ELSEVIER

Journal of Mathematical Economics 33 (2000) 463–473

JOURNAL OF Mathematical ECONOMICS

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A note on the decomposition (at a point) of aggregate excess demand on the Grassmannian [1]

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Received 3 May 1999; accepted 24 June 1999

Abstract

This paper analyzes the properties of aggregate excess demand functions for economies with an arbitrary finite set of N commodities where agents face trading restrictions of a general, abstract form: their budget set is defined by K-dimensional planes in $\Re^N$. It is shown that, if there are at least K agents in the economy, the only general property satisfied by the value of aggregate excess demand and its derivative, at any arbitrary point, is Walras Law. The result is established by considering an economy where agents' preferences are of a 'generalized Leontief' type. © 2000 Elsevier Science S.A. All rights reserved.

JEL classification: D52; C62

Keywords: Market demand; Disaggregation; Missing markets

1. Introduction

In this note we examine the properties of aggregate demand functions when agents face trading restrictions of the ''missing markets'' type. Formally, the

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[1] Most of the work for this paper was carried out while the first author was visiting Universitat Pompeu Fabra, in January 1997 and again in January 1998. He wishes to thank that institution for the warm hospitality and the support.

PII: S0304-4068(99)00032-4

agents' budget set will be defined by K-dimensional planes in $\Re^N$. This abstract formulation is quite general. For example, when financial markets are incomplete, the budget equations each agent faces can always be reduced to the condition that his net trades in commodities have to lie on (or below) a K-plane. The specification of these planes is a rich parameter space, describing both the restrictions arising from the value of prices and from the form of the trading constraints (thus, in the case of incomplete markets the parameters will include both prices and the level of asset returns).

When markets are complete and agents face no restriction in their trades, Sonnenschein (1973), Debreu (1974), and Mantel (1974) showed that if there are at least N agents in the economy, the aggregate excess demand has no properties, on a compact set of prices, besides continuity, homogeneity and Walras Law. [2] The validity of a similar result for the case in which the agents' choice problem is subject to trading restrictions of the general form mentioned above was offered as a problem in Mas-Colell (1986), and progress has not been obtained until recently.

Some first steps towards a solution were made by Bottazzi and Hens (1996) and Gottardi and Hens (1999), who showed that, in the presence of incomplete markets, when agents' demand is written as a function only of commodity and asset prices, aggregate demand again has no structure. [3] Chiappori and Ekeland (1999) have then shown that any analytic function satisfying the above mentioned restrictions describing the trading constraints can be decomposed, in the neighborhood of an arbitrary point, as the aggregate excess demand of an economy with K agents.

In the work reported in this note we offer a result which is less general than Chiappori and Ekeland. Yet our method of proof is rather different, and is simpler (for a simpler result). Also, the preferences we rely on to rationalize the given function as an excess demand function are quite distinct and may be of independent interest.

Specifically, what we do here is to solve the linearization-at-a-point problem. That is, in the context of the general Grassmannian problem, we show that, if there are at least K agents in the economy, the value of aggregate excess demand and of its derivative, at a prespecified point, can be arbitrary (except for the restrictions imposed by Walras Law). We do this by considering a collection of agents whose preferences are of the ''generalized Leontief'' type, as they are defined, effectively, on an affine subspace of $\Re^N$ of dimension $N-K$. The characterization of the agents' preferences and demand in such case, and more generally the analysis

[2] See Shafer and Sonnenschein (1982) for a survey of the further developments of this literature.

[3] In particular, Bottazzi and Hens (1996) look at the properties of aggregate excess demand, over a compact set of prices, for the case of real assets, while Gottardi and Hens (1999) examine the properties of its linear approximation at a point, in the presence of incomplete markets with nominal assets.

of how the properties of a collection of agents of this type give rise to aggregate demand, are, we believe, of independent interest.

2. The economy

We consider a pure exchange economy with N commodities and H consumers. Each consumer $h = 1, \dots, H$ is characterized by preferences described by the function $U^h(.)$, defined on $\Re^N$ and assumed increasing and quasi-concave, and by endowments $\omega^h \in \Re^N$.

Let $K < N$ and $\mathscr{G}^{N,K}$ denote the Grassmannian manifold of K-planes in $\Re^N$, defined by $\{L \subset R^N$: L is a K-dimensional linear subspace$\}$; also $\mathscr{G}_+^{N,K} \equiv \{L \in \mathscr{G}^{N,K} : L \cap \Re_+^N = \{0\}\}$.

The choice problem faced by an arbitrary agent h, for $L \in \mathscr{G}_+^{N,K}$, is the following

$$\max U^h(x) \qquad (P^h)$$

$$\text{s.t. } x - \omega^h \in L - \Re_+^N$$

Let $x^h(L)$ denote the solution set of the above problem, describing the agent's demand at L; similarly the agent's excess demand set is $z^h(L) = x^h(L) - \omega^h$. Note that if the utility function is strictly increasing, then $z^h(L) \subset L$.

Remark. The specification of the budget constraints in (P^h) allows us to capture various kinds of market structures. When $K = N - 1$, we obtain the case of complete markets, and L is simply the budget hyperplane defined by the price vector; with free disposal the budget set is then the half space lying below this hyperplane. On the other hand, when $K < N - 1$ the budget constraints correspond to the case in which markets are incomplete (and $N - K - 1$ is the number of 'missing markets'), [4] or more generally in which agents face a set of linear restrictions on the level of their net trades. In this case the set L reflects both the level of prices and the specification of asset returns (more generally of the trading constraints); thus changes in L may correspond to changes both in the level of prices and in asset returns.

[4] See Mas-Colell (1986), Balasko and Cass (1989) for a more complete argument illustrating how the budget equations with incomplete markets can be reduced to the more abstract form considered here.

Adding individual agents' excess demands we obtain the expression of the aggregate excess demand

$$z(L) = \sum_h z^h(L)$$

which also satisfies the condition: $z(L) \subset L - \Re_+^N$ for all $L \in \mathscr{G}_+^{N,K}$. Of course, with strictly increasing utility functions we have $z(L) \subset L$ for all $L \in \mathscr{G}_+^{N,K}$, which we view as the expression of Walras Law in our set-up.

3. The problem and the result

We want to examine the problem of whether the optimizing behavior of the agents, under the above general specification of their budget equations, imposes any restriction on the form of an aggregate differentiable excess demand function, in addition to Walras Law. We will, however, address this issue in a more restricted way, by limiting our attention to the values of aggregate excess demand and of its derivative, at an arbitrary point $\bar{L} \in \mathscr{G}_+^{N,K}$.

Assume that, in a neighborhood of $\bar{L}$, an arbitrary differentiable function, satisfying Walras Law, is given to us. To provide a more precise formal definition of the problem, it is convenient to introduce and make reference to a local coordinate system on $\mathscr{G}_+^{N,K}$.

For each $L \in \mathscr{G}^{N,K}$ we can always find a $N \times N$ permutation matrix $\mathbf{P}^\sigma$ and a $N \times K$ matrix of the form $\begin{bmatrix} \mathbf{I}_K \\ \mathbf{A} \end{bmatrix}$, such that $Sp\left[\mathbf{P}^\sigma \begin{pmatrix} \mathbf{I}_K \\ \mathbf{A} \end{pmatrix}\right] = L$, where $Sp[.]$ denotes the linear space generated by the columns of a matrix, $\mathbf{I}_K$ is the K-dimensional identity matrix and $\mathbf{A}$ is a matrix of dimension $(N-K) \times K$. Hence, $\bar{L}$ is identified by a pair $(\bar{P}^\sigma, \bar{A})$, and a finite parameterization of a small neighborhood $\mathscr{N}(\bar{L}) \subset \mathscr{G}_+^{N,K}$ of $\bar{L}$ is induced by the elements of the $(N-K) \times K$ matrix $\mathbf{A}$ in a neighborhood of $\bar{A}, \mathscr{V}(\bar{A}) \equiv \{\mathbf{A} \in \Re^{(N-K) \times K} : Sp\left[\bar{P}^\sigma \begin{pmatrix} \mathbf{I}_K \\ \mathbf{A} \end{pmatrix}\right] \in \mathscr{N}(\bar{L})\}$.

Using this parameterization, an aggregate excess demand function, in the neighborhood $\mathscr{N}(\bar{L})$ of $\bar{L}$, can also be written as a function of $\mathbf{A}$, $z^h(\mathbf{A})$, for $\mathbf{A} \in \mathscr{V}(\bar{A})$. In addition, if the agents' utility function is strictly increasing, Walras Law assures us that $z^h(\mathbf{A}) = \left[\bar{P}^\sigma \begin{pmatrix} \mathbf{I}_K \\ \mathbf{A} \end{pmatrix}\right] \zeta^h(\mathbf{A})$, for some $\zeta^h : \mathscr{V}(\bar{A}) \to \Re^K$. Hence, we can naturally take derivatives of $z^h(.)$ with respect to $\mathbf{A}$, whenever $\zeta^h(.)$ is differentiable. Let $D_{\mathbf{A}} z^h$ denote the derivative of $z^h(\mathbf{A})$ with respect to $\mathbf{A}$.

The problem we intend to analyze is then formally stated as follows:

Definition 3.1. Let $(\bar{A}, \bar{P}^\sigma)$ be arbitrarily given and $F : \mathscr{V}(\bar{A}) \to \Re^N$ be a differentiable function satisfying Walras Law: $F(\mathbf{A}) \in Sp\left[\bar{P}^\sigma \begin{pmatrix} \mathbf{I}_K \\ \mathbf{A} \end{pmatrix}\right]$ for all $\mathbf{A} \in \mathscr{V}(\bar{A})$. Then we say that *the function F(.) can be rationalized, at $(\bar{A}, \bar{P}^\sigma)$, by H* *agents* if there exists an economy, as described above, with H agents whose excess demand is differentiable, at $\bar{A}$, and satisfies:

$$\begin{cases} \sum_h z^h(\bar{A}) = F(\bar{A}) \\ \sum_h D_{\mathbf{A}} z^h(\bar{A}) = D_{\mathbf{A}} F(\bar{A}) \end{cases}$$

Our main result is the following:

Theorem 3.2. *For an arbitrary pair $(\bar{A},\bar{P}^\sigma)$, any differentiable function $F:\mathscr{V}(\bar{A}) \to \Re^N$ satisfying Walras Law can be rationalized, at $(\bar{A},\bar{P}^\sigma)$, by K agents if $F(\bar{A}) \neq 0$, or by $K+1$ agents if $F(\bar{A}) = 0$.*

Hence, the rational behavior of agents does not impose any restriction on the value and the derivative of aggregate excess demand, at a point, even with the present very general specification of the budget set, and the large set of parameters with respect to which demand is defined (which includes, as we saw, not only prices but also the description of trading constraints, or asset returns in the case of incomplete markets).

Note also that the minimal number of required consumers is not related to the dimension of the parameter space but to the dimension of the individual budget sets. Thus, for $K=1$ and $K=N-1$ the parameter space has the same dimension $(N-1)$ but, for $K=1$, one consumer suffices to rationalize demand, while with $K=N-1$ we need $N-1$ consumers (if $F(\bar{A}) \neq 0$).

4. A class of preference relations

We describe in this section a family $\mathscr{U}_K$ of monotone, quasi-concave utility functions in $\Re^N$. This family can be viewed as a generalization of the class of Leontief utility functions. The decomposition result will be established by finding economies whose agents' utility functions all lie in $\mathscr{U}_K$.

Let E be an arbitrary affine linear subspace of $\Re^N$ of dimension $(N-K)$ and let $\boldsymbol{q}$ be an arbitrary N-dimensional vector. The pair $(E,\boldsymbol{q})$ completely determines a utility function in $\mathscr{U}_K$, $U(.;E,\boldsymbol{q})$, as follows:

$$U(x;E,\boldsymbol{q}) = \begin{cases} \max\limits_{\left\{ \substack{x' \leqslant x \\ x' \in E} \right\}} \boldsymbol{q}\cdot x', & \text{if there is } x' \in E \text{ such that } x' \leqslant x: \\ -\infty, & \text{otherwise} \end{cases}.$$

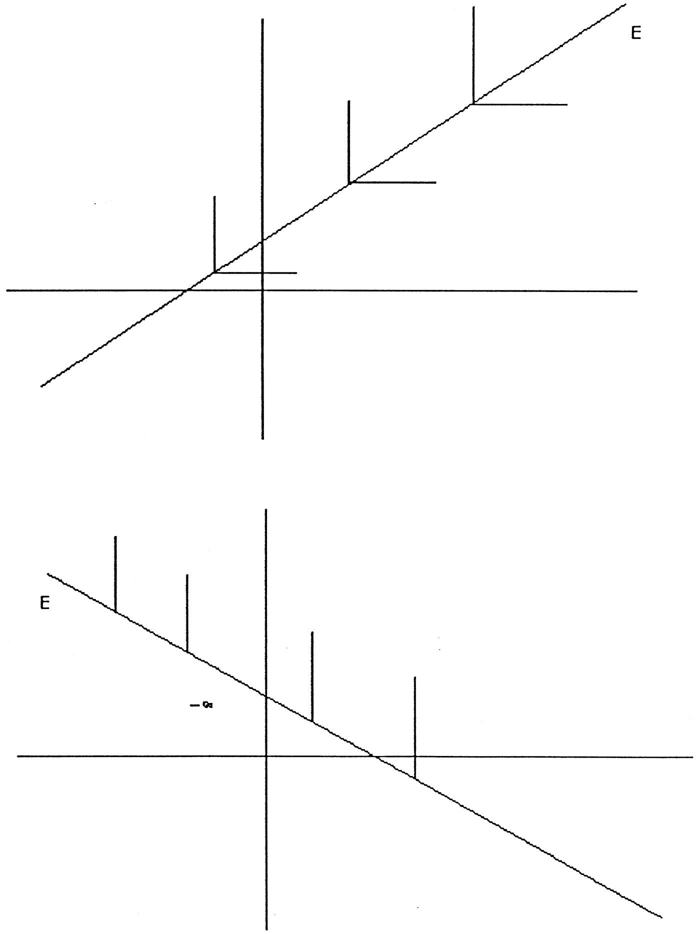

Fig. 1.

It is easily verified that the utility function $U(.;E,\boldsymbol{q})$ is monotonic and quasi-concave. We can also see that it is continuous when E is such that for all $x \in \Re^N$ there exists a vector $x' \in E$ such that $x' \leqslant x$; evidently, this property does not hold for all E, [5] and in fact not all the utility functions in the class $\mathscr{U}_K$ we are considering are continuous. [6] In the special case in which $K = N - 1$ the affine

[5] See also Fig. 1, where the shapes of some possible indifference curves in this class are illustrated for the case $N = 2$, $K = 1$.

[6] At the cost of a more elaborate description of the value of the agents' utility functions outside the space E, we could have also ensured that all preferences in the class we consider are continuous. All the results we present in this and the following section extend to such case.

subspace E has dimension 1 and the class $\mathscr{U}_K$ reduces to the class of Leontief utility functions.

When agents' preferences are described by utility functions lying in the class $\mathscr{U}_K$, the solution of the agents' choice problem has a simple form, as we show next. We will focus our attention on the values of demand in a neighborhood of $\bar{L}$.

Notice first that under the required conditions on the dimensionality of E and $\bar{L}$, the intersection of these two subspaces, $\bar{L} \cap E$, is a singleton as long as E is complementary to $\bar{L}$, or dim $(\bar{L} \oplus E) = N$ (a condition which, given $\bar{L}$, is satisfied by almost all affine subspace E of dimension $N - K$). This property extends then to all L sufficiently close to $\bar{L}$. Let $\varphi(\bar{L},E) \equiv \bar{L} \cap E$.

We prove in the following Lemma that, for any $(N-K)$-dimensional affine subspace E which is complementary to $\bar{L}$, we can find a vector $\boldsymbol{q}$ such that the value of the excess demand of an agent with utility function $U(.;E,\boldsymbol{q})$, and endowments $\omega = 0$ is given, in a sufficiently small neighborhood of $\bar{L}$, by the (unique) intersection point of L and E, $\varphi(L,E)$: [7]

Lemma 4.1. *Given $\bar{L}$, and E complementary to $\bar{L}$, there exists a vector $\boldsymbol{q} \in \Re^N$ and a neighborhood $\mathscr{N}(\bar{L},E) \subset \mathscr{N}(\bar{L})$ of $\bar{L}$, such that, for all $L \in \mathscr{N}(\bar{L},E)$*

$$\varphi(L,E) \in \underset{\{x \in L - \Re^N_+\}}{\arg\max}\ U(x;E,\boldsymbol{q})$$

Moreover, the demand set $\arg\max_{\{x \in L - \Re^N_+\}} U(x;E,\boldsymbol{q})$ *is a singleton.*

Proof. Under the assumed specification of the agent's utility function, for all $x \in L - \Re^N_+$ either there exists $x' \in E$, $x' \leqslant x$, such that $U(x';E,\boldsymbol{q}) = U(x;E,\boldsymbol{q})$, or $U(x;E,\boldsymbol{q}) = -\infty$. Hence, when the utility function is in the class $\mathscr{U}_K$, to find the solution of the agent's maximization problem (P^h), we can always limit our attention to the vectors of excess demand which are not only attainable, i.e., lie in the budget set $L - \Re^N_+$, but also belong to the subspace E.

We characterize next the properties of the convex set $E \cap (\bar{L} - \Re^N_+) = \{\boldsymbol{e} \in E : \exists \boldsymbol{l} \in \bar{L}, \boldsymbol{l} \geqslant \boldsymbol{e}\}$. We will show that this set is a displaced pointed cone, whose vertex is the (unique) intersection point $\bar{e} \in E \cap \bar{L}$.

Let $\boldsymbol{e}$ be an arbitrary vector in $E \cap (\bar{L} - \Re^N_+)$ and $\boldsymbol{l} \in \bar{L}$ be such that $\boldsymbol{l} \geqslant \boldsymbol{e}$. Since $\bar{\boldsymbol{e}} \in E \cap \bar{L}$, and E is an affine linear subspace, for any $\lambda \in \Re_+$ the vector $\bar{\boldsymbol{e}} + \lambda(\boldsymbol{e} - \bar{\boldsymbol{e}})$ is an element of E; moreover, $\bar{\boldsymbol{e}} + \lambda(\boldsymbol{l} - \bar{\boldsymbol{e}}) \in \bar{L}$ and the inequality $\bar{\boldsymbol{e}} + \lambda(\boldsymbol{l} - \bar{\boldsymbol{e}}) \geqslant \bar{\boldsymbol{e}} + \lambda\ (\boldsymbol{e} - \bar{\boldsymbol{e}})$ implies that the vector $\bar{\boldsymbol{e}} + \lambda(\boldsymbol{e} - \bar{\boldsymbol{e}})$ also belongs to $\bar{L} - \Re^N_+$. Hence, $\bar{\boldsymbol{e}} + \lambda(\boldsymbol{e} - \bar{\boldsymbol{e}}) \in E \cap (\bar{L} - \Re^N_+)$, $\forall \lambda \in \Re_+$. Thus, $E \cap (\bar{L} - \Re^N_+)$ is a displaced cone with vertex $\bar{\boldsymbol{e}}$.

[7] Evidently, it is always possible to find a vector $\omega^h \in \Re^N_{++}$ such that $\varphi(L,E) + \omega^h \in \Re^N_{++}$. It is then easy to verify that $\varphi(L,E)$ defines also the excess demand for an agent with utility function $U(.;E,\boldsymbol{q})$, endowment ω^h, and consumption space defined by the non-negative orthant $\Re^N_+$. All our results extend then immediately to the case in which the agents' consumption plans are restricted to be non-negative.

To conclude that $E \cap (\bar{L} - \Re^N_+)$ is a displaced pointed cone, with vertex $\bar{e}$, it remains to show that if $\bar{e} + v \in E \cap (\bar{L} - \Re^N_+)$ and $(\bar{e} - v) \in E \cap (\bar{L} - \Re^N_+)$, then $v = 0$. Suppose that $\bar{e} + v \in E$, $\bar{e} - v \in E$ and there exists a pair $\boldsymbol{l}, l' \in \bar{L}$ such that $\boldsymbol{l} \geqslant \bar{e} + v$ and $l' \geqslant \bar{e} - v$. Rewrite the second inequality as $v \geqslant \bar{e} - l'$. Hence, we get $\boldsymbol{l} + l' - 2\bar{e} \geq 0$. The three vectors $\boldsymbol{l}$, $-l'$, $\bar{e}$ lie in $\bar{L}$ and, since $\bar{L}$ is a linear subspace, this yields $(\boldsymbol{l} + l' - 2\bar{e}) \in \bar{L}$. In addition, since $\bar{L}$ was chosen to be in $\mathscr{G}^{N,K}_+$ (i.e., $\bar{L} \cap \Re^N_+ = \{0\}$), this implies then that $\boldsymbol{l} + l' - 2\bar{e} = 0$. But $\boldsymbol{l} - \bar{e} \geq v$, $l' - \bar{e} \geq -v$. Hence, $\bar{e} + v = \boldsymbol{l}$, $\bar{e} - v = l'$, and so it must be that $\boldsymbol{l}$, $l' \in E$. However, $E \cap L$ has a single solution, $\bar{e}$, so we get $\boldsymbol{l} = l' = \bar{e}$ and thus $v = 0$.

Since $E \cap (\bar{L} - \Re^N_+)$ is then a displaced pointed cone, with vertex $\bar{e}$, we can find a vector $\boldsymbol{q} \in \Re^N$ such that $\boldsymbol{q} \cdot \boldsymbol{e} < \boldsymbol{q} \cdot \bar{e}$, for all $\boldsymbol{e} \in E \cap (\bar{L} - \Re^N_+)$, $\boldsymbol{e} \neq \bar{e}$. Hence, for the utility function $U(.; E, \boldsymbol{q})$ in $\mathscr{U}_K$, defined by this value of $\boldsymbol{q}$ and the given subspace E, the maximum utility over $\bar{L} - \Re^N_+$ is uniquely achieved at the point $\bar{e} \in E \cap \bar{L}$.

For all L in a sufficiently small neighborhood of $\bar{L}$, $E \cap (L - \Re^N_+)$ remains a pointed cone and the maximum of $U(.; E, \boldsymbol{q})$ over $L - \Re^N_+$ is also uniquely achieved at $E \cap L$. ■

5. Proof of the main result

We prove in this section the result stated in Theorem 3.2. More precisely we will show that, for an arbitrary choice of $(\bar{A}, \bar{P}^\sigma)$, any differentiable function $F: \mathscr{V}(\bar{A}) \to \Re^N$ satisfying Walras Law can be rationalized, at the point $(\bar{A}, \bar{P}^\sigma)$, by an economy with K agents ($K+1$ if $F(\bar{A}) = 0$), all with zero endowments and utility functions belonging to the class $\mathscr{U}_K$.

We begin by changing the coordinates of the space $\Re^N$ so that in the proof, without loss of generality, we can restrict our attention to the case $\bar{A} = 0$, $\bar{P}^\sigma = \mathbf{I}_N$.

The excess demand function $z^h(\mathbf{A})$ of agent h defined with respect to these new coordinates in the neighborhood $\mathscr{V}(0)$ of $\bar{A} = 0$, implicitly defines, as we saw, the function $\zeta^h(\mathbf{A}): z^h(\mathbf{A}) = \left[\begin{pmatrix} \mathbf{I}_K \\ \mathbf{A} \end{pmatrix}\right] \zeta^h(\mathbf{A})$. Similarly, since $F(\mathbf{A}) \in Sp\left[\begin{pmatrix} \mathbf{I}_K \\ \mathbf{A} \end{pmatrix}\right] \forall \mathbf{A} \in \mathscr{V}(0)$, there exists a function $f: \mathscr{V}(0) \to \Re^K$ such that $F(\mathbf{A}) = \left[\begin{pmatrix} \mathbf{I}_K \\ \mathbf{A} \end{pmatrix}\right] f(\mathbf{A})$. Hence, to prove the result it suffices to show that we can find K agents such that the function $\Sigma_h \zeta^h(\mathbf{A})$ satisfies $\Sigma_h \zeta^h(0) = f(0)$, and $D_{\mathbf{A}}(\Sigma_h \zeta^h(0)) = D_{\mathbf{A}} f(0)$.

Let us associate to each agent $h, h = 1, \ldots, H$, a $(N-K)$-dimensional affine linear subspace of $\Re^N$, E^h, also described by the pair $(\mathbf{V}^h, \boldsymbol{v}^h)$, where $\mathbf{V}^h$ is a matrix, of dimension $N \times (N-K)$, and $\boldsymbol{v}^h$ a N-dimensional vector, such that for any $\boldsymbol{e} \in E^h$ there exists a vector $\gamma \in \Re^{N-K}$: $\boldsymbol{e} = \mathbf{V}^h\gamma + \boldsymbol{v}^h$. The intersection of E^h, identified by $(\mathbf{V}^h,\boldsymbol{v}^h)$, with $Sp\left[\begin{pmatrix}\mathbf{I}_K\\ \mathbf{A}\end{pmatrix}\right]$, which with some abuse of notation we will still denote by $\varphi(\mathbf{A},(\mathbf{V}^h,\boldsymbol{v}^h))$, is then obtained as follows:

$$\varphi\big(\mathbf{A},(\mathbf{V}^h,\boldsymbol{v}^h)\big) = \left\{\begin{matrix}\boldsymbol{e} \in \Re^N : \exists\gamma \in \Re^{N-K}, \zeta \in \Re^K \text{ such that}\\ \boldsymbol{e} = \mathbf{V}^h\gamma + \boldsymbol{v}^h\begin{pmatrix}\mathbf{I}_K\\ \mathbf{A}\end{pmatrix}\zeta\end{matrix}\right\}. \tag{5.1}$$

As long as E^h is complementary to $Sp\left[\begin{pmatrix}\mathbf{I}_K\\ 0\end{pmatrix}\right]$, or equivalently $rank\left[\mathbf{V}^h \vdots \begin{matrix}\mathbf{I}_K\\ 0\end{matrix}\right] = N$, the set $\varphi(0,(\mathbf{V}^h,\boldsymbol{v}^h))$ is a singleton. Hence, from Lemma 4.1 it follows that we can find a vector $\boldsymbol{q}^h \in \Re^N$ such that $\varphi(\mathbf{A},(\mathbf{V}^h,\boldsymbol{v}^h))$ describes the excess demand of an agent with endowment $\omega^h = 0$ and preferences described by a utility function $U(.;(\mathbf{V}^h,\boldsymbol{v}^h),\boldsymbol{q}^h)$ in $\mathscr{U}_K$, for all $\mathbf{A}$ in a sufficiently small neighborhood of $\bar{A} = 0$. Therefore, under this specification of preferences, the behavior of agent h is completely determined by the pair $(\mathbf{V}^h,\boldsymbol{v}^h)$.

It is immediate to see that any matrix $\mathbf{V}^h$ of the form $\begin{pmatrix}-\mathbf{B}_h\\ \mathbf{I}_{N-K}\end{pmatrix}$, where $\mathbf{B}^h$ is an arbitrary matrix of dimension $K \times (N-K)$, satisfies the condition $rank\left[\mathbf{V}^h \vdots \begin{matrix}\mathbf{I}_K\\ 0\end{matrix}\right] = N$. Moreover, we will now show that when $\mathbf{V}^h$ takes this form, an explicit expression for the unique element of $\varphi(\mathbf{A},(\mathbf{V}^h,\boldsymbol{v}^h))$ can be easily derived. Substituting $\begin{pmatrix}-\mathbf{B}_h\\ \mathbf{I}_{N-K}\end{pmatrix}$ for $\mathbf{V}^h$ in Eq. (5.1), we obtain

$$\begin{pmatrix}-\mathbf{B}_h\\ \mathbf{I}_{N-K}\end{pmatrix}\gamma + \boldsymbol{v}^h = \begin{pmatrix}\mathbf{I}_K\\ \mathbf{A}\end{pmatrix}\zeta. \tag{5.2}$$

Premultiplying then by $[\mathbf{I}_K \vdots \mathbf{B}^h]$ both sides of Eq. (5.2), yields

$$\left[\mathbf{I}_K \vdots \mathbf{B}^h\right]\boldsymbol{v}^h = \left(\mathbf{I}_K + \mathbf{B}^h\mathbf{A}\right)\zeta. \tag{5.3}$$

Since for $\mathbf{A}$ in a sufficiently small neighborhood of $\bar{A} = 0$ the matrix $(\mathbf{I}_K + \mathbf{B}^h\mathbf{A})$ is invertible, solving Eq. (5.3) for ζ we get $(\mathbf{I}_K + \mathbf{B}^h\mathbf{A})^{-1}[\mathbf{I}_K \vdots \mathbf{B}^h]\boldsymbol{v}^h$. We see therefore that, for $\mathbf{V}^h = \begin{pmatrix}-\mathbf{B}_h\\ \mathbf{I}_{N-K}\end{pmatrix}$, the expression in (5.1) simplifies as follows:

$$\varphi\big(\mathbf{A},(\mathbf{V}^h,\boldsymbol{v}^h)\big) = \left[\begin{pmatrix}\mathbf{I}_K\\ \mathbf{A}\end{pmatrix}\right]\left(\mathbf{I}_K + \mathbf{B}^h\mathbf{A}\right)^{-1}\left(\boldsymbol{v}^{1,h} + \mathbf{B}^h\boldsymbol{v}^{2,h}\right) \tag{5.4}$$

where the vector $\boldsymbol{v}^h$ has been partitioned so that its first K components are denoted by $\boldsymbol{v}^{1,h}$ and the last $(N-K)$ components by $\boldsymbol{v}^{2,h}$. The expression of

$\varphi(\mathbf{A},(\mathbf{V}^h,\boldsymbol{v}^h))$ in Eq. (5.4) constitutes then the excess demand function of agent h, with utility function $U(.;(\mathbf{V}^h,\boldsymbol{v}^h),\boldsymbol{q}^h)$, for $\mathbf{V}^h = \begin{pmatrix} -\mathbf{B}_h \\ \mathbf{I}_{N-K} \end{pmatrix}$.

From the above argument we readily obtain also the expression of the function $\zeta^h(\mathbf{A})$:

$$\zeta^h(\mathbf{A}) = \left(\mathbf{I}_K + \mathbf{B}^h\mathbf{A}\right)^{-1}\left(\boldsymbol{v}^{1,h} + \mathbf{B}^h\boldsymbol{v}^{2,h}\right) \tag{5.5}$$

so that its value at $\bar{A}=0$ is given by

$$\zeta^h(0) = \boldsymbol{v}^{1,h} + \mathbf{B}^h\boldsymbol{v}^{2,h}. \tag{5.6}$$

Differentiating then Eq. (5.5) with respect to $\mathbf{A}$, we get:

$$D_{\mathbf{A}}\,\zeta^h(\mathbf{A}) + D_{\mathbf{A}}\left(\mathbf{B}^h\mathbf{A}\,\zeta^h(\mathbf{A})\right) = 0. \tag{5.7}$$

Hence, $D_{\mathbf{A}}\,\zeta^h(\mathbf{A})$ is implicitly defined by Eq. (5.7); to find $D_{\mathbf{A}}\,\zeta^h(\underline{0})$ we simply have to develop the expression $D_{\mathbf{A}}(B^h\mathbf{A}\,\zeta^h(\mathbf{A}))$, and evaluate it at $\bar{A}=0$.

Let $b^h_{i,j}$ be the (i,j)-th element of the $(K\times(N-K))$ matrix $\mathbf{B}^h$, $a_{j,k}$ the (j,k)-th element of the $((N-K)\times K)$ matrix $\mathbf{A}$, and $\zeta^h_k(\mathbf{A})$ the k-th component of the K-dimensional vector describing the value of the function $\zeta(\mathbf{A})$ at $\mathbf{A}$. The i-th element of $B^h\mathbf{A}\,\zeta^h(\mathbf{A})$, a vector of dimension K, is then given by the following expression: $[\sum_{k=1}^{K}\sum_{j=1}^{N-K} b^h_{i,j}\,a_{j,k}\,\zeta^h_k(\mathbf{A})]$. The derivative of this term with respect to $a_{j,k}$, evaluated at $\bar{A}=0$, is simply $b^h_{i,j}\,\zeta^h_k(0)$, and the derivative with respect to all the $(N-K)K$ elements of $\mathbf{A}$ is obtained by letting j and k vary: $[b^h_{i,j}\,\zeta^h_k(0)]_{j=1,\dots,N-K,k=1,\dots,K}$. By the same argument we see that, as i varies, this expression also describes the derivatives of the other components of the vector $B^h\mathbf{A}\,\zeta^h(\mathbf{A})$.

We get so

$$D_{\mathbf{A}}\,\zeta^h(0) = -\left[\,\zeta^h_1(0)\mathbf{B}^h,\dots,\zeta^h_k(0)\mathbf{B}^h,\dots,\zeta^h_K(0)\mathbf{B}^h\right]. \tag{5.8}$$

The matrix $D_{\mathbf{A}}\,\zeta^h(0)$ is of dimension $K\times(N-K)K$. In Eq. (5.8) its terms have been rearranged so that the k-th block of this matrix, $-\zeta^h_k(0)\mathbf{B}^h$, is the derivative of the k-th component of $\zeta^h(\mathbf{A})$ with respect to all the elements of $\mathbf{A}$. [8]

On the basis of the expressions of $\zeta^h(0)$ and $D_{\mathbf{A}}\,\zeta^h(0)$ derived in Eqs. (5.6) and (5.8), we can then compute $\sum_h \zeta^h(0)$ and $D_{\mathbf{A}}(\sum_h \zeta^h(0))$. To complete the proof we simply have to show that we can always find a collection of K $(K+1$ when

[8] The special structure of $D_{\mathbf{A}}\,\zeta^h(0)$ in Eq. (5.8) depends on the particular specification of preferences we adopted, and on the change in coordinates we made. With utility functions in $\mathscr{U}_K$, as we saw, the substitution effect is null and demand is simply determined by the level of consumption an agent can afford, given his income and the form of the trading constraints described by $\mathbf{A}$, in the subspace E^h describing the agent's direction of preferences. Hence, a change in $\mathbf{A}$ will only influence demand by its effect on the value of the intersection point $\varphi(E^h, \mathbf{A})$ and, for any choice of one of the K coordinates of ζ, its derivative with respect to $\mathbf{A}$ will be an arbitrary matrix. On the other hand the symmetry of the point, in the new coordinate space, at which the derivative is evaluated $(\bar{A}=0)$ implies that the derivative will be the same for each of the K generating coordinates.

$f(0)=0)$ pairs $(B^h, \boldsymbol{v}^h) \in \Re^{K\times(N-K)} \times \Re^N$ which satisfy the following conditions:

$$\begin{cases} \sum_h (\boldsymbol{v}^{1,h} + \mathbf{B}^h \boldsymbol{v}^{2,h}) = f(0) \\ -\sum_h [(\boldsymbol{v}^{1,h} + \mathbf{B}^h \boldsymbol{v}^{2,h})_1 \mathbf{B}^h, \dots, (\boldsymbol{v}^{1,h} + \mathbf{B}^h \boldsymbol{v}^{2,h})_k \mathbf{B}^h, \dots, (\boldsymbol{v}^{1,h} + \mathbf{B}^h \boldsymbol{v}^{2,h})_K \mathbf{B}^h] = D_{\mathbf{A}} f(0) \end{cases} \tag{5.9}$$

where we used Eq. (5.6) to substitute for $\zeta_k^h(0)$, and $(\boldsymbol{v}^{1,h} + \mathbf{B}^h \boldsymbol{v}^{2,h})_k$ denotes the k-th element of the vector $(\boldsymbol{v}^{1,h} + \mathbf{B}^h \boldsymbol{v}^{2,h})$, $k=1,\dots,K$.

Consider first the case in which $f(0) \neq 0$; without loss of generality suppose $f_k(0)=0$ for $k=1,\dots,\bar{k}-1$ and $f_k(0) \neq 0$ for $k=\bar{k},\dots K$. For $h=\bar{k}+1,\dots,K$, set the matrix $\mathbf{B}^h$ equal to $-1/f_h(0)[D_{\mathbf{A}} f(0)]_h$, where $[D_{\mathbf{A}} f(0)]_h$ denotes the h-th block, of dimension $K \times (N-K)$, of the matrix $[D_{\mathbf{A}} f(0)]$, and the vector $\boldsymbol{v}^h$ such that $(\boldsymbol{v}^{1,h} = f_h(0)\boldsymbol{i}_h, \boldsymbol{v}^{2,h}=0)$, where $\boldsymbol{i}_h$ denotes the K- dimensional vector $(0,\dots 0,1,0,\dots 0)$ with 1 in the h-th place. Let then $\mathbf{B}^{\bar{k}} = -1/f_{\bar{k}}(0)[D_{\mathbf{A}} f(0)]_{\bar{k}}$, $(v^{1,\bar{k}} = f_{\bar{k}}(0)\boldsymbol{i}_{\bar{k}} - \sum_{h=1}^{\bar{k}-1} \boldsymbol{i}_h, v^{2,\bar{k}}=0)$ and, for $h=1,\dots,\bar{k}-1$, $\mathbf{B}^h = -[D_{\mathbf{A}} f(0)]_h - 1/f_{\bar{k}}(0)[D_{\mathbf{A}} f(0)]_{\bar{k}}$, and $(\boldsymbol{v}^{1,h} = \boldsymbol{i}_h, \boldsymbol{v}^{2,h}=0)$. It is immediate to see that when these values are substituted for $(B^h, \boldsymbol{v}^h)_{h=1,\dots,K}$ in system (5.9), all the equations are satisfied.

Finally, when $f(0)=0$, to solve system (5.9) $(K+1)$ agents are needed. In such case we can set $\mathbf{B}^{K+1}=0$, $(v^{1,(K+1)} = -\sum_{h=1}^{K} \boldsymbol{i}_h, v^{2,K+1}=0)$ and, for all other agents $h=1,\dots,K$, $\mathbf{B}^h = -[D_{\mathbf{A}} f(0)]_h$, and $(\boldsymbol{v}^{1,h} = \boldsymbol{i}_h, \boldsymbol{v}^{2,h}=0)$, to see that all conditions in system (5.9) hold.

This completes the proof of Theorem 3.2.

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