In honour of Martin Shubik

Generic finiteness of equilibrium payoffs for bimatrix games

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ABSTRACT

It is shown that in any affine space of payoff matrices the equilibrium payoffs of bimatrix games are generically finite.

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The paper that follows is based on notes written in December of 1990. I am indebted to A. McLennan and W. Zame for their helpful comments (see also Govindan and McLennan, 2001). The paper was written up and presented at the Cowles Foundation in February 1994. The text below is the text of 1994. The topic belongs to a general area that has always been of interest to Martin Shubik and, thus, it is for me a pleasure and an honour to offer this contribution to him.

1. Introduction

It is well known that if the entries of the normal form of a bimatrix game (in fact, of games with any number of players) can be perturbed independently then generically there is a finite number of equilibria (see van Damme, 1983; Fudenberg and Tirole, 1991).

Kreps and Wilson (1982) criticized this result as not being very helpful when the normal form is derived from an extensive form since then many strategies lead to the same final node. In fact, even if the payoffs of the final nodes can be perturbed independently the finiteness of the number of equilibria is not a generic property, yet Kreps–Wilson showed that the finiteness of the number of *equilibrium payoffs* is a generic property in this case. In many applications this suffices.

However, the Kreps–Wilson criticism to the normal form result can be reiterated: it is very strong to assume that final nodes utilities can be perturbed independently. Typically, many final nodes will correspond to the same final position of a game and the most natural assumption may be that the final utilities depend on the position and not on the particular history of play (i.e. the node).

In this note it is showed that for bimatrix games it is still true that the number of equilibrium payoffs is generically finite, in whatever way the payoffs of final nodes are tied together by (linear) constraints.

The interest of this result is limited (or enhanced?) by the fact that it does not extend to more than two players. McLennan (1990) has a clever example to that effect.

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2. The result

Proposition.   *Let $\mathcal{U}$ be an arbitrary affine space of matrices. Then for a.e. payoff matrices $U_1, U_2 \in \mathcal{U}$ for two players the number of equilibrium payoffs of the corresponding bimatrix game is finite.*

Remark.   It is indispensable for this result that U be the same for the two players (interpretation: if two final nodes correspond to two different positions then one can perturb the payoffs of both players). This is clear: take two payoffs matrices U_1, U_2 with a continuum of payoff equilibrium. Then $\mathcal{U}_1 = \{U_1\}$ and $\mathcal{U}_2 = \{U_2\}$ are affine spaces violating the desired property.

3. Proof of the proposition

Without loss of generality it is enough to show that almost every pair of payoff matrices has at most a unique completely mixed equilibrium payoff. In general, we should choose U_1, U_2 such that all of its submatrices belong to the generic set for the projection of U on the corresponding coordinates.

Following are the first two preliminary lemmas.

Lemma 1.   *Let $\mathcal{A}$ be a convex set of matrices. Then the function $A \to$ rank A is constant a.e. on $\mathcal{A}$.*

Proof.   This follows from the analyticity of the determinant function. Let k be the maximal rank over the matrices in $\mathcal{A}$. Suppose that we fix a $A \in \mathcal{A}$ with maximal rank. Say that its first k columns are linearly independent. By Fubini's theorem, it is enough to show that for all $B \in \mathcal{A}$ and a.e. $\alpha \in [0,1]$ we have that rank$(\alpha A + (1-\alpha)B) = k$. Let C_α be the matrix formed by the first k columns of $\alpha A + (1-\alpha)B$. Define $f(\alpha) = \det(C_\alpha^{\mathrm{T}} C_\alpha)$. Note that $f(1) \neq 0$ and that f is a polynomial on α. Hence, $f(\alpha) \neq 0$ for a.e. α and so rank$(\alpha A + (1-\alpha)B) = k$ for a.e. α.   □

Lemma 2.   *Let $\mathcal{A}$ be a convex set of matrices. Then either* (1) $Av = e(= (1, \ldots, 1))$ *has a solution for a.e. $A \in \mathcal{A}$, or* (2) $Av = e$ *does not have a solution for a.e. $A \in \mathcal{A}$.*

Proof.   Let k be the rank that prevails a.e. over $\mathcal{A}$ (Lemma 1). Consider the (isometric to $\mathcal{A}$) convex set of matrices $\mathcal{A}' = \{[A, e] : A \in \mathcal{A}\}$. Then either the a.e. rank of $\mathcal{A}'$ is k and then (1) holds, or it is $k+1$ and then (2) holds.   □

We now prove the proposition.

Consider any pair of payoff matrices U_1, U_2 with the property that, first, they belong to the generic set of Lemma 2 and, second, $U_1^{\mathrm{T}}, U_2^{\mathrm{T}}$ belong to the generic set of Lemma 2 for $\mathcal{U}^T = \{A^{\mathrm{T}} : A \in U\}$.

If possibility (2) of Lemma 2 holds for U_1 and U_2^{T}, then either there is no completely mixed solution or the unique equilibrium payoff is $(0, 0)$.

Suppose that possibility (1) of Lemma 2 holds for U_1 (hence for U_2). Then there is v_2 such that $U_2 v_2 = e$. Take now *any* solution (p_1, α) to $U_2^{\mathrm{T}} p_1 = \alpha e$, $\ p_1 \cdot e = 1$. Then:

$$1 = e^{\mathrm{T}} \cdot p_1 = v_2 \cdot U_2^{\mathrm{T}} p_1 = \alpha(v_2 \cdot e),$$

and so the only (completely mixed) equilibrium payoff for player 2 is

$$\alpha = \frac{1}{v_2 \cdot e}.$$

Since then $\alpha \neq 0$ this also implies that possibility (1) of Lemma 2 holds for U_2^{T} (hence for U_1^{T}). We now repeat the argument. Choose v_1 such that $U_1^{\mathrm{T}} v_1 = e$. Consider any solution (p_2, β) to $U_1 p_2 = \beta e$, $p_2 \cdot e = 1$. Then:

$$1 = e \cdot p_2 = v_1 \cdot U_1 p_2 = \beta(v_1 \cdot e),$$

and so the only (completely mixed) equilibrium payoff for player 1 is

$$\beta = \frac{1}{v_1 \cdot e}.$$

Finally, the argument is completely symmetric (in fact just a matter of labelling) if possibility (1) of Lemma 2 holds for U_2^{T}.

References

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