Bargaining and Value

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BARGAINING AND VALUE[1]

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We present and analyze a model of noncooperative bargaining among n participants, applied to situations describable as games in coalitional form. This leads to a unified solution theory for such games that has as special cases the Shapley value in the transferable utility (TU) case, the Nash bargaining solution in the pure bargaining case, and the recently introduced Maschler–Owen consistent value in the general nontransferable utility (NTU) case. Moreover, we show that any variation (in a certain class) of our bargaining procedure which generates the Shapley value in the TU setup must yield the consistent value in the general NTU setup.

KEYWORDS: n-person bargaining, coalitional games, noncooperative implementation, Shapley value, NTU-value, consistent value.

1. INTRODUCTION

IN THIS PAPER WE CONSIDER THE PROBLEM of distributing the benefits or costs of a cooperative endeavor among n participants. In doing so it is important to permit this distribution to be influenced by the possibility of partial cooperation, that is by the possibility that the final outcome of the cooperative process involves only a subgroup of the players. To capture this effect we describe the cooperative situation by means of a game in coalitional form.

When utility is transferable across players (the *TU case*), cooperative game theory has generated (axiomatically) a well established solution concept: the *Shapley* (1953b) *value*. For n-person *pure bargaining* models (that is, problems where the only possible final outcomes are either the complete cooperation of all players or the complete breakdown of cooperation), with or without transferable utility, a central solution concept suggested by axiomatic cooperative game theory is the *Nash* (1950) *bargaining solution* (a modern textbook reference is Myerson (1991)).

The centrality of the Nash bargaining solution for pure bargaining situations has been much reinforced by its emergence as the (limit of) equilibria of natural noncooperative bargaining procedures. The most prominent of these is the Stahl–Rubinstein alternating offers model and its variations (e.g., Osborne–Rubinstein (1990)). In the TU-case, bargaining models leading to the Shapley value have also been suggested, e.g., Harsanyi (1981), O. Hart–Moore (1990), Winter (1994), and, most especially, Gul (1989).

The following extension problem arises immediately: taking as reference points the Nash bargaining solution for pure bargaining problems and the Shapley value for TU problems, what is their proper generalization to the class of all nontransferable utility (*NTU*) games in coalitional form? There are some classical solutions, namely the Harsanyi (1963) and the Shapley (1969) NTU-values (the latter is also known as the "λ-transfer value"). The theory, however, is much less settled for this general NTU case.

In this paper we pursue the above *extension program* by means of a noncooperative bargaining approach. We first propose a simple and natural bargaining procedure (a variation of the alternating offers method) that supports both the Nash bargaining solution for pure bargaining problems and the Shapley value for TU problems. Our contention is that the bargaining procedure fits and unifies these two problems rather well and that therefore it constitutes a good launching pad for an investigation into the appropriate solution for the general case, where partial breakdown is possible and where utility is not fully transferable.[2]

Our bargaining procedure follows tradition in setting up a sequential, perfect information game, where at each stage a player becomes a proposer. The proposers are chosen at random and the meetings are multilateral (thus we depart from the pairwise meeting technology of Rubinstein–Wolinsky (1985), Gale (1986), or Gul (1989)). The requirement for agreement is unanimity. The key modeling issue is the specification of what happens if there is no agreement and, as a consequence, the game moves to the next stage. It is at this point that subgroups are made to matter by allowing for the possibility of *partial* breakdown of negotiations. Clearly, there are many ways to model such a partial breakdown. In the body of this paper we concentrate on a particular and simple class: disagreement puts only the proposer in jeopardy. That is, after his proposal is rejected, the proposer may cease to be an active participant. The noncooperative solution concept we use is that of stationary subgame perfect equilibrium. The formal model is set up and discussed in Section 2.

In Section 3 we observe that our equilibria do indeed yield, as the probability of breakdown goes to zero (that is, as the cost of delay becomes small), the Nash bargaining solution for the pure bargaining situations and the Shapley value for transferable utility situations. The heart of this paper are Sections 4 and 5 where we tackle the solution extension problem. When analyzing the limits of our noncooperative equilibria as the probability of breakdown becomes small, we come to a surprise. These limits are none of the most familiar solutions (the Harsanyi and the Shapley NTU values) but, quite remarkably, they are precisely the *consistent values* introduced recently by Maschler–Owen (1989, 1992), from completely different considerations. This was unexpected to us and we would like to think that if two sets of motivations lead to the same object then

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[2] We note that our paper can also be viewed as pertinent to the research program of implementation theory. In this context what should be emphasized is the simplicity of the mechanism we propose.

something must be right with it. In our view this demonstrates that a noncooperative approach to cooperative solutions can have a positive feedback on the cooperative theory itself, by helping to clarify the solution theory over the less established territory.

The consistent value is easy to define and analyze and we do so in Section 4. Section 5 presents the general convergence result. In Section 6 we explore a broader range of bargaining procedures: in particular, we allow for the possibility that players other than the proposer may be the victims of bargaining breakdown. However, we show that if the bargaining procedure yields the Shapley value in the TU-case, then *necessarily* the consistent value obtains in the NTU-case. Thus the consistent NTU-value is, according to our noncooperative approach, the appropriate generalization of the Shapley TU-value. In Section 7 and last we address issues of interpretation and suggest possible directions for further research.

2. THE MODEL

Our basic data is an *n-person coalitional form*[3] (N,V). To be precise, N is a finite set of n players and $V(\cdot)$ is a function that assigns a subset $V(S)$ of R^S to every *coalition* $S \subset N$. An interpretation is that $V(S)$ contains all the payoff vectors to the members of S that would be feasible if S were the group of deciding players. For a detailed discussion on this point we refer the reader to Section 7.

Note that $V(S)$ is a set, not a number. We are thus in the general case of a *nontransferable utility* (NTU) *coalitional form*. We impose on (N,V) some standard hypotheses:

(A.1) *For each coalition S, the set $V(S)$ is closed, convex and comprehensive* (i.e., $V(S) - R_+^S \subset V(S)$). *Moreover, $0 \in V(S)$ and $V(S) \cap R_+^S$ is bounded.*[4]

(A.2) *For each coalition S, the boundary of $V(S)$, denoted $\partial V(S)$, is smooth* (i.e., *at each boundary point there is a single outward normal direction*) *and nonlevel* (i.e., *the outward normal vector at any point of $\partial V(S)$ is positive in all coordinates*).[5]

(A.3) *Monotonicity*: $V(S) \times \{0^{T \setminus S}\} \subset V(T)$ *whenever $S \subset T$* (i.e., *if one completes a vector in $V(S)$ with*[6] *0's for the coordinates in $T \setminus S$, then one obtains a vector in $V(T)$*).

For each $i \in N$, let $r^i := \max \{c: c \in V(i)\} \geq 0$ (by (A.1)). Denote $r_S = (r^i)_{i \in S} \in R^S$. The form (N,V) has already been normalized (in (A.1)) and, therefore, we cannot put $r^i = 0$. The significance of the distinction between r^i and 0 will

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[3] To avoid using the term "game" with different meanings we refer to (N,V) as an *n-person form* rather than an *n-person game*.

[4] The requirement $0 \in V(S)$ for all S is just a convenient normalization that we make without loss of generality. The whole theory is invariant under translations of the utilities' origins.

[5] Actually, the smoothness and nonlevelness conditions are needed only for vectors in $\partial V(S) \cap R_+^S$.

[6] Recall the normalization in (A.1).

become clear later on in this section and in Section 7. Of course, $r^i = 0$ for all i is possible; but note that in general (A.3) *does not* even imply $r_S \in V(S)$. (A trivial counterexample is the TU game given by $v(S) = 1$ for all nonempty S.)

Two particularly simple classes of coalitional forms (N, V) are the *transferable utility* (TU) coalitional forms, i.e. those for which there is a real-valued function $v(\cdot)$ such that $V(S) = \{c \in R^S : \sum_{i \in S} c^i \leq v(S)\}$ for all[7] $S \subset N$; and the *pure bargaining* coalitional forms, which we formally define as those where[8] $r_S \in \partial V(S)$ for all $S \neq N$, and $r_N \in V(N)$.

We now describe the sequential noncooperative game to be analyzed. Let (N, V) be an NTU-game and $0 \leq \rho < 1$ be a fixed parameter. Then the *n-person noncooperative game* (associated to (N, V) and ρ) is defined as follows:

> In each round there is a set $S \subset N$ of "active" players, and a "proposer" $i \in S$. In the first round $S = N$. The proposer is chosen at random out of S, with all players in S being equally likely to be selected. The proposer makes a "proposal" which is feasible, i.e. a payoff vector in $V(S)$. If all the members of S accept it—they are asked in some prespecified order—then the game ends with these payoffs. If it is rejected by even one member of S, then we move to the next round where, with probability ρ, the set of active players is again S and, with probability $1 - \rho$, the proposer i "drops out" and the set of active players becomes[9] $S \setminus i$. In the latter case the dropped out proposer i gets a final payoff of 0.

While the game can potentially last infinitely many periods, it is plain that, whatever the strategies, with probability one the game will terminate in finite time and the expected payoffs at termination are well-defined.

It is clear that if we are dealing with a pure bargaining coalitional form and $r_N = 0$ then the above noncooperative game is the (unessential) variation of the Rubinstein bargaining model where the proposer is chosen at random at every step. It is well known that for Rubinstein style models with more than two players folk-like theorems for perfect equilibrium apply (see the example of Shaked in Chapter 3 of Osborne–Rubinstein (1990)). Therefore there is no hope for sharp predictions in our (more general) setting if the solution concept is merely (subgame) perfectness.[10] Thus we shall take the familiar route of concentrating on *stationary (subgame) perfect* (SP) *equilibria*, that is, on those subgame perfect equilibria where strategies are such that the choice at each stage only depends on the set of active players S and on the current proposer i,

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[7] In the TU-case we will use (N, V) and (N, v) interchangeably. As usual, we put for convenience $v(\varnothing) = 0$.

[8] The bargaining problem will be denoted $(r_N, V(N))$; r_N is the "disagreement point" and $V(N)$ is the set of "feasible agreements."

[9] We write $S \setminus i$ for the more cumbersome $S \setminus \{i\}$.

[10] See Jun (1989), Chae–Yang (1988), and Krishna–Serrano (1990) for some modified procedures yielding unique perfect equilibrium payoffs for a broad class of pure bargaining problems. In our model, the set of perfect equilibrium payoffs is investigated in the TU-case by Krishna–Serrano (1995).

but neither on history nor even on calendar time.[11,12,13] We remark that a stationary perfect equilibrium is first and foremost a perfect equilibrium and, in terms of its strategic basis, is therefore as "valid" as any other perfect equilibrium. However, it is the simplest type of perfect equilibrium and so it is the natural starting and reference point of the analysis.

We will now proceed to characterize the SP equilibria. To facilitate exposition we will assume that both proposers and respondents break ties in favor of quick termination of the game.[14] Given a profile of stationary strategies, let $a_{S,i} \in R^S$, for $i \in S \subset N$, denote the proposal when the set of active players is S and the proposer is i. Let also[15] $a_S := (1/|S|)\Sigma_{i \in S} a_{S,i}$ be their average. (Note: if some of the $a_{S,i}$ are random, i.e., if *mixed strategies* are used, we let a_S be the *expected* average. We will see immediately after Proposition 1 that, with tie-breaking as above, mixed strategies are not used in SP equilibria.) The following Proposition spells out the basic equations of an SP equilibrium.

PROPOSITION 1: *The proposals corresponding to an SP equilibrium are always accepted, and they are characterized by*:

(1) $a_{S,i} \in \partial V(S)$ *for all* $i \in S \subset N$; *and*

(2) $a_{S,i}^j = \rho a_S^j + (1 - \rho)a_{S \setminus i}^j$ *for all* $i, j \in S \subset N$ *with* $i \neq j$;

where $a_S = (1/|S|)\Sigma_{i \in S} a_{S,i}$. *Moreover, these proposals are nonnegative (i.e.,* $a_{S,i} \in R_+^S$ *for all* S *and* i).

In words, (2) says that j will be proposed by i the expected amount that j would get in the continuation of the game if the proposal is rejected.

PROOF: We proceed by induction. The Proposition trivially holds for the 1-player case. Suppose it holds when there are less than n players. Let $a_{S,i}$, for $i \in S \subset N$, be the proposals of a given SP equilibrium. We will show that (1) and (2) are satisfied. Denote by $c_S \in R^S$ the expected payoff vector for the members of S in the subgames where S is the set of active players. Because $V(S)$ is convex we must have $c_S \in V(S)$. The induction hypothesis implies that $a_S = c_S$ and that (1), (2) are satisfied for $S \neq N$.

Monotonicity and convexity (see (A.3) and (A.1)) imply $\rho c_N + (1 - \rho)(a_{N \setminus i}, 0) \in V(N)$ for any i. Increasing the ith coordinate until reaching the boundary $\partial V(N)$ (recall (A.1)) determines the vector d_i on the boundary $\partial V(N)$ of $V(N)$ with $d_i^j = \rho c_N^j + (1 - \rho)a_{N \setminus i}^j$ for $j \neq i$. Thus, $d_i^i \geq \rho c_N^i$. For $j \neq i$, the amount d_i^j

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[11] As the respondents reply in sequence we note that their responses may depend on what previous respondents did *in the same stage*.

[12] We emphasize that we deal with a true equilibrium, i.e., deviations may be nonstationary.

[13] In our case, S and i are also the payoff relevant variables and so the SP equilibria are the Markov perfect equilibria of Maskin–Tirole (1988).

[14] No tie-breaking assumptions are needed in our model; we use them only to get simpler statements. We thank Motty Perry for pointing this out.

[15] The number of elements of a finite set A is denoted $|A|$.

is precisely the expected payoff of j following a rejection of i's proposal. Therefore d_i is the proposal which is best for i among the proposals that will be accepted if i is the proposer (it gives to all other j's the minimum they would accept). In addition, any proposal of i which is rejected yields to i at most $\rho c_N^i + (1 - \rho) 0 \leq d_i^i$. Hence, player i will propose $a_{N,i} = d_i$ and the proposal will be accepted. From this it follows that $c_N = a_N$.

It remains to show that $a_N^i \geq 0$. To see this note that the following strategy will guarantee to i a payoff of at least 0: accept only if offered at least 0 and, when proposing, propose $0 \in V(N)$. This implies that $a_{N,i} \geq 0$.

Conversely, we show that proposals $(a_{S,i})_{S \subset N, i \in S}$ satisfying (1) and (2) can be supported as SP equilibria. Note first that they are all nonnegative. Indeed, the $a_{N,i}$ are in $V(N)$, and therefore, by convexity, so is their average a_N. Moreover $(a_{N \setminus i}, 0) \in V(N)$ (by monotonicity and $a_{N \setminus i} \in V(N \setminus i)$), implying that $b_i := \rho a_N + (1 - \rho)(a_{N \setminus i}, 0) \in V(N)$. Now $a_{N,i}$, which lies on the boundary of $V(N)$, coincides with b_i on all coordinates except the ith. Therefore, $a_{N,i} \geq b_i$. Moreover, $b_i \geq \rho a_N$ (since, by the induction hypothesis, $a_{N \setminus i} \geq 0$). Averaging over i yields $a_N \geq \rho a_N$ and we conclude that a_N is nonnegative (and therefore so are all the $a_{N,i}$).

From here it is straightforward to verify that the strategies corresponding to these proposals do form an SP equilibrium. By the induction hypothesis, this is so in any subgame after a player has dropped out. Fix a player i in N. The strategies of the other players do not allow player i to increase his payoff from proposals that are accepted—at any stage, and whether proposed by i himself or by other players. Therefore the only conceivable gain can come by managing to drop out. But this gives a payoff of 0, whereas the suggested strategy yields nonnegative payoffs.   Q.E.D.

Note that (1) implies that $a_{\{i\}, i} = r^i$. Hence $a_{\{i\}, i}$ is nonrandom, which iterating in (2) (recall that a_S has been defined to be the *expected* payoff vector, thus nonrandom), yields that $a_{S,i}$ is nonrandom. Therefore, under the given tie-breaking rule, there are no mixed strategy SP equilibria.

In general, $a_{N,i} \neq a_{N,j}$ if[16] $i \neq j$. Therefore, from $a_{N,i} \in \partial V(N)$ for all i it does not follow that their average a_N belongs to $\partial V(N)$ (it will not be if $V(N)$ is strictly convex). Hence the payoffs need not be efficient. However, we note the following important fact, which is readily implied by (2):

COROLLARY: *Let* $(M, \ldots, M) \in R_+^N$ *be an upper bound for the set* $V(N) \cap R_+^N$. *Then* $|a_{N,i}^j - a_N^j| \leq M(1 - \rho)$ *for all* i, j *in* N.

Thus, if ρ is close to 1—i.e., if the "cost of delay" is low—then there is little dispersion among individual proposals: all the $a_{N,i}$ constitute small deviations of

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[16] Note that being the proposer is not necessarily an advantage, i.e., $a_{N,i}^i \geq a_N^i$ need not hold. Consider the buyer $i = 1$ in the "2-buyers, 1-seller" game: $v(1) = v(2) = v(3) = v(12) = 0$, $v(13) = v(23) = v(123) = 1$. Here $a_{N,1}^1 = \rho/6$ and $a_N^1 = 1/6$.

a_N. This implies, first, that a_N *is almost Pareto optimal* (since the $a_{N,i}$ are Pareto optimal). And second, that *there is no substantial advantage or disadvantage to being the proposer*; the "first-mover" effect vanishes.

We conclude this section with a few remarks. First, observe that ours is a simple model where the breakdown of negotiations is not an "all or nothing" matter. When a player leaves the game, the rest continue bargaining (albeit over a diminished "pie"). Thus in our model the breakdown of negotiations is only partial, and the attainable sets of the intermediate coalitions have significant influence over the final outcome, a feature that is absent in the extreme pure bargaining case.

Second, we do not consider time discount. The cost of delay in agreement is present in the form of the breakdown probability ρ. Time discount would not add anything essential to the analysis. If so desired, however, it could be incorporated with only minor modifications of the conclusions.

Finally, from the nature of the noncooperative game and the solution concept (see Proposition 1) it makes sense that payoffs should be considered for all coalitions simultaneously. So a *payoff configuration* (*p.c.* for short) is defined to be an element $a = (a_S)_{S \subset N}$ of $\prod_{S \subset N} R^S$; that is, a list of payoff vectors, one for each subcoalition $S \subset N$.

3. THE ANALYSIS OF TWO CLASSICAL CASES

In this section we consider the simplest examples of coalitional forms: the transferable utility and the pure bargaining cases. The results of this section are special cases of the general theorem of Section 5. We begin with the TU case; the result follows from Proposition 7 and from the Corollary to Proposition 1 in the previous section.

THEOREM 2: *Let* (N, V) *be a TU form with corresponding coalitional function* v. *Then for each* $0 \leq \rho < 1$ *there is a unique SP equilibrium. Moreover, for every coalition* S: (i) *the SP equilibrium payoff vector* a_S *equals* $\mathrm{Sh}(S, v)$, *the Shapley value of the coalitional form*[17] (S, v); *and* (ii) *the SP equilibrium proposals* $a_{S,i}$ *converge as* $\rho \to 1$ *to the Shapley value* $\mathrm{Sh}(S, v)$ *for all* i *in* S.

We have thus obtained the Shapley value in a noncooperative manner. For ρ close to 1, the "equilibrium path" consists of the first proposer proposing a payoff vector that is close to the Shapley value of the game, and everyone accepting it (which ends the game).

Suppose that the coalitional form is generated from a standard (quasi-linear) convex economy, e.g., participants own inputs which enter into a constant-returns concave production function for utility. Then the value-equivalence theorems (starting with Shapley (1964); see Cheng (1996) for a survey) tell us

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[17](S, v) denotes the restriction of (N, v) to S; i.e., the player set is S and the coalitional function is the restriction of v to 2^S.

that if the number of participants is large, then the Shapley value allocations, hence the SP equilibrium payoffs of our noncooperative game, are nearly Walrasian. At least for the case where ρ is close to 1 we could say that our economy functions as if a "referee," chosen at random, announces a price system that is accepted by all participants and clears markets.

One way to gain intuition on the result of Theorem 2 is to consider the well-known axioms of the Shapley value (e.g., Myerson (1991)). In the TU case the payoff configurations that solve the system of equations (1)–(2) are clearly *linear* in v, *symmetric* relative to the labels of the players, and *Pareto efficient*. Hence the key issue is the fulfillment of the *null player* (or *dummy*) axiom, which asserts that if $v(S) = v(S \setminus i)$ for all coalitions S containing player i (thus also $r^i = v(i) = 0$), then i should get 0. To see the plausibility of our equilibrium payoffs satisfying this axiom, take for clarity the extreme case where[18] $\rho = 0$. Then rejected proposers are eliminated for sure from the game (which will therefore terminate in at most n steps). The null player axiom obviously holds for one-player coalitional forms. Suppose it holds for coalitional forms with less than n players. Let now i be a null player in the n-person game. If the proposer is $j \neq i$, then i will be offered (recall that $\rho = 0$) his payoff in the $N \setminus j$ continuation, which is 0 by the induction hypothesis. If i is the proposer then any proposal that gives to the remaining players in total less than $v(N \setminus i)$, which is what they will get in the next stage, will be rejected by at least one of them. Hence i can get at most 0, and we conclude that the null player axiom holds. (See Section 6 for further discussion on this issue.)

A noncooperative implementation of the Shapley value in TU game forms has been offered earlier by Gul (1989). Our procedure differs from his, first, in the meeting technology—Gul's unfolds through pairwise meetings—and second, in the nature of the results—Gul considers only those SP equilibria that entail immediate acceptance (other SP equilibria may yield outcomes different from the Shapley value). In our case, immediate agreement is guaranteed at all SP equilibria, and our result applies to all of them. Also, to guarantee existence, the monotonicity assumption (A.3) suffices in our model, as compared to the superadditivity assumption on the values of the subgames in Gul's case.

We come now to the pure bargaining case; the result is a particular case of Theorem 5 in Section 5.

THEOREM 3: *Let (N,V) correspond to an n-person pure bargaining problem $(r_N, V(N))$. Then for each $0 \leq \rho < 1$ there is at least one SP equilibrium. Moreover, any SP equilibrium payoff vectors $a_N(\rho)$ converge to the Nash Bargaining solution of $(r_N, V(N))$ as $\rho \to 1$.*

We emphasize that the convergence to the Nash bargaining solution is no surprise, given that for the pure bargaining case the noncooperative model

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[18] This case was considered in Mas-Colell (1988).

amounts to a variation of the Rubinstein alternating offers model (see Binmore (1987), for the convergence of the latter).[19]

4. THE CONSISTENT VALUE

This section is devoted to a preliminary study of the *consistent NTU value* introduced recently, under the name of *consistent Shapley value*, by Maschler–Owen (1989, 1992). We note that the assumption of the smoothness of the boundaries $\partial V(S)$ is not needed in this section.

The consistent value can be defined as follows. Let π be an order of the n players in N. For a TU form (N,v), the *marginal contributions* of the players in the order π—say the order is $\pi = (1,2,\ldots,n)$—are:

$$d^1(\pi) := v(1);$$
$$d^2(\pi) := v(12) - v(1) = v(12) - d^1(\pi);$$
$$d^3(\pi) := v(123) - v(12) = v(123) - d^1(\pi) - d^2(\pi);$$

and so on. The extension to NTU forms is straightforward:

$$d^1(\pi) := \max\{c^1 : c^1 \in V(1)\} = r^1;$$
$$d^2(\pi) := \max\{c^2 : (d^1(\pi), c^2) \in V(12)\};$$
$$d^3(\pi) := \max\{c^3 : (d^1(\pi), d^2(\pi), c^3) \in V(123)\};$$

and so on. Thus $d^i(\pi)$ is the most that i can get (in $V(\{1,2,\ldots,i\})$) after all the previous players j got their own $d^j(\pi)$'s. Consider now the vector of *expected marginal contributions*, where each one of the $n!$ orders is equally likely: player i's payoff is $(1/n!)\Sigma_\pi d^i(\pi)$. (In the TU case, this is the Shapley value.) By definition, the payoff vectors $d(\pi)$ are efficient for every order π. However, their average will in general be inefficient—unless the boundary of $V(N)$ happens to be flat.

The above suggests to consider first the case of a *hyperplane coalitional form* (*H-form* for short). These forms (which will play a very important technical role in our analysis[20]) are defined by the property that each $V(S)$ is a half space in R^S (and thus $\partial V(S)$ is a hyperplane). For an *H*-form the payoff vector $\Psi(N,V) := (1/n!)\Sigma_\pi d(\pi)$ is efficient. Moreover, $\Psi(N,V)$ is precisely *the consistent value for hyperplane forms* introduced by Maschler–Owen (1989). Note that for TU games it coincides with the Shapley value. Observe also that

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[19] There is also a close connection to the "Nashlike solutions" N^λ of Thomson–Lensberg (1989, Section 8), which converge to the Nash solution as $\lambda \to 1$. Indeed, it can be checked that, when $0 < \rho < 1$, the set of SP equilibrium payoffs is precisely $(1/\rho)N^\lambda$, where $\lambda := \rho/[n - (n-1)\rho]$. Note that, if $n \geq 3$ and the set $V(N)$ is not generated by a one-dimensional "pie," then one may have multiple SP equilibrium payoffs, and that without the smoothness assumption on $\partial V(N)$, the convergence to the Nash solution may fail (again, see Thomson–Lensberg (1989)). Both these phenomena arise in the standard Rubinstein alternating offers model as well.

[20] A model that generates the hyperplane coalitional forms is that of the "prize games"; see Hart (1994).

efficiency obtains for every subcoalition: $\Psi(S,V) \in \partial V(S)$ for all S. We refer to $(\Psi(S,V))_{S \subset N}$ as the *consistent value payoff configuration*[21] of the hyperplane form (N,V). It is obviously unique.

For a general NTU form (N,V), choose for each coalition S a supporting normal vector $\lambda_S \in R_{++}^S$ to $\partial V(S)$. Let (N,V') be the resulting hyperplane game (i.e., $V'(S) := \{c \in R^S \colon \lambda_S \cdot c \leq v(S; \lambda_S)\} \supset V(S)$ for all S, where $v(S; \lambda_S) := \max\{\lambda_S \cdot c \colon c \in V(S)\}$). Let $\boldsymbol{a} := (\Psi(S,V'))_{S \subset N}$ be the (unique) consistent value payoff configuration of V'. If $\boldsymbol{a}$ is actually feasible in the original form V, i.e., if $a_S \in V(S)$ *for all S*, then $\boldsymbol{a}$ is called a *consistent value payoff configuration* of the game form (N,V); see Maschler–Owen (1992). We observe that in the pure bargaining case this gives precisely the Nash bargaining solution (it follows for instance from Proposition 4 below).

Note that the normal vectors λ_S will typically be different for distinct coalitions S. Taking them all equal (more precisely, we would let λ_S be the restriction of λ_N to S) would *not* work in general, since we require that $\Psi(S,V') \in V(S)$ for all S, not only for N. The consistent value treats all coalitions in the same way, thus exhibiting a property that we may call "subcoalition perfectness."[22] In contrast, the Shapley NTU-value requires the feasibility condition only for the grand coalition N and takes all the λ_S equal to λ_N. Also, for the Harsanyi NTU-value, the solutions for all subcoalitions are based on the weights λ_N determined from the grand coalition.

Under our hypotheses consistent values exist, by a standard fixed-point argument; see Maschler–Owen (1992). Existence is also a corollary of Theorem 5 of the next section (since the SP equilibrium payoff configurations lie in the compact set $\prod_{S \subset N} V(S) \cap R_+^S$ by Proposition 1). For nonhyperplane forms consistent values need not be unique: see Owen (1994) and the next paragraph.

Suppose we had a "reverse pure bargaining problem," i.e., $r_S \in \partial V(S)$ for $S \neq N$ and[23] $r_N \notin V(N)$. Then the consistent value payoff vectors c are characterized (apply Proposition 4 below) by the *first order conditions* (and no more than that) of the optimization problem:

$$\max \prod_{i \in N} (r^i - c^i), \quad \text{subject to } c \in \partial V(N) \text{ and } c < r_N.$$

For example, take $n = 2$. Then (see Figure 1(a)), $c = (c^1, c^2) \in \partial V(12)$ must be the midpoint of PQ, the segment of the tangent to $\partial V(12)$ at c that lies below

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[21] Recall that a payoff configuration is an element of $\prod_{S \subset N} R^S$.

[22] Observe that there is an intimate relation between the concept of a "subcoalition perfect" solution and the concept of a "subgame perfect" equilibrium. The fact that, in the bargaining procedure, one requires the same equilibrium conditions in each subgame, thus for every remaining coalition S, implies that the solution concept must be the same for all subcoalitions.

[23] Say $n = 2$. The monotonicity assumption (A.3) implies that both $(r^1, 0)$ and $(0, r^2)$ belong to $V(12)$, but (r^1, r^2) could well be outside. In terms of the bargaining model, this corresponds to the case where the breakdown outcome depends on which player made the last (rejected) proposal before breakdown. Here, $(r^1, 0)$ and $(0, r^2)$ are the payoff vectors if breakdown occurred after player 2 or player 1, respectively, was the last proposer. See Section 7 for a more concrete discussion on this dependence.
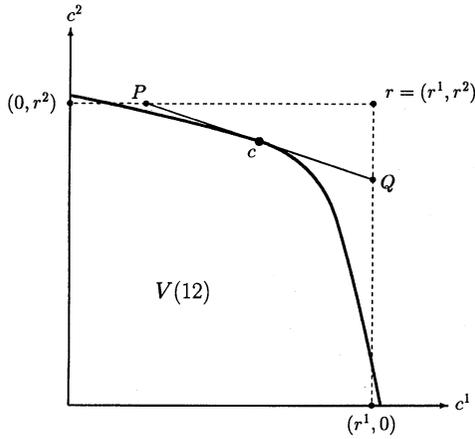
FIGURE 1(A)
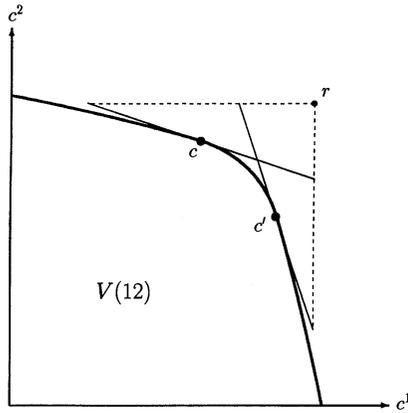
FIGURE 1(B)

$r = (r^1, r^2)$. Observe that, for this type of problem, nonuniqueness can easily occur: see Figure 1(b) for an example where both c and c' are consistent value payoff vectors.

A characterization of the consistent values is as follows:

PROPOSITION 4: *Let* (N, V) *be an NTU form, and* $a = (a_S)_{S \subset N}$ *a payoff configuration. Then* a *is a consistent value payoff configuration of* (N, V) *if and only if for each* $S \subset N$ *there exists a vector* $\lambda_S \in R_{++}^S$ *such that:*
(a) $a_S \in \partial V(S)$;
(b) $\lambda_S \cdot a_S = v(S; \lambda_S) := \max\{\lambda_S \cdot c : c \in V(S)\}$; *and*
(c) $\sum_{j \in S \setminus i} \lambda_S^i (a_S^i - a_{S \setminus j}^i) = \sum_{j \in S \setminus i} \lambda_S^j (a_S^j - a_{S \setminus i}^j)$ *for all* $i \in S$.

Conditions (a) and (b) say that the payoff vector a_S is efficient for the coalition S, and that λ_S is an outward normal to the boundary of $V(S)$ at a_S (or, stated in familiar economic terms, λ_S represents the local marginal rates of efficient utility transfers between the players at a_S). As for condition (c), it may be viewed as a "preservation of average differences" requirement. The term "preservations of differences" has been used in Hart–Mas-Colell (1989); it was introduced (under the name "balanced contributions") by Myerson (1980). In the TU-case, the jth terms on both sides are equal for the Shapley value. We could say that the contribution of j to i, measured by $a_S^i - a_{S \setminus j}^i$, equals the contribution of i to j, measured symmetrically by $a_S^j - a_{S \setminus i}^j$. It is known that in the TU case this preservation of differences principle does actually characterize the Shapley value payoff configuration and is also equivalent to an appropriate notion of consistency (see Hart–Mas-Colell (1989)). In the NTU case we cannot expect the jth terms on both sides[24] to be equal in general.[25] It turns out however that they are equal in average: the average contribution to i from the other players equals the average contribution of i to the other players. (One may of course replace "average" by "total").[26]

PROOF OF PROPOSITION 4: The proof is by induction. Assume the result holds for all $S \subset N$, $S \neq N$. We have to show that condition (c) for $S = N$ is equivalent to $a_N = \Psi(N, V')$, where $V'(S) := \{c_S \in R^S : \lambda_S \cdot c_S \leq v(s; \lambda_S)\}$ for all S (the associated H-form). By definition, $\Psi^i(N, V')$ is player i's expected marginal contribution over all $n!$ orders of the players. We classify these orders into n groups according to the last player j in the order. If $j \neq i$ then the (conditional) expectation of i's marginal contribution is the same as in $(N \setminus j, V')$, which equals $a_{N \setminus j}^i$ by the induction hypothesis. When i comes last, the expected marginal contribution of every $j \neq i$ is $a_{N \setminus i}^j$ by the same argument; the remainder $(v(N; \lambda_N) - \sum_{j \neq i} \lambda_N^j a_{N \setminus i}^j)/\lambda_N^i$ is i's contribution (recall the definition of $V'(N)$). Therefore

$$(3) \qquad \Psi^i(N, V') = (1/n)\left(\sum_{j \neq i} a_{N \setminus j}^i + \left[v(N; \lambda_N) - \sum_{j \neq i} \lambda_N^j a_{N \setminus i}^j \right]\bigg/ \lambda_N^i \right).$$

Taking into account that $v(N; \lambda_N) = \sum_{j \in N} \lambda_N^j a_N^j$ we have

$$n \lambda_N^i \Psi^i(N, V') = \sum_{j \neq i} \lambda_N^i a_{N \setminus j}^i + \lambda_N^i a_N^i + \sum_{j \neq i} \lambda_N^j \left(a_N^j - a_{N \setminus i}^j \right),$$

[24] Rescaled according to the rates λ_S in order to bring them to the common "local unit of account."

[25] The reason is that the normal vectors λ_S are usually distinct for different coalitions. Dropping condition (b) and taking them all equal leads to the egalitarian solutions (which constitute the first step in the construction of the Harsanyi NTU value); see Hart–Mas-Colell (1989). (Indeed, in this case one may easily prove inductively that condition (c) implies the preservation of differences condition.)

[26] For hyperplane forms, condition (c) is equivalent to the "weak 2-consistency" of Maschler–Owen (1989). For nonhyperplane forms it is however different.

hence

$$n\lambda_N^i\left(\Psi^i(N,V')-a_N^i\right)=\sum_{j\neq i}\lambda_N^i\left(a_{N\setminus j}^i-a_N^i\right)+\sum_{j\neq i}\lambda_N^j\left(a_N^j-a_{N\setminus i}^j\right).$$

This completes the proof, since the equality of the right-hand side to 0 is precisely condition (c).   Q.E.D.

REMARK: Formula (3) (also (5.1) in Maschler–Owen (1989)) is useful for computing consistent values recursively (see, for instance, the two examples below): Assume that $\partial V(N)$ is a hyperplane and that $a_{N\setminus i}$ is given for all i. Complete $a_{N\setminus i}$ into an efficient N-vector [27] $b_i=(a_{N\setminus i},\alpha^i)\in\partial V(N)$. Then the average $(1/n)\Sigma_i b_i$ of the b_i's is a consistent value for (N,V). Note that in the case of TU-games, this yields the formula

$$\mathrm{Sh}(N,v)=(1/n)\sum_{i\in N}(\mathrm{Sh}(N\setminus i,v),v(N)-v(N\setminus i)).$$

In order to develop some intuition it may be useful to compute the consistent value for some classical NTU examples.

EXAMPLE (Owen (1972)): Let $n=3$; $V(i)=\{c:c\leq 0\}$ for all i, $V(13)=V(23)=\{c:c\leq 0\}$, $V(12)=\{c:c^1+4c^2\leq 1,\ c^1\leq 1,\ c^2\leq 1/4\}$ and $V(123)=\{c:c^1+c^2+c^3\leq 1,\ c^i\leq 1,\ c^i+c^j\leq 1,\ \text{all }i,j\}$. Then the consistent value with $\lambda_N=(1,1,1)$ yields $a_{12}=(1/2,1/8)$ and $a_N=(1/2,3/8,1/8)$. Thus player 3 (the "banker") gets a positive amount (player 3 is not a null player since he indeed eases the utility transfer possibilities between 1 and 2; this is in contrast to the Shapley NTU value, which is $(1/2,1/2,0)$, for which 3 effectively becomes a null player). Moreover, 1 gets more than 2 due to the asymmetry of $V(12)$ which favors player 1 (in contrast, the egalitarian-based Harsanyi NTU value, which is $(2/5,2/5,1/5)$, does not recognize this). Thus the consistent value captures better the influence of the subcoalition $\{1,2\}$ (this is due to the "subcoalition perfectness" property we have already mentioned; namely, that $\{1,2\}$ is treated no differently than the grand coalition).

EXAMPLE (Roth (1980)): Let $n=3$; $V(i)=\{c:c\leq 0\}$ for all i,

$$V(12)=\{c:c\leq(1/2,1/2)\},$$

$$V(13)=\{c:c\leq(\varepsilon,1-\varepsilon)\},$$

$$V(23)=\{c:c\leq(\varepsilon,1-\varepsilon)\},\quad\text{and}$$

$$V(123)=\mathrm{conv}\{(1/2,1/2,0),(\varepsilon,0,1-\varepsilon),(0,\varepsilon,1-\varepsilon)\}-R_+^3$$

("conv" denotes "convex hull"). Then there is a unique consistent value for $\lambda_N=(1,1,1)$, namely $a_N=(1/6+\varepsilon/3,\ 1/6+\varepsilon/3,\ (2/3)(1-\varepsilon))$. It may appear,

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[27] α^i is uniquely defined since the boundary is nonlevel.

as argued by Roth, that since there is no conflict between 1 and 2 the outcome should be $(1/2, 1/2, 0)$. But this is only if the participation of player 3 is not needed for a final agreement. If some form of unaminous consent is required, then the consistent value makes a lot of sense. Players 1 and 2 cannot get $1/2$ each unless 3 either agrees to it or drops out of the game. This natural interpretation of $V(\cdot)$ is intimately related to our noncooperative game and thus to the consistent value (see the next two sections). Indeed, in terms of the bargaining model, the danger to players 1 and 2 is that one of them may drop out before 3, in which case their payoffs are either 0 or ε. This is the source of the power of player 3 to extract a considerable amount of utility.

5. THE GENERAL RESULT

In this section we study the equilibria of the noncooperative game in the general NTU case.

THEOREM 5: *Suppose that (N, V) is an NTU coalitional form satisfying the assumptions $(A.1)$, $(A.2)$, and $(A.3)$. Then for each $0 \le \rho < 1$ there is an SP equilibrium. Moreover, as $\rho \to 1$ every limit point of SP equilibrium payoff configurations is a consistent value payoff configuration of (N, V).*

Theorem 5 will be proved in three steps: Proposition 6 deals with the existence of SP equilibria. Proposition 7 proves the result for the case of hyperplane coalitional forms. Finally, Proposition 8 provides the general convergence argument.

PROPOSITION 6: *Let (N, V) be an NTU form. Then for each $0 \le \rho < 1$ there is an SP equilibrium.*

PROOF: A straightforward fixed-point argument will take care of this. We can proceed recursively. Clearly, the result is true for $n = 1$. Suppose now that we have a_S for all $S \ne N$ with the property that for any $T \subset N$, $T \ne N$, $(a_S)_{S \subset T}$ is an SP equilibrium payoff configuration for (T, V). By Proposition 1, $a_S \ge 0$ for all S. We now specify n functions $\alpha_i(b)$ from the compact convex set $V(N) \cap R_+^N$ into itself by letting $\alpha_i(b)$ be defined by: $\alpha_i(b) \in \partial V(N)$ and $\alpha_i^j(b) := \rho b^j + (1 - \rho) a_{N \setminus i}^j$ for all $j \ne i$. Because of the nonlevelness part of (A.2) the functions $\alpha_i(\cdot)$ are well-defined and continuous. By the convexity of the domain, $(1/n) \sum_{i \in N} \alpha_i(b)$ maps also into $V(N) \cap R_+^N$. Hence, by Brouwer's fixed point theorem, there is a vector $a_N \in V(N) \cap R_+^N$ satisfying $a_N = (1/n) \sum_i \alpha_i(a_N)$. By Proposition 1, a_N are equilibrium payoffs for N (and $a_{N,i} = \alpha_i(a_N)$ for all i). By the recursion hypothesis $(a_S)_{S \subset N}$ are the payoffs of an overall SP equilibrium for (N, V).   Q.E.D.

REMARK: The Proof of Proposition 6 does not make use of the smoothness hypothesis.

PROPOSITION 7: *Let (N, V) be a hyperplane form. Then for each $0 \le \rho < 1$ there is a unique SP equilibrium. Moreover the SP equilibrium payoff configuration equals the unique consistent value payoff configuration of (N, V).*

PROOF: We proceed by induction and assume that the statement is correct for hyperplane forms with less than n participants. Let $\lambda \in R_{++}^N$ and $V(N) = \{c \in R^N: \sum_i \lambda^i c^i \le w\}$. For every i,

$$n\lambda^i a_N^i = \lambda^i a_{N,i}^i + \sum_{j \ne i} \lambda^i a_{N,j}^i$$

$$= \left[w - \sum_{j \ne i} \lambda^j a_{N,i}^j \right] + \sum_{j \ne i} \lambda^i a_{N,j}^i$$

$$= \left[w - \sum_{j \ne i} \lambda^j \left(\rho a_N^j + (1-\rho) a_{N \setminus i}^j \right) \right]$$

$$\qquad + \sum_{j \ne i} \lambda^i \left(\rho a_N^i + (1-\rho) a_{N \setminus j}^i \right).$$

Now $w = \sum_j \lambda^j a_N^j$; also $b(i) := [w - \sum_{j \ne i} \lambda^j a_{N \setminus i}^j]/\lambda^i$ is the expected marginal contribution of i in a random order, conditional on i being last. Indeed (as in the proof of Proposition 4), classify the orders according to the last player. The (conditional) expected marginal contribution of each $j \ne i$ when i comes last is the same as the expected marginal contribution of j in $(N \setminus i, V)$, which by the induction hypothesis is precisely $a_{N \setminus i}^j$. The remainder $b(i)$ is then i's contribution when he is last. Hence, the last equality becomes, after dividing by λ^i,

$$na_N^i = \rho a_N^i + (1-\rho)b(i) + \sum_{j \ne i} \left(\rho a_N^i + (1-\rho) a_{N \setminus j}^i \right)$$

$$= n\rho a_N^i + (1-\rho)b(i) + (1-\rho) \sum_{j \ne i} a_{N \setminus j}^i.$$

Therefore $na_N^i = b(i) + \sum_{j \ne i} a_{N \setminus j}^i$, which yields the result: a_N^i is the expected marginal contribution of i (this contribution is $b(i)$ when i comes last, and is $a_{N \setminus j}^i$—again by the induction hypothesis—when $j \ne i$ comes last).

Finally, to show the existence of the SP equilibrium we choose $a_S := \Psi(S, V)$ for all S and then define $a_{S,i}$ by equations (1) and (2). It is easy to check (use the computation above) that, indeed, $a_S = (1/|S|)\sum_{i \in S} a_{S,i}$. Hence these proposals form an SP equilibrium by Proposition 1. Q.E.D.

PROPOSITION 8: *Let (N, V) be an NTU form. If $a(\rho)$ is an SP equilibrium payoff configuration for each ρ and a is a limit point of $a(\rho)$ as $\rho \to 1$, then a is a consistent value payoff configuration of (N, V).*

PROOF: Let $a = (a_S)_{S \subset N}$ and define λ_S to be the outward unit length normal to $\partial V(S)$ at a_S. We begin by associating with every ρ a hyperplane form (N, V_ρ).

To this effect let $\lambda_S(\rho)$ be the outward unit normal to the hyperplane passing through the vectors $\{a_{S,i}: i \in S\}$, and let $V_\rho(S)$ be the resulting half-space; if the hyperplane is not unique, choose $\lambda_S(\rho)$ closest possible to λ_S. Since $a_{S,i}(\rho) \to a_S$ (by the Corollary to Proposition 1) we have $\lambda_S(\rho) \to \lambda_S$; the smoothness of $\partial V(S)$ is essential here. Therefore $V_\rho(S) \to V'(S) := \{c \in R^S: \lambda_S \cdot c \le \lambda_S \cdot a_S\}$. Because of the characterization in Proposition 1, the p.c. $a(\rho)$ remains an SP equilibrium p.c. for (N, V_ρ), thus $a(\rho)$ is the consistent value p.c. of (N, V_ρ) by Proposition 7. The continuity of the marginal contributions with respect to the hyperplanes implies that the p.c. a is the consistent value p.c. of (N, V'), thus also of (N, V).   Q.E.D.

REMARK: The contrast between Proposition 7 and Theorem 5 (actually, Proposition 8) illustrates the role of assuming that ρ is close to 1: it localizes the arguments, in the sense that it guarantees that, for every S, all the proposals $a_{S,i}$ of an SP equilibrium are clustered together (Corollary to Proposition 1) so closely that, up to a second order effect, it is as if we could replace every $V(S)$ by a linear approximation.

6. GENERALIZING THE BARGAINING PROCEDURE

In this section we present and study some extensions of the bargaining procedure of Section 2. The aim is not complete generality, but rather to have a set-up that allows us to perform some comparative analysis. To this effect we keep the main structure of rounds, proposers, and the possible dropping out of a player after the proposal is rejected. However, it is no longer necessarily the proposer that drops out, and the probabilistic structure contemplated is more general.

As before, the noncooperative games we consider consist of (potentially) infinitely many rounds of bargaining. In each round there is a set $S \subset N$ of active players—starting with $S = N$ in the first round—out of which a proposer $i \in S$ is chosen; this is done now according to a given probability distribution[28] $\sigma = (\sigma_i)_{i \in S}$. The proposer makes a proposal, which is a payoff vector feasible for S (i.e., it belongs to $V(S)$). The members of S are then asked (in some prespecified order) whether or not they accept the proposal. If they all accept it, then the game ends with these payoffs. Otherwise, if the proposal is rejected by even one member of S, then the game moves to the next round. With probability ρ_i the set of active players does not change (call this case "repeat"), while with probability $1 - \rho_i$ one of the active players drops out and gets a final payoff of 0 (call this case "breakdown"). More precisely, the set of active players in the next round is S with probability ρ_i, and it is $S \setminus k$ with probability $\tau_{k|i}$ for each $k \in S$ (thus $\rho_i + \sum_{k \in S} \tau_{k|i} = 1$, for all $i \in S$). Note that all the probabilities above may depend on S (when necessary we will write $\sigma_i(S)$, $\rho_i(S)$, and so on).

[28] I.e., $\sum_{i \in S} \sigma_i = 1$ and $\sigma_i \ge 0$ for all $i \in S$.

Thus the player that drops out is no longer necessarily the proposer. Moreover, the various probabilities may depend on the set of active players and the proposer. The procedure we have used throughout this paper corresponds to $\sigma_i = 1/s$; $\rho_i = \rho$; $\tau_{i|i} = 1 - \rho$ and $\tau_{k|i} = 0$ for $k \neq i$ (as usual, $s := |S|$).

The analytical tools of the previous sections may be used to study this more general procedure. We start with the TU-case. The result will be stated in terms of recursive equations: the solutions for the subcoalitions of S determine the solution for S. Let us introduce some notations:

$$\rho(S) := \sum_{i \in S} \sigma_i(S)\rho_i(S);$$

$$\gamma_{i,k}(S) := \sigma_i(S)\tau_{k|i}(S)/(1 - \rho(S)); \quad \text{and}$$

$$\beta_k(S) := \sum_{i \in S} \gamma_{i,k}(S).$$

Thus, $\rho(S)$ is the total probability of "repeat" (i.e., no player drops out following a rejected proposal); $\gamma_{i,k}(S)$ is the conditional probability, given "breakdown" (i.e., some player drops out following a rejection), that the proposer was i and the dropped out player was k; and $\beta_k(S)$ is the total probability that k dropped out, given "breakdown."

PROPOSITION 9: *Let (N,V) be a TU form with corresponding coalitional function v. If $0 \leq \rho(S) < 1$ for all S, then there is a unique SP equilibrium, whose payoffs $(a_S)_{S \subset N}$ satisfy*

(4) $$a_S^i = \sum_{k \in S \setminus i} \beta_k(S)a_{S \setminus k}^i + \sum_{k \in S} \gamma_{i,k}(S)[v(S) - v(S \setminus k)]$$

for all $i \in S \subset N$.

Formula (4) may be understood as follows. The first term is the expectation $\sum_{k \in S} \beta_k(a_{S \setminus k}, 0)$ of the payoff vectors of the subcoalitions $S \setminus k$, completed into S-vectors by giving 0 to the dropped out player k (note that $\sum_{k \in S} \beta_k = 1$). For the second term, the expected[29] marginal contribution $\beta_k[v(S) - v(S \setminus k)]$ of each player k is divided between all the members i of S, in proportion to the probabilities that i was the proposer when k dropped out.

Before proving Proposition 9, we will analyze a number of interesting cases. For the first four, (a)–(d), we assume that all players have the same probability of being the proposer (thus $\sigma_i = 1/s$ for all $i \in S$) and the probability of "repeat" is the same whoever was the proposer (i.e., $\rho_i = \rho$ for all $i \in S$). This implies $\beta_k = 1/s$ for all $k \in S$.

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[29] The marginal contribution $v(S) - v(S \setminus k)$ of player k is weighted by β_k since it can matter only to the extent that k drops out—whence his marginal contribution is lost—and this has overall probability β_k.

(a) *Only the proposer drops out*: This is the model of the previous sections. Here $\gamma_{k,k} = 1/s$ and $\gamma_{i,k} = 0$ for $i \neq k$, yielding the Shapley value. Note that (4) in this model becomes formula (3) for the TU-case (see the Proof of Proposition 4, and the Remark following it).

(b) *Only the responders (but not the proposer) drop out, all with equal probability*: Here $\gamma_{k,k} = 0$ and $\gamma_{i,k} = 1/[s(s-1)]$ for $i \neq k$. It is easy to check that $a_S^i = v(S)/s$ for all S and all $i \in S$ satisfies the resulting recursion (4). We thus obtain the "equal split" solution ES (of course, relative to $(0,\ldots,0)$). This means, in particular, that—in contrast to case (a)—the solution is not sensitive to the worth of subcoalitions.

(c) *The proposer drops out with probability $(1 - \rho)\theta$, and all responders drop out with equal probability (of $(1 - \rho)(1 - \theta)/(s - 1)$ each), for some $0 < \theta < 1$* (Dagan (1992)[30]): Here the probabilities β_k and $\gamma_{i,k}$ are the average of the corresponding probabilities in the previous two cases (with weights θ and $1 - \theta$, respectively). The linearity of the formula (4) in these probability coefficients implies that the solution is $\theta \, \mathrm{Sh} + (1 - \theta)\mathrm{ES}$ (where Sh is the Shapley value and ES is the equal split value of (b) above).

(d) *All players drop out with equal probability*: Here $\gamma_{i,k} = 1/s^2$ for all i and k. The resulting solution is different from the previous ones (thus, it is neither the Shapley value nor the equal split solution). However, for large n, it is easy to see that it is close to the equal split solution of (b) (a minor boundedness condition is needed here).

(e) *Unequal probabilities of being the proposer and of dropping out, and only the proposer drops out* (Gomes (1991)[31]): Here $\beta_k = \gamma_{k,k}$ and $\gamma_{i,k} = 0$ for $i \neq k$. If each player $i \in N$ has his own probability w_i of being chosen the proposer (these are updated as the game proceeds by conditioning on the set of active players; i.e., $\sigma_i(S) = w_i \backslash \sum_{j \in S} w_j$ for all S) and his own survival probability ρ_i (*independent of S*), then the solution resulting from (4) is precisely the *weighted Shapley* (1953a) *value* relative to the weights' vector $(w_i(1 - \rho_i))_{i \in N}$ (see also Kalai–Samet (1985) and Hart–Mas-Colell (1989)).

Proposition 9 enables us to characterize which of these bargaining procedures lead to the Shapley value.

COROLLARY: *The SP equilibrium payoffs coincide with the Shapley value for all TU-games if and only if the bargaining procedure satisfies $\beta_k(S) = \gamma_{k,k}(S) = 1/|S|$ and $\gamma_{i,k}(S) = 0$ when $i \neq k$, for all $k \in S \subset N$.*

PROOF: Applying (4) recursively yields a_N^i as an average of terms of the form $v(S) - v(S \backslash k)$, for various S and $k \in S$, possibly distinct from i. To obtain the Shapley value, only the marginal contributions of i can matter; therefore $\gamma_{i,k} = 0$ whenever $i \neq k$. The equality of the β_k's is implied by symmetry.
                                                                                              Q.E.D.

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[30] This case has been studied by Nir Dagan; he obtained the characterization of the solution directly, without using Proposition 9.

[31] Again, the characterization of the solution in this case has been obtained by Armando Gomes directly.

Recall the discussion of Section 3 on the axioms of the Shapley value. All the bargaining procedures of this section clearly lead to solutions that satisfy the efficiency and the linearity axioms. The Corollary says that to obtain the Shapley value one needs, first, that only proposers (but not responders) may drop out; and second, that the probabilities $\sigma_i(1 - \rho_i)$ of dropping out should be equalized across the players. The first condition is related to the null player axiom,[32] and the second to the symmetry axiom.

PROOF OF PROPOSITION 9: The line of proof parallels that of Proposition 7, and we will not repeat the arguments here. The recursion proceeds as follows: A proposer $i \in S$ proposes to each other player j in $S \setminus i$ the expected payoff of next round, namely

$$a_{S,i}^j = \rho_i a_S^j + \sum_{k \in S} \tau_{k|i} a_{S \setminus k}^j$$

(where for convenience we define $a_{S \setminus j}^j$ as 0; also, recall that all the probabilities may depend on S). The proposer then takes all the surplus (so the total is $v(S)$), namely

$$a_{S,i}^i = \rho_i a_S^i + \sum_{k \in S} \tau_{k|i} \left[a_{S \setminus k}^i + v(S) - v(S \setminus k) \right]$$

(recall that the coordinates of a_S add up to at most $v(S)$, and those of $a_{S \setminus k}$ to $v(S \setminus k)$, which is at most $v(S)$ by the monotonicity assumption (A.3)). Fix i in S. Taking expectation of $a_{S,j}^i$ over j (with probabilities σ_j), we obtain

$$a_S^i = \sum_{j \in S} \sigma_j \rho_j a_S^i + \sum_{j \in S} \sum_{k \in S} \sigma_j \tau_{k|j} a_{S \setminus k}^i + \sum_{k \in S} \sigma_i \tau_{k|i} [v(S) - v(S \setminus k)].$$

Moving the a_S^i term to the left-hand side, dividing throughout by $1 - \rho = 1 - \sum_j \sigma_j \rho_j = \sum_j \sigma_j (1 - \rho_j)$, and finally changing the order of summation in the middle term yields (4).   Q.E.D.

Consider now the NTU-case. It can be easily checked that formula (4) of Proposition 9 may be extended to hyperplane games (in the same way that Proposition 7 extends Theorem 2) and, when all the $\rho_i(S)$ approach 1, also to the general NTU case. One just needs to replace the term $\gamma_{i,k}(S)[v(S) - v(S \setminus k)]$ in formula (4) by $\gamma_{i,k}(S)[\sum_{j \in S} \lambda_S^j a_S^j - \sum_{j \in S \setminus k} \lambda_S^j a_{S \setminus k}^j]/\lambda_S^i$, where $\lambda_S \in R_{++}^S$ is the unique supporting normal to the boundary of $V(S)$ at a_S. It is easily seen that this is just equation (c) of Proposition 4. Recalling the previous Corollary, one obtains the following important implication.

If the bargaining procedure yields the Shapley value in the TU-case, then it yields the consistent value in the NTU-case.

[32] If a player other than the proposer may drop out after rejection, then a null player, when he is the proposer, has bargaining power. Indeed, it is the proposer that essentially gets the marginal contribution of the dropped out player.

So, from the noncooperative viewpoint espoused in this paper, the consistent value is the appropriate NTU generalization of the Shapley TU-value.

Finally, we note that one could also consider models where, after rejection, each player has a certain probability of dropping out, independently of the other players. If these probabilities are small and of comparable size, then the terms corresponding to more than one player dropping out become relatively negligible, and we are back, essentially, to models where only one player may drop out.

7. INTERPRETATION AND DISCUSSION

In this section we discuss some interpretative issues concerning our setup, both in terms of the coalitional form and of the noncooperative bargaining game used. Plainly, the two are strongly interconnected. The particular conflict situation that underlies the coalitional form being studied is an essential factor in judging the appropriateness of any bargaining procedure. In order to fix ideas, it may be useful to consider some examples. We will do this within the familiar framework of economic models, but it should be clear to the reader how to interpret them more generally.

Suppose that we have a freely transferable consumption good and that utility functions depend only on this good and are linear in it (that is, we are in a TU setup). There is also a production funtion $f(x)$ for the consumption good, where x is a vector of inputs (which yield no utility). Every member i of society owns a vector of inputs ω^i. As a first example we assume that the "technology" $f(x)$ is freely available and replicable, i.e., every individual or group can use it without limit or congestion. Then $v(S) = f(\sum_{i \in S} \omega^i)$ is the amount of utility that a group of people S could get "by themselves." This can mean what they get if they separate and form their own economy or, indistinctly, what they get if they were alone in the world (i.e., if the rest of people had separated, or did not show up). The bargaining procedure can be applied to this economy without difficulty and it yields a perfectly sensible outcome: the Shapley value.

Our noncooperative bargaining game (or at least its stationary equilibria) does not really allow for strategic coalition formation. When applied to the economy in the previous example, where groups of agents can actually separate (and still have access to the technology), this may be seen as restrictive.[33] We are thus led to consider the following variation of the example. Suppose now that the technology is not replicable: the function $f(x)$ captures the productivity not only of the vector x of resources, but also of some underlying, indivisible, jointly owned resource. There is then a distinction between what a group S can

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[33]Allowing for strategic coalition formation leads, on the cooperative side, to domination and core related ideas, and, on the noncooperative side, to the models of Harsanyi (1974), Selten (1981), Baron–Ferejohn (1987), Chatterjee–Dutta–Ray–Sengupta (1993), Bennett (1991), Perry–Reny (1994), Okada (1991), and others.

get if they are the members of society (and, therefore, control the common resource), which is $f(\sum_{i \in S} \omega^i)$, and what they would get if they had left society, a lower amount. This is a model where, because separating coalitions cannot take the common resource with them, their strategic significance is much diminished. Interpreting $v(S)$ as $f(\sum_{i \in S} \omega^i)$ we have, in consequence, an instance that our noncooperative game fits very naturally. In particular, it allows us to exploit the distinction we have made between the level of utility r^i (which is what agent i gets if he is the only one left in society, hence in full control of the joint resource; in this case, $r^i = f(\omega^i)$) and the level 0, which is what player i gets if he separates (what really matters is that he gets a level of utility of at most r^i). The current example requires this distinction, which our model allows.

To elaborate further we emphasize that from the standpoint of our bargaining procedure the number $v(S)$ (or the set $V(S)$ in the NTU case) is the utility accruing to the members of S if they are the remaining players at the end of the game. Accordingly, r^i is the payoff if only player i is left. As indicated, our formal model allows for the utility of a player that leaves the game to be less than r^i (it is this amount that we have normalized to be zero). As we have seen, the scope of possible applications is thus increased. We should add that we could go further and specify also the utility of groups of players leaving the game. We do not do so, however, because in our bargaining procedure players leave the game, if at all, one at a time. Also, we are assuming that the "coalition" of expelled players does not form; the utility of a dropped-out agent is fixed at zero when he leaves the game.

A limit instance of the last example is when there are no individual resources, but only the jointly owned, indivisible, resource. Then, normalizing, the coalitional form has $v(S) = 1$ for all nonempty S, i.e., $v(S) = 1$ is the utility that S gets if S has the resource; the utility of not having the resource is zero. Our mechanism applies well to this (monotonic but not supperadditive) situation. It may be useful to offer a brief discussion. The unique SP equilibrium of the game gives $1/n$ to every player. If ρ is small, say $\rho = 0$, then this is true only in expectation. In fact, being selected a proposer is "bad news" since here a proposer has no bargaining power. If he demands anything for himself he will be rejected for sure: some other player must be getting a proposal of less than $1/(n-1)$, the expected payoff in the continuation for a player that rejects. So the proposer can only get 0. If however ρ is large, i.e., close to 1, then we know that (at the SP equilibrium) proposals depend very little on who is chosen to be the proposer: the remaining players will accept an offer slightly above $1/n$ (hence leaving almost $1/n$ to the proposer). What happens is that a respondent takes into account that with high probability the same situation will repeat, and in that case he may be chosen to the "hot seat" of proposer.

In the context of the class of examples considered up to now, the meaning of the expression "player i is dropped out" is clear: it means that the player loses the benefit of the use of the joint resources, while the remaining players lose the benefit of the use of his resources. We are referring to "resources" in order to

have a specific model in front of us. But there can be others. In fact, in all generality, we would dispense with the coalitional form and simply have a strategic or normal form supplemented by a specification of commitment procedures to joint play in the normal form. The bargaining game will then be concerned with the determination of the agreed upon joint play.

Note also that the possibility of "dropping out" gives power and imposes servitude to a proposer. As for the power, we could imagine that, when chosen as proposer, a player is automatically committed to a threat of a random withdrawal (perhaps destruction) of his own resources.[34] As for the servitude, the cost of withdrawal does not need to be viewed as a physical disappearance, but merely as a loss of veto power. We could put the matter as follows. We deal with bargaining situations where in principle the unanimous consent of all the participants is required. Yet we do not want individual players to hold out on an agreement forever. Ours is a simple way to accomplish this effect: when a player is called upon to be a proposer he loses his current veto power—he must make a proposal. Of course, he can exercise veto power indirectly by formulating a proposal that will be rejected, but then he runs the risk of losing his veto power forever: he has had "his chance" and been "frivolous" about it (in the model of this paper it is important that losing veto power be costly to an individual; i.e., $0 \leq r^i$). In other words, consent can be given either actively and explicitly or by "getting out of the way." The latter will typically have collective and individual costs.

We conclude by suggesting a number of issues that deserve further investigation: (a) exploring additional noncooperative bargaining games—beyond those of Section 6; in particular, allowing the possibility of strategic coalition formation; (b) investigating the relation between the consistent value and the Walrasian equilibria in large economies;[35] (c) studying noncooperative solution concepts less strict than SP equilibrium; for example, nonstationary perfect equilibria,[36] or limits of perfect equilibria of finite horizon games (as the length of the game increases);[37] (d) providing axiomatizations for the consistent value.[38]

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[34] Hence there is no strategic choice of threats here. See Myerson (1991), for a similar point in relation to the Nash bargaining solution.

[35] See Hart–Mas-Colell (1995) for an example in a related context (the Harsanyi NTU-value), where a lack of quasilinearity (that is, a presence of strict curvature in $\partial V(S)$) leads to a breakdown of the value equivalence principle.

[36] See Krishna–Serrano (1995) for the TU-case.

[37] Some interesting preliminary results in this direction have been obtained by Gomes (1991).

[38] See Hart (1994) for an axiomatization in the hyperplane case.

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