_m is closed.

Let $g_n \in G_m$, $g_n \to g \in G$. For each n take r_n, $t_n \in g^{-1}(r_n)$, $\mu(g_n^{-1}(r_n)) \geq (1/m)$. Then $r_n = g_n(t_n) \leq \| g_n - g \| + \| g \|$. Hence, the sequence r_n is bounded and, without loss of generality, we can assume that $r_n \to r$. For all $\epsilon > 0$ if n is large enough, we have $g_n^{-1}(r_n) \subset g^{-1}([r - \epsilon, r + \epsilon])$. Therefore, $\mu(g^{-1}([r - \epsilon, r + \epsilon])) \geq \mu(g_n^{-1}(r_n)) \geq \langle 1/m \rangle$ and then $\mu(g^{-1}(r)) = \lim_{\epsilon \to 0} \mu(g^{-1}([r - \epsilon, r + \epsilon])) \geq 1/m$. Thus, $g \in G_m$.

(b) The interior (relative to G) of G_m is empty.

Fix $g \in G_m$. The set $\{r\colon \mu(g^{-1}(r)) \geq (1/m)\}$ is finite. Take one element $\bar{r}$ in the set and consider a $\delta > 0$ such that $\mu(g^{-1}([\bar{r} - \delta, \bar{r} + \delta])) - \mu(g_n^{-1}(\bar{r})) < 1/2m$. We show that for any $\epsilon > 0$ there is a g' such that $\| g - g' \| bl\epsilon$, $\mu(g'^{-1}(r)) < 1/m$ for $r \in [\bar{r} - \delta, \bar{r} + \delta]$ and $g'^{-1}(r) = g^{-1}(r)$ for $r \notin [\bar{r} - \delta, \bar{r} + \delta]$. By iterating the construction this clearly establishes the claim. Fix $\epsilon > 0$. Without loss of generality we can assume that $\delta < \epsilon/2$.

Let $C_1, \ldots, C_{4m} \subset g^{-1}(\bar{r})$ be pairwise disjoint closed sets such that $\mu(C_j) < 1/4m$ for all $j \leq 4m$ and $\mu(g^{-1}(\bar{r}) \setminus \bigcup_j C_j) < 1/4m$. Define g' first in the set $g^{-1}(R \setminus (\bar{r} - \delta, \bar{r} + \delta)) \cup \bigcup_j C_j$ by $g'(t) = g(t)$ if $g(t) \notin (\bar{r} - \delta, \bar{r} + \delta)$, and $g'(t) = \bar{r} - \delta + (j\delta/2m)$ if $t \in C_j$. By Tietze extension theorem we can then extend g' continuously to the set $g^{-1}((\bar{r} - \delta, \bar{r} + \delta))$ and in such a way that

$g'(t) \in [\bar{r} - \delta, \bar{r} + \delta]$ for all t in this region. Clearly, then, $\| g - g' \| \leq 2\delta < \epsilon$ and for any $r \in (\bar{r} - \delta, \bar{r} + \delta)$ we have $\mu(g'^{-1}(r)) \leq \mu(g^{-1}[\bar{r} - \delta, \bar{r} + \delta]) \setminus g^{-1}(r)) + \mu(g^{-1} \setminus \bigcup_j C_j) + \max_j \mu(C_j) < (1/2m) + (1/4m) + (1/4m) = 1/m$.

This concludes the proof. □

References

Araujo, A.P. and P.K. Monteiro, 1989a, Equilibrium without uniform conditions, Journal of Economic Theory 48, no. 2, 416–427.

Araujo, A.P. and P.K. Monteiro, 1989b, General equilibrium with infinitely many goods: The case of separable utilities, to be published in: M. Majumdar ed., Essays in honor of David Gale, (MacMillan, New York).

Balasko, Y., 1988, Foundations of the theory of general equilibrium (Academic Press, New York).

Cass, D., 1984, Competitive equilibrium with incomplete financial markets, CARESS Working Paper no. 84–09.

Debreu, G., 1970, Economies with a finite set of equilibria, Econometrica 38, 387–392.

Geanakoplos, J. and H. Polemarchakis, 1986, Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete, in: W. Heller, R. Starr and D. Starret, eds., Uncertainty, information and communication (Cambridge University Press, Cambridge, UK).

Green, R. and S. Spear, 1987, Equilibria in large commodity spaces with incomplete financial markets, Working Paper, Carnegie-Mellon University.

Hart, O., 1975, On the optimality of equilibrium when the market structure is incomplete, Journal of Economic Theory 11, 418–443.

Hernandez, A., 1988, Existence of equilibrium with borrowing constraints, Ph.D. Dissertation, Chapter II, University of Rochester.

Mas-Colell, A., 1985, The theory of general economic equilibrium: A differentiable approach (Cambridge University Press, Cambridge, UK).

Mas-Colell, A., 1991, Comments to session on Infinite dimensional general equilibrium, to be published in: J.J. Laffont, ed., Advances in economic theory, 5th World Congress (Cambridge University Press, Cambridge, UK).

Mas-Colell, A. and J. Nachbar, 1991, On the finiteness of the number of critical equilibria, with an application to random selections, Journal of Mathematical Economics

Radner, R., 1972, Existence of equilibrium of plans, prices and price expectations in a sequence of markets, Econometrica 40, 289–303.

Werner, J., 1985, Equilibrium in economies with incomplete financial markets, Journal of Economic Theory 36, 110–119.

Zame, W.R., 1988, Asymptotic behavior of asset markets, I: Asymptotic inefficiency, Working Paper, SUNY at Buffalo.